Communications in Mathematical Research
31(1)(2015), 81–88

Constructing the Cocyclic Structures for Crossed Coproduct Coalgebras

Chen Quan-guo[1] and Wang Ding-guo[2]

(*1. School of Mathematics and Statistics, Yili Normal University, Yining, Xinjiang, 835000*)

(*2. School of Mathematical Sciences, Qufu Normal University, Qufu, Shandong, 273165*)

Communicated by Du Xian-kun

Abstract: In this paper, we construct a cocylindrical object associated to two coalgebras and a cotwisted map. It is shown that there exists an isomorphism between the cocyclic object of the crossed coproduct coalgebra induced from two coalgebras with a cotwisted map and the cocyclic object related to the diagonal of the cocylindrical object.

Key words: Hopf algebra, crossed coproduct coalgebra, cocyclic object

2010 MR subject classification: 16W30

Document code: A

Article ID: 1674-5647(2015)01-0081-08

DOI: 10.13447/j.1674-5647.2015.01.09

1 Introduction

Getzler and Jones[1] introduced a method to compute the cyclic homology of smash product algebras $A \rtimes G$, where G is a group that acts on an algebra A by automorphisms. This method is based on constructing a cylindrical object $A\sharp G$, and shows that $\Delta(A\sharp G) \cong C_\bullet(A \rtimes G)$, where Δ is the diagonal and $C_\bullet$ the cyclic object functor. Then by using the Eilenberg-Zilber theorem for cylindrical object, they obtained a quasi-isomorphism of mixed complexes $\Delta(A\sharp G) \cong \mathrm{Tot}_\bullet(A\sharp G)$ and a spectral sequence converging to $HC_\bullet(A \rtimes G)$. This method has been used to computer the cyclic homology of some types of algebras in [2–4].

From the perspective of duality, it is necessary to consider the cocyclic structures of some coalgebras. It is the starting point of this paper to construct the cocyclic structures of crossed coproducts with invertible cotwisted maps, of which twisted smash coproducts in sense of Wang and Li[5] are special cases.

Received date: April 9, 2013.

Foundation item: The NSF (11261063, 11171183) of China, the Fund (2012ZDXK03) of the Key Disciplines in the General Colleges and Universities of Xinjiang Uygur Autonomous Region, the Foundation (2013721043) for Excellent Youth Science and Technology Innovation Talents of Xinjiang Uygur Autonomous Region and the NSF (ZR2011AM013) of Shandong Province.

E-mail address: cqg211@163.com (Chen Q G).

This paper is organized as follows: In Section 2, we recall the basic concepts of cocylindrical objects. Then crossed products with cotwisted maps are discussed in Section 3. The key content of this paper is to construct cocylindrical objects in Section 4. Finally, It is shown that there exists an isomorphism between the cocyclic object of the crossed coproduct coalgebra $A \times_T B$ and $\Delta(A\natural_T B)$ the cocyclic object related to the diagonal of $A\natural_T B$ In Section 5.

2 Cocylindrical Objects

Let us recall cocyclic objects. If $\mathcal{A}$ is any category, a paracocyclic object in $\mathcal{A}$ is a sequence of objects $\mathcal{A}_0$, $\mathcal{A}_1$, $\cdots$ together with coface operators $\partial^i : \mathcal{A}_n \to \mathcal{A}_{n+1}$ $(i = 0, 1, \cdots, n+1)$, codegeneracy operators, $\sigma^i : \mathcal{A}_n \to \mathcal{A}_{n-1}$ $(i = 0, 1, \cdots, n-1)$ and cyclic operators $\tau_n : \mathcal{A}_n \to \mathcal{A}_n$, where these operators satisfy the cosimplicial conditions and the following extra relations:

$$\tau_{n+1}\partial^i = \partial^{i-1}\tau_n, \quad 1 \le i \le n, \qquad \tau_{n+1}\partial^0 = \partial^n,$$
$$\tau_{n-1}\sigma^i = \sigma^{i-1}\tau_n, \quad 1 \le i \le n, \qquad \tau_{n-1}\sigma^0 = \sigma^n\tau_n^2.$$

If, in addition, $\tau_n^{n+1} = \mathrm{id}_n$, then we have a cocyclic object in the sense of Connes[6].

A bi-paracocyclic object in a category $\mathcal{A}$ is a double sequence $\mathcal{A}(p, q)$ of objects of $\mathcal{A}$ and operators $\partial_{p,q}$, $\sigma_{p,q}$, $\tau_{p,q}$ and $\bar{\partial}_{p,q}$, $\bar{\sigma}_{p,q}$, $\bar{\tau}_{p,q}$ such that for all $p \ge 0$, $q \ge 0$,

$$\mathcal{B}_p(q) = \{\mathcal{A}(p, q),\ \sigma^i_{p,q},\ \partial^i_{p,q},\ \tau_{p,q}\}, \qquad \mathcal{B}_q(p) = \{\mathcal{A}(p, q),\ \bar{\sigma}^i_{p,q},\ \bar{\partial}^i_{p,q},\ \bar{\tau}_{p,q}\}$$

are paracocyclic objects in $\mathcal{A}$ and every horizontal operator commutes with every vertical operator. We say that a bi-paracocyclic object is cocylindrical, if for all $p, q \ge 0$,

$$\bar{\tau}^{p+1}_{p,q}\tau^{q+1}_{p,q} = \mathrm{id}_{p,q}.$$

If A is a bi-paracocyclic object in a category $\mathcal{A}$, the paracocyclic object related to the diagonal of A is denoted by ΔA. So the paracocyclic operators on $\Delta A(n) = A(n, n)$ are $\bar{\partial}^i_{n,n+1}\partial^i_{n,n}$, $\bar{\sigma}^i_{n,n-1}\sigma^i_{n,n}$ and $\bar{\tau}^i_{n,n}\tau_{n,n}$, When A is a cocylindrical object, we conclude that $(\bar{\tau}_{n,n}\tau_{n,n})^{n+1} = \mathrm{id}_{n,n}$. So ΔA is a cocyclic object.

Throughout this paper, we works over a field k. All algebras and coalgebras are over k. The undecorated tensor product $\otimes$ means tensor product over k. Let C be a coalgebra. We use Sweedler's notation:

$\Delta(c) = c_{(1)} \otimes c_{(2)},\ \Delta^2(c) = (\Delta \otimes \mathrm{id})\Delta(c) = c_{(1)} \otimes c_{(2)} \otimes c_{(3)},\ \cdots,\ \Delta^n c = (\mathrm{id} \otimes \Delta) \circ \Delta^{n-1}(c)$, where summation is omitted.

3 Crossed Coproducts with Cotwisted Maps

Let $(A,\ \Delta_A,\ \varepsilon_A)$ be a coalgebra and $(B,\ \Delta_B,\ \varepsilon_B)$ be a coalgebra. Give a linear map $T : A \otimes B \to B \otimes A$. Then $A \otimes B$ has the coproduct:

$$\Delta(a \otimes b) = a_{(1)} \otimes b_{(1)T} \otimes a_{(2)T} \otimes b_{(2)},$$

where $b_T \otimes a_T = b_U \otimes a_U = b_V \otimes a_V = b_X \otimes a_X = \cdots = T(a \otimes b)$ for all $b \in B,\ a \in A$. We say that $A \otimes B$ is a crossed coproduct which is denoted by $A \times_T B$, if $A \otimes B$ is a coalgebra with the counit $\varepsilon_A \otimes \varepsilon_B$. In this case, the map T is called a cotwisted map.

Lemma 3.1 *Using the notation as above, $A \times_T B$ is a crossed coproduct if and only if the following conditions hold: for all $a \in A$ and $b \in B$,*

$$\varepsilon_A(a_T)b_T = \varepsilon_A(a)b, \qquad \varepsilon_B(b_T)a_T = \varepsilon_B(b)a; \tag{3.1}$$

$$b_{T_1(1)} \otimes b_{T_1(2)} \otimes a_{T_1} = b_{(1)T_1} \otimes b_{(2)T_2} \otimes a_{T_1T_2}; \tag{3.2}$$

$$b_T \otimes a_{T(1)} \otimes a_{T(2)} = b_{T_1T_2} \otimes a_{(1)T_2} \otimes a_{(2)T_1}. \tag{3.3}$$

4 The Cocylindrical Object $A\natural_T B$

In what follows, we always assume that the cotwisted map T is invertible, and its inverse is denoted by T^{-1}.

Lemma 4.1 *The cotwisted map T satisfies* (3.1)–(3.3) *if and only if its inverse T^{-1} satisfies the following identities: for all $a \in A$ and $b \in B$,*

$$\varepsilon_A(a_{T^{-1}})b_{T^{-1}} = \varepsilon_A(a)b, \qquad \varepsilon_B(b_{T^{-1}})a_{T^{-1}} = \varepsilon_B(b)a; \tag{4.1}$$

$$b_{T_1^{-1}(1)} \otimes b_{T_1^{-1}(2)} \otimes a_{T_1^{-1}} = b_{(1)T_1^{-1}} \otimes b_{(2)T_2^{-1}} \otimes a_{T_2^{-1}T_1^{-1}}; \tag{4.2}$$

$$b_{T^{-1}} \otimes a_{T^{-1}(1)} \otimes a_{T^{-1}(2)} = b_{T_1^{-1}T_2^{-1}} \otimes a_{(1)T_1^{-1}} \otimes a_{(2)T_2^{-1}}. \tag{4.3}$$

Example 4.1 Let B be a Hopf algebra with the invertible antipode S and A be a left B-comodule coalgebra. Define the cotwisted map

$$T : A \otimes B \to B \otimes A, \qquad a \otimes b \mapsto a_{\langle -1\rangle} b \otimes a_{\langle 0\rangle}.$$

Then we have the crossed coproduct coalgebra $A \times_T B$ with the coproduct

$$\Delta(a \times_T b) = a_{(1)} \times_T a_{(2)\langle -1\rangle} b_{(1)} \otimes a_{(2)\langle 0\rangle} \times_T b_{(2)}, \qquad a \in A,\ b \in B.$$

Note that the cotwisted map T is invertible, and its inverse T^{-1} is

$$T^{-1} : B \otimes A \to A \otimes B, \qquad b \otimes a \mapsto a_{\langle 0\rangle} \otimes S^{-1}(a_{\langle -1\rangle})b.$$

Example 4.2[5] Let B be a Hopf algebra with the invertible antipode S and A be a B-bicomodule coalgebra. Let

$$\rho_l : A \to B \otimes A,\ a \mapsto a_{\langle -1\rangle} \otimes a_{\langle 0\rangle} \quad \text{and} \quad \rho_r : A \to A \otimes B,\ a \mapsto a_{[0]} \otimes a_{[1]}$$

be left and right H-comodule structures on A, respectively. Then we have the crossed coproduct coalgebra $A \times_T B$ with the coproduct

$$\Delta(a \times_T b) = a_{(1)} \times_T a_{(2)\langle -1\rangle} b_{(1)} S(a_{(2)\langle 0\rangle[1]}) \otimes a_{(2)\langle 0\rangle[0]} \times_T b_{(2)}, \qquad a \in A,\ b \in B,$$

where the cotwisted map

$$T : A \otimes B \to B \otimes A, \qquad a \otimes b \mapsto a_{\langle -1\rangle} b S(a_{\langle 0\rangle[1]}) \otimes a_{\langle 0\rangle[0]}.$$

We can check that T is invertible with its inverse T^{-1} given as follows:

$$T^{-1} : B \otimes A \to A \otimes B, \quad b \otimes a \mapsto a_{[0]\langle 0\rangle} \otimes S^{-1}(a_{[0]\langle -1\rangle}) b S^2(a_{[1]}), \quad a \in A,\ b \in B.$$

Proof. For all $a \in A$ and $b \in B$, we have

$$\begin{aligned} T^{-1} \circ T(a \otimes b) &= a_{\langle 0\rangle[0][0]\langle 0\rangle} \otimes S^{-1}(a_{\langle 0\rangle[0][0]\langle -1\rangle}) a_{\langle -1\rangle} b S(a_{\langle 0\rangle[1]}) S^2(a_{\langle 0\rangle[0][1]}) \\ &= a_{\langle 0\rangle[0]\langle 0\rangle} \otimes S^{-1}(a_{\langle 0\rangle[0]\langle -1\rangle}) a_{\langle -1\rangle} b S(a_{\langle 0\rangle[1](2)}) S^2(a_{\langle 0\rangle[1](1)}) \\ &= a_{\langle 0\rangle\langle 0\rangle} \otimes S^{-1}(a_{\langle 0\rangle\langle -1\rangle}) a_{\langle -1\rangle} b \\ &= a_{\langle 0\rangle} \otimes S^{-1}(a_{\langle -1\rangle(2)}) a_{\langle -1\rangle(1)} b \\ &= a \otimes b. \end{aligned}$$

So we get $T^{-1}\circ T=\mathrm{id}$. And $T\circ T^{-1}=\mathrm{id}$ can be checked similarly.

For the sake of convenience of expression, for $a_i\in A$, $b_i\in B$, $i,j\in\mathbf{N}$, we denote

$$T(a_i\otimes b_j)=b_{jT_{ij}}\otimes a_{iT_{ij}},\qquad T^{-1}(b_j\otimes a_i)=a_{iT_{ji}^{-1}}\otimes b_{jT_{ji}^{-1}}.$$

Now we introduce the cocylindrical module

$$A\natural_T B=\{B^{\otimes(p+1)}\otimes A^{\otimes(q+1)}\}_{p,q\geq 0},$$

where A, B are both coalgebras and T is a cotwisted map. We define the operators $\partial_{p,q}$, $\sigma_{p,q}$, $\tau_{p,q}$ and $\bar\partial_{p,q}$, $\bar\sigma_{p,q}$, $\bar\tau_{p,q}$ as follows: for all $a_i\in A$ and $b_j\in B$,

$$\partial^i_{p,q}(b_0,\cdots,b_p\mid a_0,\cdots,a_q)=(b_0,\cdots,b_{i-1},\Delta_B(b_i),b_{i+1},\cdots,b_p\mid a_0,\cdots,a_q),$$

$$\partial^{p+1}_{p,q}(b_0,\cdots,b_p\mid a_0,\cdots,a_q)=(b_{0(2)},\cdots,b_p,b_{0(1)T_{q0}\cdots T_{00}}\mid a_{0T_{00}},\cdots,a_{qT_{q0}}),$$

$$\sigma^i_{p,q}(b_0,\cdots,b_p\mid a_0,\cdots,a_q)=(b_0,\cdots,b_{i-1},\varepsilon_B(b_i),b_{i+1},\cdots,b_p\mid a_0,\cdots,a_q),$$

$$\tau_{p,q}(b_0,\cdots,b_p\mid a_0,\cdots,a_q)=(b_1,\cdots,b_p,b_{0T_{q0}\cdots T_{00}}\mid a_{0T_{00}},\cdots,a_{qT_{q0}}).$$

$$\bar\partial^j_{p,q}(b_0,\cdots,b_p\mid a_0,\cdots,a_q)=(b_0,\cdots,b_p\mid a_0,\cdots,a_{i-1},\Delta_A(a_i),a_{i+1},\cdots,a_q),$$

$$\bar\partial^{q+1}_{p,q}(b_0,\cdots,b_p\mid a_0,\cdots,a_q)=(b_{0T_{00}^{-1}},\cdots,b_{pT_{p0}^{-1}}\mid a_{0(2)},\cdots,a_q,a_{0(1)T_{p0}^{-1}\cdots T_{00}^{-1}}),$$

$$\bar\sigma^i_{p,q}(b_0,\cdots,b_p\mid a_0,\cdots,a_q)=(b_0,\cdots,b_p\mid a_0,\cdots,a_{i-1},\varepsilon_A(a_i),a_{i+1},\cdots,a_q),$$

$$\bar\tau_{p,q}(b_0,\cdots,b_p\mid a_0,\cdots,a_q)=(b_{0T_{00}^{-1}},\cdots,b_{pT_{p0}^{-1}}\mid a_1,\cdots,a_q,a_{0T_{p0}^{-1}\cdots T_{00}^{-1}}).$$

where $0\leq i\leq p,\ 0\leq j\leq q$.

Theorem 4.1 *Let A and B be both coalgebras, and T an invertible cotwisted map. Then $A\natural_T B$ with the operators defined above is a cocylindrical object.*

Proof. We only check the commutativity of the cyclic operators and the cocylindrical condition. First, we check that every horizontal operator commutes with every vertical operator, that is, $\tau_{p,q}\bar\tau_{p,q}=\bar\tau_{p,q}\tau_{p,q}$. Indeed, for all $a_i\in A$ and $b_j\in B$, we have

$$\begin{aligned}&\tau_{p,q}\bar\tau_{p,q}(b_0,\cdots,b_p\mid a_0,\cdots,a_q)\\=&\tau_{p,q}(b_{0T_{00}^{-1}},\cdots,b_{pT_{p0}^{-1}}\mid a_1,\cdots,a_q,a_{0T_{p0}^{-1}\cdots T_{00}^{-1}})\\=&(b_{1T_{10}^{-1}},\cdots,b_{pT_{p0}^{-1}},b_{0T_{q0}\cdots T_{10}}\mid a_{1T_{10}},\cdots,a_{qT_{q0}},a_{0T_{p0}^{-1}\cdots T_{10}^{-1}})\end{aligned}$$

and

$$\begin{aligned}&\bar\tau_{p,q}\tau_{p,q}(b_0,\cdots,b_p\mid a_0,\cdots,a_q)\\=&\bar\tau_{p,q}(b_1,\cdots,b_p,b_{0T_{q0}\cdots T_{00}}\mid a_{0T_{00}},\cdots,a_{qT_{q0}})\\=&(b_{1T_{10}^{-1}},\cdots,b_{pT_{p0}^{-1}},b_{0T_{q0}\cdots T_{10}}\mid a_{1T_{10}},\cdots,a_{qT_{q0}},a_{0T_{p0}^{-1},\cdots,T_{10}^{-1}}).\end{aligned}$$

From above, it follows that $\tau^{p+1}_{p,q}\bar\tau^{q+1}_{p,q}=\bar\tau^{q+1}_{p,q}\tau^{p+1}_{p,q}$ for any $p,q\in\mathbf{N}$. Then, we check that $\tau^{p+1}_{p,q}\bar\tau^{q+1}_{p,q}=\mathrm{id}_{p,q}$. In fact,

$$\begin{aligned}&\tau^{p+1}_{p,q}\bar\tau^{q+1}_{p,q}(b_0,\ \cdots,\ b_p\mid a_0,\ \cdots,\ a_q)\\=&\ \tau^{p+1}_{p,q}\bar\tau^{q}_{p,q}(b_{0T_{00}^{-1}},\ \cdots,\ b_{pT_{p0}^{-1}}\mid a_1,\ \cdots,\ a_q,\ a_{0T_{p0}^{-1}\cdots T_{00}^{-1}})\\=&\ \tau^{p+1}_{p,q}\bar\tau^{q-1}_{p,q}(b_{0T_{00}^{-1}T_{01}^{-1}},\ \cdots,\ b_{pT_{p0}^{-1}T_{p1}^{-1}}\mid a_2,\ \cdots,\ a_q,a_{0T_{p0}^{-1}\cdots T_{00}^{-1}},\ a_{1T_{p1}^{-1}\cdots T_{11}^{-1}T_{01}^{-1}})\end{aligned}$$

$$
\begin{aligned}
&= \cdots \\
&= \tau_{p,q}^{p+1}\bar{\tau}_{p,q}^{q-k}(b_{0T_{00}^{-1}T_{01}^{-1}\cdots T_{0k}^{-1}},\ \cdots,\ b_{pT_{p0}^{-1}\cdots T_{pk}^{-1}} \\
&\qquad\qquad\qquad \mid a_{k+1},\ \cdots,\ a_q,\ a_{0T_{p0}^{-1}\cdots T_{10}^{-1}T_{00}^{-1}},\ \cdots,\ a_{kT_{pk}^{-1}\cdots T_{1k}^{-1}T_{0k}^{-1}}) \\
&= \cdots \\
&= \tau_{p,q}^{p+1}\bar{\tau}_{p,q}(b_{0T_{00}^{-1}T_{01}^{-1}\cdots T_{0(q-1)}^{-1}},\ \cdots,\ b_{pT_{p0}^{-1}T_{p1}^{-1}\cdots T_{p(q-1)}^{-1}} \\
&\qquad\qquad\qquad \mid a_q,\ a_{0T_{p0}^{-1}\cdots T_{10}^{-1}T_{00}^{-1}},\ \cdots,\ a_{(q-1)T_{p(q-1)}^{-1}\cdots T_{1(q-1)}^{-1}T_{0(q-1)}^{-1}}) \\
&= \tau_{p,q}^{p+1}(b_{0T_{00}^{-1}T_{01}^{-1}\cdots T_{0(q-1)}^{-1}T_{0q}^{-1}},\ \cdots,\ b_{pT_{p0}^{-1}T_{p1}^{-1}\cdots T_{pq}^{-1}} \\
&\qquad\qquad\qquad \mid a_{0T_{p0}^{-1}\cdots T_{10}^{-1}T_{00}^{-1}},\ \cdots,\ a_{(q-1)T_{p(q-1)}^{-1}\cdots T_{1(q-1)}^{-1}T_{0(q-1)}^{-1}},\ a_{qT_{pq}^{-1}\cdots T_{1q}^{-1}T_{0q}^{-1}}) \\
&= \tau_{p,q}^{p}(b_{1T_{10}^{-1}T_{11}^{-1}\cdots T_{1(q-1)}^{-1}T_{1q}^{-1}},\ \cdots,\ b_{pT_{p0}^{-1}T_{p1}^{-1}\cdots T_{pq}^{-1}},\ b_0 \\
&\qquad\qquad\qquad \mid a_{0T_{p0}^{-1}\cdots T_{10}^{-1}},\ \cdots,\ a_{(q-1)T_{p(q-1)}^{-1}\cdots T_{1(q-1)}^{-1}},\ a_{qT_{pq}^{-1}\cdots T_{1q}^{-1}}) \\
&= \cdots \\
&= \tau_{p,q}^{p-k}(b_{(k+1)T_{(k+1)0}^{-1}T_{(k+1)1}^{-1}\cdots T_{(k+1)(q-1)}^{-1}T_{(k+1)q}^{-1}},\ \cdots,\ b_{pT_{p0}^{-1}T_{p1}^{-1}\cdots T_{pq}^{-1}},\ b_0,\ \cdots,\ b_k \\
&\qquad\qquad\qquad \mid a_{0T_{p0}^{-1}\cdots T_{(k+1)0}^{-1}},\ \cdots,\ a_{(q-1)T_{p(q-1)}^{-1}\cdots T_{(k+1)(q-1)}^{-1}},\ a_{qT_{pq}^{-1}\cdots T_{(k+1)q}^{-1}}) \\
&= \cdots \\
&= \tau_{p,q}(b_{pT_{p0}^{-1}T_{(p1}^{-1}\cdots T_{p(q-1)}^{-1}T_{pq}^{-1}},\ b_0,\ \cdots,\ b_k \mid a_{0T_{p0}^{-1}},\ \cdots,\ a_{(q-1)T_{p(q-1)}^{-1}}, a_{qT_{pq}^{-1}}) \\
&= (b_0,\ \cdots,\ b_p \mid a_0,\ \cdots,\ a_q).
\end{aligned}
$$

Corollary 4.1 *For two coalgebras A and B, if the cotwisted map T is invertible, then $\Delta(A\natural_T B)$ is a cocylic object.*

5 Relation of $\Delta(A\natural_T B)$ with the Cocylic Object of the Crossed Coproduct $A \times_T B$

Let B be a coalgebra. The cocyclic object $C^\bullet(B)$ is defined by $C^n(B) = B^{\otimes(n+1)}$, $n \geq 0$, with coface, codegeneracy and cyclic operators given by:

$$\partial_i(b_0 \otimes b_1 \otimes \cdots \otimes b_n) = b_0 \otimes b_1 \otimes \cdots \otimes b_{i(1)} \otimes b_{i(2)} \otimes \cdots \otimes b_n, \qquad 0 \leq i \leq n,$$

$$\partial_{n+1}(b_0 \otimes b_1 \otimes \cdots \otimes b_n) = b_{0(2)} \otimes b_1 \otimes \cdots \otimes b_n \otimes b_{0(1)},$$

$$\sigma_i(b_0 \otimes b_1 \otimes \cdots \otimes b_n) = b_0 \otimes b_1 \otimes \cdots \otimes b_i \otimes \varepsilon(b_{i+1}) \otimes \cdots \otimes b_n, \qquad 0 \leq i \leq n-1,$$

$$\tau_n(b_0 \otimes b_1 \otimes \cdots \otimes b_n) = b_1 \otimes \cdots \otimes b_n \otimes b_0.$$

Applying the above operations to the crossed coproduct $A \times_T B$, we have a cocyclic object $C^\bullet(A \times_T B)$.

For $n \in \mathbf{N}$, we define a map $\phi_n : A\natural_T B(n,n) \to C^n(A \times_T B)$,

$\phi_n(b_0, \cdots, b_n \mid a_0, \cdots, a_n)$

$$= a_{0T_{n0}^{-1}\cdots T_{10}^{-1}T_{00}^{-1}} \otimes b_{0T_{00}^{-1}} \otimes a_{1T_{n1}^{-1}\cdots T_{11}^{-1}} \otimes b_{1T_{10}^{-1}T_{11}^{-1}} \otimes \cdots \otimes a_{nT_{nn}^{-1}} \otimes b_{nT_{n0}^{-1}T_{n1}^{-1}\cdots T_{nn}^{-1}}.$$

The following theorem reveals the relations between $\Delta(A\natural_T B)$ and $C^\bullet(A \times_T B)$.

Theorem 5.1 *$\phi = \{\phi_n\}_{n\geq 0}$ defines a cocyclic map between $\Delta(A\natural_T B)$ and $C^\bullet(A \times_T B)$.*

Proof. We show that ϕ commutes with the cylic and cosimplical operators:

(1) $\tau\phi_n = \phi_n\tau_{n,n}\bar{\tau}_{n,n}$. In fact, for all $a_i \in A$ and $b_j \in B$, we have

$$\begin{aligned}
&\phi_n\tau_{n,n}\bar{\tau}_{n,n}(b_0, \cdots, b_n \mid a_0, \cdots, a_n)\\
&= \phi_n\tau_{n,n}(b_{0T_{00}^{-1}}, \cdots, b_{nT_{n0}^{-1}} \mid a_1, \cdots, a_n, a_{0T_{n0}^{-1},\cdots,T_{10}^{-1}T_{00}^{-1}})\\
&= \phi_n(b_{1T_{10}^{-1}}, \cdots, b_{nT_{n0}^{-1}}, b_{0T_{n0}\cdots T_{10}} \mid a_{1T_{10}}, \cdots, a_{nT_{n0}}, a_{0T_{n0}^{-1},\cdots,T_{10}^{-1}})\\
&= a_{1T_{n1}^{-1}\cdots T_{11}^{-1}} \otimes b_{1T_{10}^{-1}T_{11}^{-1}} \otimes a_{2T_{n2}^{-1}\cdots T_{22}^{-1}} \otimes b_{2T_{20}^{-1}T_{21}^{-1}T_{22}^{-1}}\\
&\quad \otimes a_{3T_{n3}^{-1}\cdots T_{33}^{-1}} \otimes b_{3T_{30}^{-1}T_{31}^{-1}T_{32}^{-1}T_{33}^{-1}} \otimes \cdots \otimes a_{kT_{nk}^{-1}\cdots T_{kk}^{-1}} \otimes b_{kT_{k0}^{-1}T_{k1}^{-1}T_{k2}^{-1}\cdots T_{kk}^{-1}}\\
&\quad \otimes \cdots \otimes a_{nT_{nn}^{-1}} \otimes b_{nT_{n0}^{-1}T_{n1}^{-1}T_{n2}^{-1}\cdots T_{nn}^{-1}} \otimes a_{0T_{n0}^{-1},\cdots,T_{10}^{-1}T_{00}^{-1}} \otimes b_{0T_{00}^{-1}}
\end{aligned}$$

and

$$\begin{aligned}
&\tau\phi_n(b_0, \cdots, b_n \mid a_0, \cdots, a_n)\\
&= \tau(a_{0T_{n0}^{-1}\cdots T_{10}^{-1}T_{00}^{-1}} \otimes b_{0T_{00}^{-1}} \otimes a_{1T_{n1}^{-1}\cdots T_{11}^{-1}} \otimes b_{1T_{10}^{-1}T_{11}^{-1}} \otimes \cdots \otimes a_{nT_{nn}^{-1}} \otimes b_{nT_{n0}^{-1}T_{n1}^{-1}\cdots T_{nn}^{-1}})\\
&= a_{1T_{n1}^{-1}\cdots T_{11}^{-1}} \otimes b_{1T_{10}^{-1}T_{11}^{-1}} \otimes \cdots \otimes a_{nT_{nn}^{-1}} \otimes b_{nT_{n0}^{-1}T_{n1}^{-1}\cdots T_{nn}^{-1}} \otimes a_{0T_{n0}^{-1}\cdots T_{10}^{-1}T_{00}^{-1}} \otimes b_{0T_{00}^{-1}}.
\end{aligned}$$

(2) $\partial^{n+1}\phi_n = \phi_{n+1}\bar{\partial}^{n+1}_{n+1,n}\partial^{n+1}_{n,n}$. For all $a_i \in A$ and $b_j \in B$, one has

$$\begin{aligned}
&\phi_{n+1}\bar{\partial}^{n+1}_{n+1,n}\partial^{n+1}_{n,n}(b_0, \cdots, b_n \mid a_0, \cdots, a_n)\\
&= \phi_{n+1}(b_{0(2)V_{00}^{-1}}, b_{1U_{10}^{-1}}, \cdots, b_{nU_{n0}^{-1}}, b_{0(1)T_{n0}\cdots T_{10}T_{00}U_{00}^{-1}}\\
&\qquad \mid a_{0T_{00}(2)}, a_{1T_{10}}, \cdots, a_{nT_{n0}},\ a_{0T_{00}(1)U_{00}^{-1}U_{n0}^{-1}\cdots U_{10}^{-1}V_{00}^{-1}})\\
&= a_{\underline{0T_{00}(2)T_{00}^{-1}}T_{n0}^{-1}\cdots T_{10}^{-1}X_{00}^{-1}} \otimes b_{0(2)V_{00}^{-1}X_{00}^{-1}} \otimes a_{1T_{10}T_{01}^{-1}T_{n1}^{-1}\cdots T_{11}^{-1}} \otimes b_{1(2)U_{10}^{-1}T_{10}^{-1}T_{11}^{-1}}\\
&\quad \otimes a_{2T_{20}T_{02}^{-1}T_{n2}^{-1}\cdots T_{22}^{-1}} \otimes b_{2U_{20}^{-1}T_{20}^{-1}T_{21}^{-1}T_{22}^{-1}} \otimes \cdots \otimes a_{nT_{n0}T_{0n}^{-1}T_{nn}^{-1}} \otimes b_{nU_{n0}^{-1}T_{n0}^{-1}T_{n1}^{-1}\cdots T_{nn}^{-1}}\\
&\quad \otimes a_{\underline{0T_{00}(1)U_{00}^{-1}}U_{n0}^{-1}\cdots U_{10}^{-1}V_{00}^{-1}Y_{00}^{-1}} \otimes b_{\underline{0(1)T_{n0}\cdots T_{10}T_{00}U_{00}^{-1}T_{00}^{-1}}T_{01}^{-1}\cdots T_{0n}^{-1}Y_{00}^{-1}}\\
&\overset{(4.3)}{=} a_{\underline{0(2)T_{n0}^{-1}}\cdots T_{10}^{-1}X_{00}^{-1}} \otimes b_{0(2)V_{00}^{-1}X_{00}^{-1}} \otimes a_{1T_{10}T_{01}^{-1}T_{n1}^{-1}\cdots T_{11}^{-1}} \otimes b_{1(2)U_{10}^{-1}T_{10}^{-1}T_{11}^{-1}}\\
&\quad \otimes a_{2T_{20}T_{02}^{-1}T_{n2}^{-1}\cdots T_{22}^{-1}} \otimes b_{2U_{20}^{-1}T_{20}^{-1}T_{21}^{-1}T_{22}^{-1}} \otimes \cdots \otimes a_{nT_{n0}T_{0n}^{-1}T_{nn}^{-1}} \otimes b_{n\underline{U_{n0}^{-1}T_{n0}^{-1}}T_{n1}^{-1}\cdots T_{nn}^{-1}}\\
&\quad \otimes a_{\underline{0(1)U_{n0}^{-1}}\cdots U_{10}^{-1}V_{00}^{-1}Y_{00}^{-1}} \otimes b_{0(1)T_{n0}\cdots T_{10}T_{01}^{-1}\cdots T_{0n}^{-1}Y_{00}^{-1}}\\
&\overset{(4.3)}{=} a_{0T_{n0}^{-1}(2)T_{(n-1)0}^{-1}\cdots T_{10}^{-1}X_{00}^{-1}} \otimes b_{0(2)V_{00}^{-1}X_{00}^{-1}} \otimes a_{1T_{10}T_{01}^{-1}T_{n1}^{-1}\cdots T_{11}^{-1}} \otimes b_{1(2)U_{10}^{-1}T_{10}^{-1}T_{11}^{-1}}\\
&\quad \otimes a_{2T_{20}T_{02}^{-1}T_{n2}^{-1}\cdots T_{22}^{-1}} \otimes b_{2U_{20}^{-1}T_{20}^{-1}T_{21}^{-1}T_{22}^{-1}} \otimes \cdots \otimes a_{nT_{n0}T_{0n}^{-1}T_{nn}^{-1}} \otimes b_{nT_{n0}^{-1}T_{n1}^{-1}\cdots T_{nn}^{-1}}\\
&\quad \otimes a_{0T_{n0}^{-1}(1)U_{(n-1)0}^{-1}\cdots U_{10}^{-1}V_{00}^{-1}Y_{00}^{-1}} \otimes b_{0(1)T_{n0}\cdots T_{10}T_{01}^{-1}\cdots T_{0n}^{-1}Y_{00}^{-1}}\\
&= \cdots
\end{aligned}$$

$$\overset{(4.3)}{=} a_{0T_{n0}^{-1}T_{(n-1)0}^{-1}\cdots T_{10}^{-1}(2)X_{00}^{-1}} \otimes b_{0(2)V_{00}^{-1}X_{00}^{-1}} \otimes a_{1\underline{T_{10}T_{01}^{-1}}T_{n1}^{-1}\cdots T_{11}^{-1}} \otimes b_{1(2)T_{10}^{-1}T_{11}^{-1}}$$
$$\otimes a_{\underline{2T_{20}T_{02}^{-1}}T_{n2}^{-1}\cdots T_{22}^{-1}} \otimes b_{2U_{20}^{-1}T_{20}^{-1}T_{21}^{-1}T_{22}^{-1}} \otimes \cdots \otimes a_{n\underline{T_{n0}T_{0n}^{-1}}T_{nn}^{-1}} \otimes b_{nT_{n0}^{-1}T_{n1}^{-1}\cdots T_{nn}^{-1}}$$
$$\otimes a_{0T_{n0}^{-1}T_{(n-1)0}^{-1}\cdots T_{10}^{-1}(1)V_{00}^{-1}Y_{00}^{-1}} \otimes b_{0(1)\underline{T_{n0}\cdots T_{10}T_{01}^{-1}\cdots T_{0n}^{-1}}Y_{00}^{-1}}$$
$$= a_{0T_{n0}^{-1}T_{(n-1)0}^{-1}\cdots T_{10}^{-1}\underline{(2)X_{00}^{-1}}} \otimes b_{0(2)\underline{V_{00}^{-1}X_{00}^{-1}}} \otimes a_{1T_{n1}^{-1}\cdots T_{11}^{-1}} \otimes b_{1(2)T_{10}^{-1}T_{11}^{-1}}$$
$$\otimes a_{T_{n2}^{-1}\cdots T_{22}^{-1}} \otimes b_{2U_{20}^{-1}T_{20}^{-1}T_{21}^{-1}T_{22}^{-1}} \otimes \cdots \otimes a_{nT_{nn}^{-1}} \otimes b_{nT_{n0}^{-1}T_{n1}^{-1}\cdots T_{nn}^{-1}}$$
$$\otimes a_{0T_{n0}^{-1}T_{(n-1)0}^{-1}\cdots T_{10}^{-1}\underline{(1)V_{00}^{-1}}Y_{00}^{-1}} \otimes b_{0(1)Y_{00}^{-1}}$$
$$= a_{0T_{n0}^{-1}T_{(n-1)0}^{-1}\cdots T_{10}^{-1}X_{00}^{-1}(2)} \otimes b_{0(2)X_{00}^{-1}} \otimes a_{1T_{n1}^{-1}\cdots T_{11}^{-1}} \otimes b_{1(2)W_{10}^{-1}T_{11}^{-1}}$$
$$\otimes a_{T_{n2}^{-1}\cdots T_{22}^{-1}} \otimes b_{2T_{20}^{-1}T_{21}^{-1}T_{22}^{-1}} \otimes \cdots \otimes a_{nT_{nn}^{-1}} \otimes b_{nT_{n0}^{-1}T_{n1}^{-1}\cdots T_{nn}^{-1}}$$
$$\otimes a_{0T_{n0}^{-1}T_{(n-1)0}^{-1}\cdots T_{10}^{-1}X_{00}^{-1}(1)Y_{00}^{-1}} \otimes b_{0(1)Y_{00}^{-1}}.$$

Also

$$\partial^{n+1}\phi_n(b_0,\cdots,b_n \mid a_0,\cdots,a_n)$$
$$= a_{\underline{0T_{n0}^{-1}\cdots T_{10}^{-1}T_{00}^{-1}}(2)U_{00}} \otimes b_{\underline{0T_{00}^{-1}(2)}} \otimes a_{1T_{n1}^{-1}\cdots T_{11}^{-1}} \otimes b_{1T_{10}^{-1}T_{11}^{-1}}$$
$$\otimes a_{2T_{n2}^{-1}\cdots T_{22}^{-1}} \otimes b_{2T_{20}^{-1}T_{21}^{-1}T_{22}^{-1}} \otimes \cdots$$
$$\otimes a_{(n-1)T_{n(n-1)}^{-1}T_{(n-1)(n-1)}^{-1}} \otimes b_{(n-1)T_{(n-1)0}^{-1}T_{(n-1)1}^{-1}\cdots T_{(n-1)(n-1)}^{-1}}$$
$$\otimes a_{nT_{nn}^{-1}} \otimes b_{nT_{n0}^{-1}T_{n1}^{-1}\cdots T_{nn}^{-1}} \otimes a_{\underline{0T_{n0}^{-1}\cdots T_{10}^{-1}T_{00}^{-1}}(1)} \otimes b_{\underline{0T_{00}^{-1}(1)}U_{00}}$$
$$\overset{(4.2)}{=} a_{0T_{n0}^{-1}\cdots T_{10}^{-1}T_{00}^{-1}X_{00}^{-1}(2)U_{00}} \otimes b_{0(2)T_{00}^{-1}} \otimes a_{1T_{n1}^{-1}\cdots T_{11}^{-1}} \otimes b_{1T_{10}^{-1}T_{11}^{-1}}$$
$$\otimes a_{2T_{n2}^{-1}\cdots T_{22}^{-1}} \otimes b_{2T_{20}^{-1}T_{21}^{-1}T_{22}^{-1}} \otimes \cdots$$
$$\otimes a_{(n-1)T_{n(n-1)}^{-1}T_{(n-1)(n-1)}^{-1}} \otimes b_{(n-1)T_{(n-1)0}^{-1}T_{(n-1)1}^{-1}\cdots T_{(n-1)(n-1)}^{-1}}$$
$$\otimes a_{nT_{nn}^{-1}} \otimes b_{nT_{n0}^{-1}T_{n1}^{-1}\cdots T_{nn}^{-1}} \otimes a_{0T_{n0}^{-1}\cdots T_{10}^{-1}T_{00}^{-1}X_{00}^{-1}(1)} \otimes b_{0(1)X_{00}^{-1}U_{00}}$$
$$\overset{(4.3)}{=} a_{0T_{n0}^{-1}\cdots T_{10}^{-1}T_{00}^{-1}(2)Z_{00}^{-1}U_{00}} \otimes b_{0(2)T_{00}^{-1}} \otimes a_{1T_{n1}^{-1}\cdots T_{11}^{-1}} \otimes b_{1T_{10}^{-1}T_{11}^{-1}}$$
$$\otimes a_{2T_{n2}^{-1}\cdots T_{22}^{-1}} \otimes b_{2T_{20}^{-1}T_{21}^{-1}T_{22}^{-1}} \otimes \cdots$$
$$\otimes a_{(n-1)T_{n(n-1)}^{-1}T_{(n-1)(n-1)}^{-1}} \otimes b_{(n-1)T_{(n-1)0}^{-1}T_{(n-1)1}^{-1}\cdots T_{(n-1)(n-1)}^{-1}}$$
$$\otimes a_{nT_{nn}^{-1}} \otimes b_{nT_{n0}^{-1}T_{n1}^{-1}\cdots T_{nn}^{-1}} \otimes a_{0T_{n0}^{-1}\cdots T_{10}^{-1}T_{00}^{-1}(1)X_{00}^{-1}} \otimes b_{0(1)X_{00}^{-1}Z_{00}^{-1}U_{00}}$$
$$= a_{0T_{n0}^{-1}\cdots T_{10}^{-1}T_{00}^{-1}(2)} \otimes b_{0(2)T_{00}^{-1}} \otimes a_{1T_{n1}^{-1}\cdots T_{11}^{-1}} \otimes b_{1T_{10}^{-1}T_{11}^{-1}}$$
$$\otimes a_{2T_{n2}^{-1}\cdots T_{22}^{-1}} \otimes b_{2T_{20}^{-1}T_{21}^{-1}T_{22}^{-1}} \otimes \cdots$$
$$\otimes a_{(n-1)T_{n(n-1)}^{-1}T_{(n-1)(n-1)}^{-1}} \otimes b_{(n-1)T_{(n-1)0}^{-1}T_{(n-1)1}^{-1}\cdots T_{(n-1)(n-1)}^{-1}}$$
$$\otimes a_{nT_{nn}^{-1}} \otimes b_{nT_{n0}^{-1}T_{n1}^{-1}\cdots T_{nn}^{-1}} \otimes a_{0T_{n0}^{-1}\cdots T_{10}^{-1}T_{00}^{-1}(1)X_{00}^{-1}} \otimes b_{0(1)X_{00}^{-1}}.$$

By comparing the above equations, we can get the desired result. With the same argument, using Lemma 4.1 and with direct computations, we can check that

$$\partial^i\phi_n = \phi_{n+1}\bar{\partial}^i_{n+1,n}\partial^i_{n,n}, \quad \sigma^i\phi_n = \phi_{n-1}\bar{\partial}^i_{n-1,n}\sigma^i_{n,n}, \qquad 0 \le i \le n.$$

Theorem 5.2 *Let A and B be two coalgebras and T an invertible cotwisted map. Then we have an isomorphism of cocyclic objects*

$$\varDelta(A\natural_T B) \cong C^\bullet(A \times_T B).$$

Proof. For $n \in \mathbf{N}$, we define a map $\varphi_n : C^n(A \times_T B) \to A\natural_T B(n,n)$,

$$\begin{aligned}&\varphi_n(a_0 \otimes b_0 \otimes \cdots \otimes a_n \otimes b_n)\\ &= (b_{0T_{00}}, b_{1T_{11}T_{10}}, b_{2T_{22}T_{21}T_{20}}, \cdots b_{nT_{nn}T_{n(n-1)}\cdots T_{n0}}\\ &\qquad \mid a_{0T_{00}T_{10}\cdots T_{n0}}, a_{1T_{11}T_{21}\cdots T_{n1}}, a_{2T_{22}T_{32}\cdots T_{n2}}, \cdots, a_{nT_{nn}}).\end{aligned}$$

Since

$$\begin{aligned}&\varphi_n\phi_n(b_0, \cdots, b_n \mid a_0, \cdots, a_n)\\ &= \varphi_n(a_{0T_{n0}^{-1}\cdots T_{10}^{-1}T_{00}^{-1}} \otimes b_{0T_{00}^{-1}} \otimes a_{1T_{n1}^{-1}\cdots T_{11}^{-1}} \otimes b_{1T_{10}^{-1}T_{11}^{-1}} \otimes \cdots \otimes a_{nT_{nn}^{-1}} \otimes b_{nT_{n0}^{-1}T_{n1}^{-1}\cdots T_{nn}^{-1}})\\ &= (b_{0T_{00}^{-1}T_{00}}, b_{1T_{10}^{-1}T_{11}^{-1}T_{11}T_{10}}, \cdots, b_{nT_{n0}^{-1}T_{n1}^{-1}\cdots T_{nn}^{-1}T_{nn}T_{n(n-1)}\cdots T_{n0}}\\ &\qquad \mid a_{0T_{n0}^{-1}\cdots T_{10}^{-1}T_{00}^{-1}T_{00}T_{10}\cdots T_{n0}}, a_{1T_{n1}^{-1}\cdots T_{11}^{-1}T_{11}T_{21}\cdots T_{n1}}, \cdots, a_{nT_{nn}^{-1}T_{nn}})\\ &= (b_0, \cdots, b_n \mid a_0, \cdots, a_n).\end{aligned}$$

So it follows that $\varphi_n\phi_n = \mathrm{id}_{A\natural_T B(n,n)}$.

$\phi_n\varphi_n = \mathrm{id}_{C^n(A\times_T B)}$ can be checked similarly.

References

[1] Getzler E, Jones J D S. The cyclic homology of crossed product algebras. *J. Reine Angew. Math.*, 1993, **445**: 163–174.

[2] Akbarpour R, Khalkhali M. Hopf algebra equivariant cyclic homology and cyclic homology of crossed product algebras. *J. Reine Angew. Math.*, 2003, **559**: 137–152.

[3] Zhang J, Hu N H. Cyclic homology of strong smash product algebras. *J. Reine Angew. Math.*, 2012, **663**: 177–207.

[4] Carboni G, Guccione J A, Guccione J J, *et al.* Cyclic homology of Brzeziński's crossed products and of braided Hopf crossed products. *Adv. Math.*, 2012, **231**: 3502–3568.

[5] Wang S H, Li J Q. On twisted smash products for bimodule algebras and the drinfeld double. *Comm. Algebra*, 1998, **26**: 2435–2444.

[6] Connes A. Noncommutative Geometry. New York: Academic Press, 1994.