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A New Generalized FB Complementarity Function for Symmetric Cone Complementarity Problems

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Abstract: We establish that the generalized Fischer-Burmeister(FB) function and penalized Generalized Fischer-Burmeister (FB) function defined on symmetric cones are complementarity functions (C-functions), in terms of Euclidean Jordan algebras, and the Generalized Fischer-Burmeister complementarity function for the symmetric cone complementarity problem (SCCP). It provides an affirmative answer to the open question by Kum and Lim (Kum S H, Lim Y. Penalized complementarity functions on symmetric cones. *J. Glob. Optim.*. 2010, **46**: 475–485) for any positive integer.
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1 Introduction

The symmetric cone complementarity problem (SCCP) is defined to find $x, y \in V$ such that

$$\boldsymbol{x} \in K, \quad \boldsymbol{y} = f(x) \in K, \quad \langle \boldsymbol{x}, \ \boldsymbol{y} \rangle = 0, \tag{1.1}$$

where K is the cone of squares in Euclidean Jordan algebra V, and $F : V \to V$ is a continuously differentiable mapping (see [1]–[2]). This class of problem provides a unified framework for the classical nonlinear and complementarity problem (NCP), the second-order cone optimization and complementarity problem (SCOCP), and the semi-definite programming and complementarity problem (SDCP), and has attracted much attention due to its various applications in operations research, economics and engineering.

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A popular and powerful approach to solve the complementarity problem is to reformulate each problem as an equivalent system of non-smooth equations by a complementarity function (C-function) (see [3]–[4]) or as an unconstrained minimization problem by merit function (M-function) (see [5]). A function $\phi : V \times V \to V$ is called a C-function for SCCP if

$$\phi(\boldsymbol{x},\ \boldsymbol{y}) = 0 \Leftrightarrow \boldsymbol{x} \in K,\ \boldsymbol{y} = f(\boldsymbol{x}) \in K,\ \langle \boldsymbol{x},\ \boldsymbol{y}\rangle = 0. \tag{1.2}$$

Various C-functions for the standard NCP functions ware extend to the SCCP. For instance, Gowda *et al.*[6] showed that the Fischer-Burmeister function

$$\boldsymbol{\phi}_{\rm FB}(\boldsymbol{x},\ \boldsymbol{y}) = \boldsymbol{x} + \boldsymbol{y} - (\boldsymbol{x}^2 + \boldsymbol{y}^2)^{1/2} \tag{1.3}$$

are C-function for any Euclidean Jordan algebra.

A function that can constitute an equivalent unconstrained minimization problem for the SCCP is called an M-function. In other words, a merit function is a function whose global minima is coincident with the solutions of the original SCCP. For constructing an M-function, the C-function severs an important role.

In order to solve (1.1), we only need to find the solution of the nonlinear equations $\boldsymbol{\phi}(\boldsymbol{x},\ \boldsymbol{F}(\boldsymbol{x})) = 0$ induced via the C-function associated with the symmetric cone. Take FB function as a example, the SCCP is equivalent to a system of nonlinear equations:

$$\boldsymbol{\Phi}(\boldsymbol{x}) = \begin{pmatrix} \boldsymbol{\phi}_{\rm FB}(\boldsymbol{x}_1,\ F(\boldsymbol{x}_1)) \\ \vdots \\ \boldsymbol{\phi}_{\rm FB}(\boldsymbol{x}_n,\ F(\boldsymbol{x}_n)) \end{pmatrix} = 0. \tag{1.4}$$

For each C-function, there is a natural merit function $\boldsymbol{\Psi}_{\rm FB}$ given by

$$\boldsymbol{\Psi}_{\rm FB} := \frac{1}{2}\|\boldsymbol{\Phi}_{\rm FB}\|^2 = \frac{1}{2}\sum_{i=1}^{n} \boldsymbol{\phi}_{\rm FB}(\boldsymbol{x}_i,\ F(\boldsymbol{x}_i))^2, \tag{1.5}$$

from which the SCCP can be recast as an unconstrained minimization

$$\min_{x\in\mathbf{R}^n} \boldsymbol{\Psi}_{\rm FB}(\boldsymbol{x}). \tag{1.6}$$

In this paper, we are particularly interested in the generalized FB, which is presented in a recent paper to deal with NCP by Chen[7]–[8]. The definition of the generalized FB function is as follows.

Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{R}^n$. For $p > 1$,

$$\boldsymbol{\phi}_p(\boldsymbol{x},\ \boldsymbol{y}) = \boldsymbol{x} + \boldsymbol{y} - (|\boldsymbol{x}|^p + |\boldsymbol{y}|^p)^{1/p} \tag{1.7}$$

is called the generalized FB function of NCP.

Shortly afterwards, Pan *et al.*[9] developed the M-function method for SOCCP based on the generalized FB function and Kum *et al.*[10] proved that generalized FB function and penalized generalized FB function are complementarity functions for SOCCP. Nowadays, Kum[11] extends the generalized FB function to the SCCP when $p = 1, 2, 3, 4$ and proposes a question that "Is the function a C-function for any positive integer $n \geq 2$?" Motivated by the above mentioned work, we are trying to extend the generalized FB function to the SCCP when p>1. Moreover, under suitable conditions, we derive the boundedness of level set of the natural M-function induced by the penalized generalized FB function from a trace inequality in Euclidean Jordan algebras, which is very useful toward an entire development of the M-function theory for SCCP based on the penalized version as a future research.

2 Preliminaries

In this section, we mention some concepts and materials of Euclidean Jordan algebras that will be used throughout this paper. For more detailed expositions of Euclidean Jordan algebras, the readers can find in the monograph[12].

A Euclidean Jordan algebra is a triple $(V,\ \circ,\ \langle\ \cdot\ ,\ \cdot\ \rangle)$, where V is a finite-dimensional inner product space over the real field $\mathbf{R}$, and $(\boldsymbol{x},\ \boldsymbol{y}) \mapsto \boldsymbol{x} \circ \boldsymbol{y} :\ V \times V \to V$ is a bilinear mapping which satisfies the following conditions:

(1) $\boldsymbol{x} \circ \boldsymbol{y} = \boldsymbol{y} \circ \boldsymbol{x}$ for all $\boldsymbol{x}, \boldsymbol{y} \in V$;

(2) $\boldsymbol{x} \circ (\boldsymbol{x}^2 \circ \boldsymbol{y}) = \boldsymbol{x}^2 \circ (\boldsymbol{x} \circ \boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in V$, where $\boldsymbol{x}^2 := \boldsymbol{x} \circ \boldsymbol{x}$;

(3) $\langle \boldsymbol{x} \circ \boldsymbol{y},\ \boldsymbol{z} \rangle = \langle \boldsymbol{x},\ \boldsymbol{y} \circ \boldsymbol{z} \rangle$ for all $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in V$.

We call $\boldsymbol{x} \circ \boldsymbol{y}$ the Jordan product of $\boldsymbol{x}$ and $\boldsymbol{y}$. In addition, we assume that there is an element $\boldsymbol{e}$, called the unit element, such that $\boldsymbol{x} \circ \boldsymbol{e} = \boldsymbol{x}$ for all $\boldsymbol{x} \in V$. An element $\boldsymbol{c} \in V$ is idempotent if $\boldsymbol{c}^2 = \boldsymbol{c}$, and two idempotents $\boldsymbol{c}$ and $\boldsymbol{c}'$ are orthogonal if $\boldsymbol{c} \circ \boldsymbol{c}' = 0$. If an idempotent cannot be written by a sum of two non-zero idempotents, then $\boldsymbol{c}$ is called primitive. A complete system of orthogonal idempotents is finite set $\{\boldsymbol{c}_1,\ \boldsymbol{c}_2,\ \cdots,\ \boldsymbol{c}_k\}$ of idempotents with $\boldsymbol{c}_i \circ \boldsymbol{c}_j = 0$ and $\sum\limits_{i=1}^{k} \boldsymbol{c}_i = \boldsymbol{e}$. A complete system of orthogonal primitive idempotents is called a Jordan frame of V. The important spectral decomposition theorems are stated as follows.

Theorem 2.1[12] (spectral theorem, first version) *For an element of an Euclidean Jordan algebra, there exist unique real numbers* $\lambda_1(\boldsymbol{x}),\ \lambda_2(\boldsymbol{x}),\ \cdots,\ \lambda_k(\boldsymbol{x})$ *and a unique complete system of orthogonal idempotents* $\{\boldsymbol{c}_1,\ \boldsymbol{c}_2,\ \cdots,\ \boldsymbol{c}_k\}$ *such that*

$$\boldsymbol{x} = \lambda_1(\boldsymbol{x})\boldsymbol{c}_1 + \lambda_2(\boldsymbol{x})\boldsymbol{c}_2 + \cdots + \lambda_k(\boldsymbol{x})\boldsymbol{c}_k. \tag{2.1}$$

The uniqueness is in the following sense: if there exist a complete system of orthogonal idempotents $\{\boldsymbol{e}_1,\ \boldsymbol{e}_2,\ \cdots,\ \boldsymbol{e}_s\}$ and distinct real numbers $\eta_1(\boldsymbol{x}),\ \eta_2(\boldsymbol{x}),\ \cdots,\ \eta_s(\boldsymbol{x})$ such that $\boldsymbol{x} = \eta_1(\boldsymbol{x})\boldsymbol{e}_1 + \eta_2(\boldsymbol{x})\boldsymbol{e}_2 + \cdots + \eta_s(\boldsymbol{x})\boldsymbol{e}_s$, then $k = s$, $\eta_i = \lambda_i$ and $\boldsymbol{e}_i = \boldsymbol{c}_i$ for all $1 \leq i \leq k$.

Theorem 2.2[12] (spectral theorem, second version) *Let V be a Euclidean Jordan algebra with rank r. Then for every $\boldsymbol{x} \in V$, there exist a Jordan frame $\{\boldsymbol{c}_1,\ \boldsymbol{c}_2,\ \cdots,\ \boldsymbol{c}_r\}$ and real numbers $\{\boldsymbol{c}_1,\ \boldsymbol{c}_2,\ \cdots,\ \boldsymbol{c}_r\}$ such that*

$$\boldsymbol{x} = \lambda_1(\boldsymbol{x})\boldsymbol{c}_1 + \lambda_2(\boldsymbol{x})\boldsymbol{c}_2 + \cdots + \lambda_r(\boldsymbol{x})\boldsymbol{c}_r, \tag{2.2}$$

where the numbers $\lambda_i(\boldsymbol{x})$ $(i = 1, 2, \cdots, r)$ are the eigenvalues of $\boldsymbol{x}$.

Let $\mathrm{tr}(\boldsymbol{x}) = \sum\limits_{i=1}^{r} \lambda_i(\boldsymbol{x})$ be the trace of $\boldsymbol{x} = \lambda_1(\boldsymbol{x})\boldsymbol{c}_1 + \lambda_2(\boldsymbol{x})\boldsymbol{c}_2 + \cdots + \lambda_r(\boldsymbol{x})\boldsymbol{c}_r$ in the second spectral theorem. Note that the trace is associative, i.e., $\mathrm{tr}(\boldsymbol{x},\ \boldsymbol{y} \circ \boldsymbol{z}) = \mathrm{tr}(\boldsymbol{x} \circ \boldsymbol{y},\ \boldsymbol{z})$, we define the inner product $\mathrm{tr}(\boldsymbol{x},\ \boldsymbol{y} \circ \boldsymbol{z}) = \mathrm{tr}(\boldsymbol{x} \circ \boldsymbol{y},\ \boldsymbol{z})$ by $\mathrm{tr}(\boldsymbol{x},\ \boldsymbol{y} \circ \boldsymbol{z}) = \mathrm{tr}(\boldsymbol{x} \circ \boldsymbol{y},\ \boldsymbol{z})$. Thus, we may define norm on V by

$$\|\boldsymbol{x}\| = \sqrt{\langle \boldsymbol{x},\ \boldsymbol{x} \rangle} = \sqrt{\mathrm{tr}(\boldsymbol{x}^2)} = \sqrt{\sum_{i=1}^{r} \lambda_i^2(\boldsymbol{x})}, \qquad x \in V. \tag{2.3}$$

Given a Euclidean Jordan algebra V, we define the set of squares as $K = \{\boldsymbol{x}^2 : \ \boldsymbol{x} \in V\}$. K, if closed, convex, homogeneous and self-dual cone, is the symmetric cone. Recall the partial order on V defined by $\boldsymbol{x} \preceq \boldsymbol{y} :\Leftrightarrow \boldsymbol{y} - \boldsymbol{x} \in \bar{K}$, and $\boldsymbol{x} \prec \boldsymbol{y} :\Leftrightarrow \boldsymbol{y} - \boldsymbol{x} \in \bar{K}$, where $\bar{K}$ denotes the interior of K.

Lemma 2.1 *Let p be a positive real number.*

(1) *Each element $\boldsymbol{x} \succeq 0$ has a unique p-th root denote by $\boldsymbol{x}^{1/p}$ in $\bar{K}$. If $\boldsymbol{x} \in \bar{K}$ has a spectral decomposition, then*

$$\boldsymbol{x}^{1/p} = \sum_{i=1}^{r} (\lambda_i(\boldsymbol{x}))^{1/p} \boldsymbol{c}_i;$$

(2) (The Löwner-Heinz inequality[13])

$$0 \preceq \boldsymbol{x} \preceq \boldsymbol{y} \Rightarrow \boldsymbol{x}^p \preceq \boldsymbol{y}^p, \qquad 0 \leq p \leq 1. \tag{2.4}$$

Let $g : \mathbf{R} \to \mathbf{R}$ be a real-valued function. Define the corresponding Löwner operator $G(\boldsymbol{x}) : J \to J$ as

$$G(\boldsymbol{x}) := \sum_{j=1}^{r} g(\lambda_j(\boldsymbol{x})) \boldsymbol{c}_j(\boldsymbol{x}) = g(\lambda_1(\boldsymbol{x}) \boldsymbol{c}_1(x) + \lambda_2(\boldsymbol{x}) \boldsymbol{c}_2(\boldsymbol{x}) + \cdots + \lambda_r(\boldsymbol{x}) \boldsymbol{c}_r(\boldsymbol{x})). \tag{2.5}$$

In particular, letting $t_+ = \max\{0, \ t\}$, $t_- = \max\{0, \ -t\}$, and noting $|t| = t_+ + t_-$, we define $\boldsymbol{x}_+ = \sum\limits_{i=1}^{r} (\lambda_i(\boldsymbol{x}))_+ \boldsymbol{c}_i(\boldsymbol{x})$, $\boldsymbol{x}_- = \sum\limits_{i=1}^{r} (\lambda_i(\boldsymbol{x}))_- \boldsymbol{c}_i(\boldsymbol{x})$ and $|\boldsymbol{x}| = \sum\limits_{i=1}^{r} |\lambda_i(\boldsymbol{x})| \boldsymbol{c}_i(\boldsymbol{x})$. Note that $\boldsymbol{x} \in K$ ($\boldsymbol{x} \in \operatorname{int}(K)$) if and only if $\lambda_i(\boldsymbol{x}) \geq 0$ ($\lambda_i(\boldsymbol{x}) > 0$) for all $i \in \{1, 2, \cdots, r\}$, where $\operatorname{int}(K)$ denotes the interior of K. It is obvious that $\boldsymbol{x} \in K$, $\boldsymbol{x} = \boldsymbol{x}_+ - \boldsymbol{x}_-$ and $|\boldsymbol{x}| = \boldsymbol{x}_+ + \boldsymbol{x}_-$. We also can define $\boldsymbol{x}^{-1} = \sum\limits_{i=1}^{r} \lambda_i(\boldsymbol{x})^{-1} \boldsymbol{c}_i(\boldsymbol{x})$, $\boldsymbol{x}^{1/2} = \sum\limits_{i=1}^{r} \lambda_i(\boldsymbol{x})^{1/2} \boldsymbol{c}_i(\boldsymbol{x})$, and denote $|\boldsymbol{x}|$ by $|\boldsymbol{x}| = (\boldsymbol{x}^2)^{1/2}$ and $\boldsymbol{x}_+ = \dfrac{\boldsymbol{x} + |\boldsymbol{x}|}{2}$, $\boldsymbol{x}_- = \dfrac{|\boldsymbol{x}| - \boldsymbol{x}}{2}$. It is easy to verify that $\boldsymbol{x}_+ \circ \boldsymbol{x}_- = 0$ and $\langle \boldsymbol{x}_+, \ \boldsymbol{x}_- \rangle = 0$.

3 The Main Result

In this section, we present two complementarity functions for SCCP. For this purpose, we need to show the following useful proposition.

Proposition 3.1 *The followings are equivalent:*

(1) $\boldsymbol{x}, \boldsymbol{y} \succeq 0$ *and* $\langle \boldsymbol{x}, \ \boldsymbol{y} \rangle = 0$;

(2) $\boldsymbol{x}, \boldsymbol{y} \succeq 0$ *and* $\boldsymbol{x} \circ \boldsymbol{y} = 0$;

(3) $\boldsymbol{x} + \boldsymbol{y} = (\boldsymbol{x}^2 + \boldsymbol{y}^2)^{1/2}$;

(4) $\boldsymbol{x} + \boldsymbol{y} \succeq 0$ *and* $\boldsymbol{x} \circ \boldsymbol{y} = 0$;

(5) $\boldsymbol{x}, \boldsymbol{y} \succeq 0$ *and* $\boldsymbol{x}^t \circ \boldsymbol{y}^s = 0$ *for all nonnegative real numbers* s, t.

Proof. The equivalences from (1) to (4) appear in [14, Proposition 6] and (5) in [11, Proposition 3.1(vii)]. Suppose that $\boldsymbol{x}, \boldsymbol{y} \succeq 0$ and $\boldsymbol{x} \circ \boldsymbol{y} = 0$. Let $A := \{t \geq 0 \mid \boldsymbol{x}^t \circ \boldsymbol{y} = 0\}$ by continuity of the Jordan product. It is a closed subset of $[0, \ \infty)$. We claim that contains all positive dyadic powers $\left\{ \dfrac{m}{2^n} \,\middle|\, n, m \in N \right\}$.

From

$$\begin{aligned}0 &= \langle \boldsymbol{x},\ \boldsymbol{y}\rangle\\ &= \langle \boldsymbol{x},\ (\boldsymbol{y}^{1/2})^2\rangle\\ &= \langle \boldsymbol{x}\circ \boldsymbol{y}^{1/2},\ \boldsymbol{y}^{1/2}\rangle\\ &= \langle L(\boldsymbol{x})\circ \boldsymbol{y}^{1/2},\ \boldsymbol{y}^{1/2}\rangle\\ &= \langle L(\boldsymbol{x})^{1/2}\circ \boldsymbol{y}^{1/2},\ L(\boldsymbol{x})^{1/2}\circ \boldsymbol{y}^{1/2}\rangle,\end{aligned}$$

where we used the fact that $l(\boldsymbol{x})$ is positive semi-definite and hence it has the square root $L(\boldsymbol{x})^{1/2}$, we have

$$L(\boldsymbol{x})^{1/2}\circ \boldsymbol{y}^{1/2} = 0$$

or

$$L(\boldsymbol{x})\boldsymbol{y}^{1/2}\circ \boldsymbol{y}^{1/2} = \boldsymbol{x}\circ \boldsymbol{y}^{1/2} = 0.$$

Similarly,

$$\boldsymbol{y}\circ \boldsymbol{x}^{1/2} = 0.$$

By induction, one has

$$\boldsymbol{x}^{\frac{1}{2^n}}\circ \boldsymbol{y} = 0$$

for all positive integers n.

For positive integer $m \geq 2$,

$$0 = \langle \boldsymbol{x}\circ \boldsymbol{y},\ \boldsymbol{x}^{m-1}\rangle = \langle \boldsymbol{y},\ \boldsymbol{x}^{m}\rangle.$$

This implies

$$\boldsymbol{y}\circ \boldsymbol{x}^{m} = 0.$$

Therefore,

$$\boldsymbol{x}^{\frac{m}{2^n}}\circ \boldsymbol{y} = 0$$

for all positive integers n and m.

By the density of A in the space of non-negative real numbers, we conclude that

$$\boldsymbol{x}^{t}\circ \boldsymbol{y} = 0$$

for all nonnegative real numbers t.

Theorem 3.1 *The function $\phi_p(\boldsymbol{x},\ \boldsymbol{y}) : V\times V \to V$ defined as* (1.7) *is a C-function, where V is any Euclidean Jordan algebra.*

Proof. First, suppose that $\boldsymbol{x}, \boldsymbol{y} \succeq 0$ and $\boldsymbol{x}\circ \boldsymbol{y} = 0$. It is obvious that $\boldsymbol{x} = |\boldsymbol{x}|$ and $\boldsymbol{y} = |\boldsymbol{y}|$. For $t \geq 1$, we have, by means of Proposition 3.1(5), $\boldsymbol{x}^t\boldsymbol{y} = 0$ and $\boldsymbol{y}^t\boldsymbol{x} = 0$ for all nonnegative real numbers t. This implies by induction that

$$(\boldsymbol{x}+\boldsymbol{y})^p = \boldsymbol{x}^p + \boldsymbol{y}^p.$$

Indeed, letting $p > 1$, one has

$$\begin{aligned}(\boldsymbol{x}+\boldsymbol{y})^p &= (\boldsymbol{x}+\boldsymbol{y})^{p-1}\circ(\boldsymbol{x}+\boldsymbol{y})\\ &= (\boldsymbol{x}^{p-1}+\boldsymbol{y}^{p-1})\circ(\boldsymbol{x}+\boldsymbol{y})\\ &= \boldsymbol{x}^p + \boldsymbol{y}^p + \boldsymbol{x}^{p-1}\circ \boldsymbol{y} + \boldsymbol{y}^{p-1}\circ \boldsymbol{x}\\ &= \boldsymbol{x}^p + \boldsymbol{y}^p.\end{aligned}$$

Therefore,

$$\boldsymbol{x}+\boldsymbol{y}=(\boldsymbol{x}^p+\boldsymbol{y}^p)^{1/p}=(|\boldsymbol{x}|^p+|\boldsymbol{y}|^p)^{1/p},$$

that is,

$$\phi_p(\boldsymbol{x},\ \boldsymbol{y})=\boldsymbol{x}+\boldsymbol{y}-(|\boldsymbol{x}|^p+|\boldsymbol{y}|^p)^{1/p}=0.$$

Second, suppose that

$$\phi_p(\boldsymbol{x},\ \boldsymbol{y})=\boldsymbol{x}+\boldsymbol{y}-(|\boldsymbol{x}|^p+|\boldsymbol{y}|^p)^{1/p}=0,$$

that is,

$$\boldsymbol{x}+\boldsymbol{y}=(|\boldsymbol{x}|^p+|\boldsymbol{y}|^p)^{1/p}.$$

Setting

$$\omega=(|\boldsymbol{x}|^p+|\boldsymbol{y}|^p)^{1/p},$$

we have

$$\omega^p=(|\boldsymbol{x}|^p+|\boldsymbol{y}|^p)\succeq|\boldsymbol{x}|$$

and

$$\omega^p=(|\boldsymbol{x}|^p+|\boldsymbol{y}|^p)\succeq|\boldsymbol{y}|.$$

By Löwner-Heinz inequality, we know that $\omega\succeq|\boldsymbol{x}|$ and $\omega\succeq|\boldsymbol{y}|$. Since $|\boldsymbol{x}|\succeq\boldsymbol{x}$ and $|\boldsymbol{y}|\succeq\boldsymbol{y}$, we have

$$\boldsymbol{x}=\omega-\boldsymbol{y}\succeq\omega-|\boldsymbol{y}|\succeq 0,\qquad \boldsymbol{y}=\omega-\boldsymbol{x}\succeq\omega-|\boldsymbol{x}|\succeq 0.$$

Thus,

$$|\boldsymbol{x}|=\boldsymbol{x},\qquad |\boldsymbol{y}|=\boldsymbol{y},$$

and so

$$(\boldsymbol{x}+\boldsymbol{y})^p=(|\boldsymbol{x}|^p+|\boldsymbol{y}|^p)=\boldsymbol{x}^p+\boldsymbol{y}^p.$$

Therefore,

$$(\boldsymbol{x}+\boldsymbol{y})^{p-1}=\boldsymbol{x}^{p-1}+\boldsymbol{y}^{p-1}.$$

From

$$\begin{aligned}(\boldsymbol{x}+\boldsymbol{y})^p&=(\boldsymbol{x}+\boldsymbol{y})^{p-1}\circ(\boldsymbol{x}+\boldsymbol{y})\\&=(\boldsymbol{x}^{p-1}+\boldsymbol{y}^{p-1})\circ(\boldsymbol{x}+\boldsymbol{y})\\&=\boldsymbol{x}^p+\boldsymbol{y}^p+\boldsymbol{x}^{p-1}\circ\boldsymbol{y}+\boldsymbol{y}^{p-1}\circ\boldsymbol{x}\\&=\boldsymbol{x}^p+\boldsymbol{y}^p,\end{aligned}$$

we have

$$\boldsymbol{x}^{p-1}\circ\boldsymbol{y}+\boldsymbol{y}^{p-1}\circ\boldsymbol{x}=0.$$

The associative property of the inner product yields

$$0=\langle\boldsymbol{x}^{p-1}\circ\boldsymbol{y}+\boldsymbol{y}^{p-1}\circ\boldsymbol{x},\ \boldsymbol{e}\rangle=\langle\boldsymbol{x}^{p-1},\ \boldsymbol{y}\rangle+\langle\boldsymbol{y}^{p-1},\ \boldsymbol{x}\rangle.$$

Since $\boldsymbol{x},\boldsymbol{y}\succeq 0$, the terms $\langle\boldsymbol{x}^{p-1},\ \boldsymbol{y}\rangle$ and $\langle\boldsymbol{y}^{p-1},\ \boldsymbol{x}\rangle$ are non-negative and hence must be zero. It follows from Proposition 3.1 together with its proof that

$$\boldsymbol{x}^{p-1}\circ\boldsymbol{y}=0.$$

Hence

$$\boldsymbol{x} \circ \boldsymbol{y} = 0.$$

We conclude that $\phi_p(\boldsymbol{x},\ \boldsymbol{y}) : V \times V \to V$ defined as (1.7) is a C-function.

Similarly, we show that a penalized version of the generalized FB function is also a C-function of SCCP. For $p > 1$ and $\alpha > 0$, define a function as

$$\psi_p(\boldsymbol{x},\ \boldsymbol{y}) = \boldsymbol{x} + \boldsymbol{y} - (|\boldsymbol{x}|^p + |\boldsymbol{y}|^p)^{1/p} + \alpha \boldsymbol{x}_+ \circ \boldsymbol{y}_+. \tag{3.1}$$

We call $\psi_p(\boldsymbol{x},\ \boldsymbol{y})$ the penalized version of the generalized FB function.

Theorem 3.2 *For $p > 1$ and $\alpha > 0$, the penalized version of the generalized FB function*

$$\psi_p(\boldsymbol{x},\ \boldsymbol{y}) = \boldsymbol{x} + \boldsymbol{y} - (|\boldsymbol{x}|^p + |\boldsymbol{y}|^p)^{1/p} + \alpha \boldsymbol{x}_+ \circ \boldsymbol{y}_+$$

is still a C-function of SCCP.

Proof. First, assume that $\boldsymbol{x}, \boldsymbol{y} \succeq 0$ and $\boldsymbol{x} \circ \boldsymbol{y} = 0$. Then $\boldsymbol{x}_+ = \boldsymbol{x}$ and $\boldsymbol{y}_+ = \boldsymbol{y}$. We have

$$\boldsymbol{x}_+ \circ \boldsymbol{y}_+ = \boldsymbol{x} \circ \boldsymbol{y} = 0.$$

Hence, from Theorem 3.1, one has

$$\psi_p(\boldsymbol{x},\ \boldsymbol{y}) = \boldsymbol{x} + \boldsymbol{y} - (|\boldsymbol{x}|^p + |\boldsymbol{y}|^p)^{1/p} = \phi_p(\boldsymbol{x},\ \boldsymbol{y}) = 0.$$

Second, suppose that

$$\psi_p(\boldsymbol{x},\ \boldsymbol{y}) = \boldsymbol{x} + \boldsymbol{y} - (|\boldsymbol{x}|^p + |\boldsymbol{y}|^p)^{1/p} + \alpha \boldsymbol{x}_+ \circ \boldsymbol{y}_+ = 0.$$

We decompose $\boldsymbol{x}$ as $\boldsymbol{x} = \boldsymbol{x}_+ - \boldsymbol{x}_-$, where $\boldsymbol{x}_- = (-\boldsymbol{x})_+$. Taking the inner product with $-\boldsymbol{x}_-$, we have

$$\begin{aligned}
0 &= \langle -\boldsymbol{x}_-,\ \boldsymbol{x} + \boldsymbol{y} - (|\boldsymbol{x}|^p + |\boldsymbol{y}|^p)^{1/p} + \alpha \boldsymbol{x}_+ \circ \boldsymbol{y}_+ \rangle \\
&= \langle -\boldsymbol{x}_-,\ \boldsymbol{x}_+ - \boldsymbol{x}_- - [(|\boldsymbol{x}|^p + |\boldsymbol{y}|^p)^{1/p} - \boldsymbol{y}] + \alpha \boldsymbol{x}_+ \circ \boldsymbol{y}_+ \rangle \\
&= \langle -\boldsymbol{x}_-,\ \boldsymbol{x}_+ \rangle + \langle \boldsymbol{x}_-,\ \boldsymbol{x}_- \rangle + \langle \boldsymbol{x}_-,\ (|\boldsymbol{x}|^p + |\boldsymbol{y}|^p)^{1/p} - \boldsymbol{y} \rangle + \langle -\boldsymbol{x}_-,\ \alpha \boldsymbol{x}_+ \circ \boldsymbol{y}_+ \rangle \\
&= 0 + \|\boldsymbol{x}_-\|^2 + \langle \boldsymbol{x}_-,\ (|\boldsymbol{x}|^p + |\boldsymbol{y}|^p)^{1/p} - \boldsymbol{y} \rangle + \langle -\boldsymbol{x}_- \circ \boldsymbol{x}_+,\ \alpha \boldsymbol{y}_+ \rangle \\
&= \|\boldsymbol{x}_-\|^2 + \langle \boldsymbol{x}_-,\ (|\boldsymbol{x}|^p + |\boldsymbol{y}|^p)^{1/p} - \boldsymbol{y} \rangle + 0.
\end{aligned}$$

It follows from Löwner-Heinz inequality that

$$(|\boldsymbol{x}|^p + |\boldsymbol{y}|^p)^{1/p} - \boldsymbol{y} \in K.$$

Hence $\|\boldsymbol{x}_-\|^2$ and $\langle \boldsymbol{x}_-,\ (|\boldsymbol{x}|^p + |\boldsymbol{y}|^p)^{1/p} - \boldsymbol{y} \rangle$ are both nonnegative. Since

$$0 = \|\boldsymbol{x}_-\|^2 + \langle \boldsymbol{x}_-,\ (|\boldsymbol{x}|^p + |\boldsymbol{y}|^p)^{1/p} - \boldsymbol{y} \rangle,$$

we obtain $\boldsymbol{x}_- = 0$. So $\boldsymbol{x} \succeq 0$.

Similarly, we can get $\boldsymbol{y} \succeq 0$.

Thus

$$\boldsymbol{x} + \boldsymbol{y} - (\boldsymbol{x}^p + \boldsymbol{y}^p)^{1/p} + \alpha \boldsymbol{x} \circ \boldsymbol{y} = 0.$$

So we have

$$(\boldsymbol{x} + \boldsymbol{y} + \alpha \boldsymbol{x} \circ \boldsymbol{y})^p = (\boldsymbol{x}^p + \boldsymbol{y}^p),$$

that is,

$$\begin{aligned}
(\boldsymbol{x} + \boldsymbol{y} + \alpha \boldsymbol{x} \circ \boldsymbol{y})^p &= (\boldsymbol{x} + \boldsymbol{y} + \alpha \boldsymbol{x} \circ \boldsymbol{y})^{p-1} \circ (\boldsymbol{x} + \boldsymbol{y} + \alpha \boldsymbol{x} \circ \boldsymbol{y}) \\
&= (\boldsymbol{x}^{p-1} + \boldsymbol{y}^{p-1}) \circ (\boldsymbol{x} + \boldsymbol{y} + \alpha \boldsymbol{x} \circ \boldsymbol{y}) \\
&= \boldsymbol{x}^p + \boldsymbol{y}^p + \boldsymbol{x}^{p-1} \circ \boldsymbol{y} + \alpha \boldsymbol{x}^{p-1} \circ (\boldsymbol{x} \circ \boldsymbol{y}) + \boldsymbol{y}^{p-1} \circ \boldsymbol{x} + \alpha \boldsymbol{y}^{p-1} \circ (\boldsymbol{x} \circ \boldsymbol{y}) \\
&= \boldsymbol{x}^p + \boldsymbol{y}^p.
\end{aligned}$$

We have

$$\boldsymbol{x}^{p-1} \circ \boldsymbol{y} + \alpha \boldsymbol{x}^{p-1} \circ (\boldsymbol{x} \circ \boldsymbol{y}) + \boldsymbol{y}^{p-1} \circ \boldsymbol{x} + \alpha \boldsymbol{y}^{p-1} \circ (\boldsymbol{x} \circ \boldsymbol{y}) = 0.$$

The associative property of the inner product yields

$$\begin{aligned} 0 &= \langle \boldsymbol{x}^{p-1} \circ \boldsymbol{y} + \alpha \boldsymbol{x}^{p-1} \circ (\boldsymbol{x} \circ \boldsymbol{y}) + \boldsymbol{y}^{p-1} \circ \boldsymbol{x} + \alpha \boldsymbol{y}^{p-1} \circ (\boldsymbol{x} \circ \boldsymbol{y}),\ \boldsymbol{e} \rangle \\ &= \langle \boldsymbol{x}^{p-1},\ \boldsymbol{y} \rangle + \alpha \langle \boldsymbol{x}^{p-1},\ \boldsymbol{x} \circ \boldsymbol{y} \rangle + \langle \boldsymbol{y}^{p-1},\ \boldsymbol{x} \rangle + \alpha \langle \boldsymbol{y}^{p-1},\ \boldsymbol{x} \circ \boldsymbol{y} \rangle \\ &= \langle \boldsymbol{x}^{p-1},\ \boldsymbol{y} \rangle + \langle \boldsymbol{y}^{p-1},\ \boldsymbol{x} \rangle + \alpha \langle \boldsymbol{x}^{p},\ \boldsymbol{y} \rangle + \alpha \langle \boldsymbol{y}^{p},\ \boldsymbol{x} \rangle. \end{aligned}$$

Since $\boldsymbol{x}, \boldsymbol{y} \succeq 0$, all the terms are non-negative and hence must be zero. It follows from Proposition 3.1(5) together with its proof that $\boldsymbol{x}^p \circ \boldsymbol{y} = 0$. Hence $\boldsymbol{x} \circ \boldsymbol{y} = 0$.

4 Concluding Remarks

In this paper, we extend the generalized FB function to SCCP and introduce a penalized generalized FB function over symmetric. In future research, the next logical step is to analyze semi-smoothness of differentiability of the generalized FB function.

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