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Normality Criteria of Meromorphic Functions Concerning Shared Analytic Function

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Abstract: In this paper, we use Pang-Zalcman lemma to investigate the normal family of meromorphic functions concerning shared analytic function, which improves some earlier related results.

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1 Introduction and Main Results

Let D be a domain in $\mathbf{C}$, and $\mathcal{F}$ be a family of meromorphic functions defined in the domain D. $\mathcal{F}$ is said to be normal in D if any sequence $\{f_n\} \subset \mathcal{F}$ contains a subsequence f_{n_j}, which converges spherically locally uniformly in D to a meromorphic function or ∞ (see [1]–[5]).

Let $f(z)$ be a mermorphic function in a domain D and $z_0 \in D$. If $f(z_0) = z_0$, we say that z_0 is the fixed-point of $f(z)$. Let $f(z)$ and $g(z)$ denote two meromorphic functions in D. If $f(z) - \psi(z)$ and $g(z) - \psi(z)$ have the same zeros (or ignoring multiplicity), then we say that $f(z)$ and $g(z)$ share $\psi(z)$ CM (or IM).

In 1998, Wang and Fang[6] proved the following result:

Theorem 1.1 *Let k and $n \geq k + 1$ be two positive integers, and f be a transcendental merimorphic function. Then $(f^n)^{(k)}$ assumes every finite nonzero value infinitely often.*

Corresponding to Theorem 1.1, there are the following theorems about normal families.

Theorem 1.2[7] *Let k and $n \geq k + 3$ be two positive integers and $\mathcal{F}$ be a family of meromorphic functions defined in a domain D. If $(f^n)^{(k)} \neq 1$ for every function $f \in \mathcal{F}$, then $\mathcal{F}$ is normal in D.*

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In 2009, Li and Gu[8] proved:

Theorem 1.3 *Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D. Let $k, n \geq k+2$ be positive integers and $a \neq 0$ be a finite complex number. For each pair $(f, g) \in \mathcal{F}$, if $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share a in D, then $\mathcal{F}$ is normal in D.*

Lately, many authors studied the functions of the form $f(f^{(k)})^n$. Hu and Meng[9] proved:

Theorem 1.4 *Take positive integers n and k with $n, k \geq 2$, and take a non-zero complex number a. Let $\mathcal{F}$ be a family of meromorphic functions in the plane domain D such that each $f \in \mathcal{F}$ has all its zeros of multiplicity at least k. For each pair $(f,\ g) \in \mathcal{F}$, if $f(f^{(k)})^n$ and $g(g^{(k)})^n$ share a IM, then $\mathcal{F}$ is normal in D.*

Recently, Jiang and Gao[10] extended Theorem 1.4 as follows:

Theorem 1.5 *Let $m \geq 0$, $n \geq 2m+2$ and $k \geq 2$ be three positive integers and m be divisible by $n+1$. Suppose that $a(z) (\not\equiv 0)$ is a holomorphic function with zeros of multiplicity m in a domain D. Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, and for each $f \in \mathcal{F}$, f has all its zeros of multiplicity $\max\{k+m,\ 2m+2\}$ at least. For each pair $(f,\ g) \in \mathcal{F}$, if $f(f^{(k)})^n$ and $g(g^{(k)})^n$ share $a(z)$ IM, then $\mathcal{F}$ is normal in D.*

A natural question is: What can be said if the function $f(f^{(k)})^n$ in Theorem 1.5 is replaced by the function $f^d(f^{(k)})^n$? In this paper, we answer this question by proving the following theorem:

Theorem 1.6 *Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D, and $m \geq 0$, $n \geq 2m+2$, $k \geq 2$, $d \geq 1$, $p \geq 1$ be five integers and m be divisible by $n+d$. Let $\psi(z) \not\equiv 0$ be an analytic function with zeros of multiplicity m in a domain D. Suppose that every $f \in \mathcal{F}$ has all its zeros of multiplicity at least $p \geq \max\left\{k + \dfrac{m}{d},\ 2m+2\right\}$. For each pair $(f,\ g) \in \mathcal{F}$, if $f^d(f^{(k)})^n$ and $g^d(g^{(k)})^n$ share $\psi(z)$ IM, then $\mathcal{F}$ is normal in D.*

Remark 1.1 Obviously, from Theorem 1.6, we can get Theorem 1.5 when $d = 1$.

2 Some Lemmas

In order to prove Theorem 1.6, we require the following results.

Lemma 2.1[11] *Let $\mathcal{F}$ be a family of meromorphic functions on the unit disc satisfying all zeros of functions in $\mathcal{F}$ have multiplicity $\geq p$ and all poles of functions in $\mathcal{F}$ have multiplicity $\geq q$. Let α be a real number satisfying $-q < \alpha < p$. Then $\mathcal{F}$ is not normal at 0 if and only if there exist*

a) *a number $0 < r < 1$;*
b) *points z_n with $|z_n| < r$;*

c) *functions* $f_n \in \mathcal{F}$;

d) *positive numbers* $\rho_n \to 0$

such that $g_n(\zeta) := \rho_n^{-\alpha} f_n(z_n + \rho_n\zeta)$ *converges spherically uniformly on each compact subset of* $\mathbf{C}$ *to a non-constant meromorphic function* $g(\zeta)$, *whose all zeros have multiplicity* $\geq p$ *and all poles have multiplicity* $\geq q$ *and order is at most* 2.

Lemma 2.2 *Let* $m \geq 0$, $k, n \geq 2$, $d \geq 1$ *be four integers,* $H(z) = a_m z^m + a_{m-1}z^{m-1} + \cdots + a_0$ *be a polynomial, where* $a_m(\neq 0)$, $a_{m-1}, \cdots, a_0$ *are constants. If* f *is a non-constant polynomial, and the multiplicity of its all zeros is at least* $k + \dfrac{m}{d}$, *then* $f^d(z)(f^{(k)}(z))^n - H(z)$ *has at least two distinct zeros, and* $f^d(z)(f^{(k)}(z))^n - H(z) \not\equiv 0$.

Proof. Since f is a non-constant polynomial with zeros of multiplicity $k + \dfrac{m}{d}$ at least, we know that the degree of f is $k + \dfrac{m}{d}$ at least, and

$$\deg(f^d(z)(f^{(k)}(z))^n) > \deg(H(z)).$$

Then $f^d(z)(f^{(k)}(z))^n - H(z)$ has at least one zero.

If $f^d(z)(f^{(k)}(z))^n - H(z)$ has only one zero, we may assume that

$$f^d(z)(f^{(k)}(z))^n - H(z) = \lambda(z - z_0)^l,$$

where λ is a non-zero constant, l is a positive integer. Compare the degrees of $H(z)$ and $f(z)$, we have

$$l = \deg(f^d(z)(f^{(k)}(z))^n) > m + 1.$$

Then

$$(f^d(z)(f^{(k)}(z))^n)^{(m)} - \lambda \cdot l \cdot (l-1) \cdots (l-m+1)(z-z_0)^{l-m} = H^{(m)}(z) = m! a_m \neq 0,$$
$$(f^d(z)(f^{(k)}(z))^n)^{(m+1)} = \lambda \cdot l \cdot (l-1) \cdots (l-m)(z-z_0)^{l-m-1}.$$

Thus z_0 is the unique zero of $(f^d(z)(f^{(k)}(z))^n)^{(m+1)}$. Since f is a non-constant polynomial with zeros of multiplicity $k + \dfrac{m}{d}$ at least, we know that z_0 is a zero of f. Thus

$$(f^d(f^{(k)})^n)^{(m)}(z_0) = 0,$$

it contradicts with

$$(f^d(f^{(k)})^n)^{(m)}(z_0) = H^{(m)}(z_0) \neq 0.$$

Thus, $f^d(z)(f^{(k)}(z))^n - H(z)$ has at least two distinct zeros.

Lemma 2.3 *Let* $m \geq 0$, $n \geq 2m+2$, $k, d \geq 1$ *be four integers,* $H(z) = a_m z^m + a_{m-1}z^{m-1} + \cdots + a_0$ *be a polynomial, where* $a_m(\neq 0)$, $a_{m-1}, \cdots, a_0$ *are constants. If* f *is a non-polynomial rational function, and the multiplicity of its all zeros is at least* $2m+2$, *then* $f^d(z)(f^{(k)}(z))^n - H(z)$ *has at least two distinct zeros, and* $f^d(z)(f^{(k)}(z))^n - H(z) \not\equiv 0$.

Proof. Since f is a non-polynomial rational function, it is obvious that

$$f^d(z)(f^{(k)}(z))^n - H(z) \not\equiv 0.$$

Let

$$f^d(f^{(k)})^n = \frac{A(z-\alpha_1)^{m_1}(z-\alpha_2)^{m_2}\cdots(z-\alpha_s)^{m_s}}{(z-\beta_1)^{n_1}(z-\beta_2)^{n_2}\cdots(z-\beta_t)^{n_t}}, \tag{2.1}$$

where A is a non-zero constant, $s, t \geq 1$, $m_i \geq 2m+2$ $(i = 1, 2, \cdots, s)$, $n_j \geq n(k+1)+d$ $(j = 1, 2, \cdots, t)$. For simplicity, we denote

$$M = m_1 + m_2 + \cdots + m_s \geq (2m+2)s, \tag{2.2}$$

$$N = n_1 + n_2 + \cdots + n_t \geq [d + n(k+1)]t > (2m+2)t. \tag{2.3}$$

By differentiating both sides of (2.1) step by step, we have

$$\begin{aligned}&(f^d(f^{(k)})^n)^{(m+1)}\\ &= \frac{A(z-\alpha_1)^{m_1-(m+1)}(z-\alpha_2)^{m_2-(m+1)}\cdots(z-\alpha_s)^{m_s-(m+1)}g_1(z)}{(z-\beta_1)^{n_1+(m+1)}(z-\beta_2)^{n_2+(m+1)}\cdots(z-\beta_t)^{n_t+(m+1)}},\end{aligned} \tag{2.4}$$

where $g_1(z)$ is a non-constant polynomial with $\deg(g_1) \leq (m+1)(s+t-1)$.

Now, we discuss two cases.

Case 1. If $f^d(z)(f^{(k)}(z))^n - H(z)$ has a unique zero z_0, then we set

$$f^d(f^{(k)})^n = H(z) + \frac{B(z-z_0)^l}{(z-\beta_1)^{n_1}(z-\beta_2)^{n_2}\cdots(z-\beta_t)^{n_t}} = \frac{P(z)}{Q(z)}, \tag{2.5}$$

where B is a non-zero constant and l is a positive integer, P and Q are polynomials with degree M and N, also P and Q have no common factors.

Here we discuss two subcases.

Subcase 1.1. $m \geq l$.

By differentiating both sides of (2.5), we have

$$\begin{aligned}&(f^d(f^{(k)})^n)^{(m+1)}\\ &= H^{(m+1)}(z) + \frac{g_2(z)}{(z-\beta_1)^{n_1+(m+1)}(z-\beta_2)^{n_2+(m+1)}\cdots(z-\beta_t)^{n_t+(m+1)}},\end{aligned} \tag{2.6}$$

where $g_2(z)$ is a polynomial with $\deg(g_2) \leq (m+1)t - (m-l+1)$. By (2.1) and (2.5), since $m \geq l$, one has

$$N + m \leq M.$$

From (2.4) and (2.6),

$$M - (m+1)s \leq (m+1)t - (m-l+1).$$

Then

$$\begin{aligned}l - m &\geq M - (m+1)(s+t) + 1\\ &> M - (m+1)\Big(\frac{M}{2m+2} + \frac{N}{2m+2}\Big) + 1\\ &> M - (m+1)\Big(\frac{M}{2m+2} + \frac{M}{2m+2}\Big) + 1\\ &= 1,\end{aligned}$$

it contradicts with $m \geq l$.

Subcase 1.2. $m < l$.

By differentiating both sides of (2.5), we have

$$\begin{aligned}&(f^d(f^{(k)})^n)^{(m+1)}\\ &= H^{(m+1)}(z) + \frac{(z-z_0)^{l-(m+1)}g_3(z)}{(z-\beta_1)^{n_1+(m+1)}(z-\beta_2)^{n_2+(m+1)}\cdots(z-\beta_t)^{n_t+(m+1)}},\end{aligned} \tag{2.7}$$

where $g_3(z)$ is a polynomial with $\deg(g_3) \le (m+1)t$.

By differentiating both sides of (2.5) step by step for m times, we can get that z_0 is a zero of $(f^d(f^{(k)})^n)^{(m)} = H^{(m)}$. Since $H^{(m)} = a_m \neq 0$, one has

$$z_0 \neq \alpha_i, \qquad i = 1, 2, \cdots, s.$$

Here we discuss in two subcases.

Subcase 1.2.1. $l \neq N + m$.

From (2.1) and (2.5), we obtain $\deg(P) \ge \deg(Q)$, that is, $M \ge N$. Since $z_0 \neq \alpha_i$ $(i = 1, 2, \cdots, s)$, (2.4) and (2.7) imply

$$\sum_{i=1}^{s}[m_i - (m+1)] = M - (m+1)s \le \deg(g_3) \le (m+1)t.$$

So

$$M \le (m+1)(s+t).$$

By using (2.2) and (2.3), we obtain

$$\begin{aligned} M &\le (m+1)(s+t) \\ &< (m+1)\Big(\frac{M}{2m+2} + \frac{N}{2m+2}\Big) \\ &\le (m+1)\Big(\frac{M}{2m+2} + \frac{M}{2m+2}\Big) \\ &= M, \end{aligned}$$

which is a contradiction.

Subcase 1.2.2. $l = N + m$.

We further distinguish two subcases.

(i) $M \ge N$.

By (2.4) and (2.7), we obtain

$$M - (m+1)s \le (m+1)t.$$

Similar to Subcase 1.2.1, we obtain a contradiction $M < M$.

(ii) $M < N$.

By using (2.4) and (2.7) again, we obtain

$$l - m - 1 \le \deg(g_1) \le (m+1)(s+t-1).$$

Hence

$$\begin{aligned} N &= l - m \\ &\le (m+1)(s+t-1) + (m+1) - m \\ &\le (m+1)(s+t) \\ &< (m+1)\Big(\frac{M}{2m+2} + \frac{N}{2m+2}\Big) \\ &\le N, \end{aligned}$$

which is a contradiction.

Case 2. If $f^d(f^{(k)})^n - H(z)$ has no zero, then $l = 0$ in (2.5). Proceeding as in the proof of Case 1, we get a contradiction.

Lemma 2.3 is proved.

Lemma 2.4[12] *Suppose that $f(z)$ is a transcendental meromorphic function, n, k, d are three positive integers. Then, when $k\geq 1$, $n,d\geq 2$, $f^d(f^{(k)})^n-\varphi(z)$ has infinitely many zeros, where $\varphi(z)\not\equiv 0$, $T(r,\varphi)=S(r,f)$.*

3 Proof of Theorem 1.6

From Theorem 1.5, when $d=1$, Theorem 1.6 holds.

Next, we prove the case $d\geq 2$.

For any point $z_0\in D$, either $\psi(z_0)=0$ or $\psi(z_0)\neq 0$.

Case 1. $\psi(z_0)=0$.

We may assume $z_0=0$ and $\psi(z)=z^m+a_{m+1}z^{m+1}+\cdots=z^mh(z)$, where a_{m+1}, a_{m+2}, $\cdots$ are constants, $h(0)=1$, and m can be divisible by $n+d$.

Let

$$\mathcal{F}_1=\left\{F_j:F_j(z)=\frac{f_j(z)}{z^{\frac{m}{n+d}}}\Big|f_j\in\mathcal{F}\right\}.$$

If $\mathcal{F}_1$ is not normal at 0, by Lemma 2.1, there exist a sequence $\{z_j\}$ of complex numbers with $z_j\to z_0$ and a sequence $\{\rho_j\}$ of positive numbers with $\rho_j\to 0$ such that

$$g_j(\xi)=\rho_j^{-\frac{kn}{n+d}}F_j(z_j+\rho_j\xi)\to g(\xi)$$

locally uniformly on compact subsets of $\mathbf{C}$, where $g(\xi)$ is a non-constant meromorphic function in $\mathbf{C}$, all of whose zeros have multiplicity at least $p\geq\max\left\{k+\dfrac{m}{d},\ 2m+2\right\}$. Moreover, $g(\xi)$ has order at most 2.

Here we distinguish two cases.

Case 1.1. Suppose that $\dfrac{z_j}{\rho_j}\to c$, c is a finite complex number. Then

$$\phi_j(\xi)=\frac{f_j(\rho_j\xi)}{\rho_j^{\frac{m+kn}{n+d}}}=\frac{F\Big(z_j+\rho_j\Big(\xi-\dfrac{z_j}{\rho_j}\Big)\Big)}{\rho_j^{\frac{kn}{n+d}}}\frac{(\rho_j\xi)^{\frac{m}{n+d}}}{\rho_j^{\frac{m}{n+d}}}\to\xi^{\frac{m}{n+d}}g(\xi-c)=H(\xi)$$

locally uniformly on compact subsets of $\mathbf{C}$ disjoint from the poles of g, where $H(\xi)$ is a non-constant meromorphic function in $\mathbf{C}$, all of whose zeros have multiplicity at least $p\geq\max\left\{k+\dfrac{m}{d},\ 2m+2\right\}$. Moreover, $H(\xi)$ has order at most 2. So

$$\phi_j^d(\xi)(\phi_j^{(k)}(\xi))^n-\frac{\psi(\rho_j\xi)}{\rho_j^m}=\frac{f_j^d(\rho_j\xi)(f_j^{(k)}(\rho_j\xi))^n-\psi(\rho_j\xi)}{\rho_j^m}\to H^d(\xi)(H^{(k)}(\xi))^n-\xi^m$$

spherically locally uniformly in $\mathbf{C}$ disjoint from the poles of g.

If $H^d(\xi)(H^{(k)}(\xi))^n\equiv\xi^m$, since H has zeros with multiplicity at least $p\geq\max\left\{k+\dfrac{m}{d},\ 2m+2\right\}$, obviously there is a contradiction. Hence $H^d(\xi)(H^{(k)}(\xi))^n\not\equiv\xi^m$.

Since the multiplicity of all zeros of H is at least $p\geq\max\left\{k+\dfrac{m}{d},\ 2m+2\right\}$, by Lemmas 2.2, 2.3 and 2.4, $H^d(\xi)(H^{(k)}(\xi))^n-\xi^m$ has at least two distinct zeros.

Suppose that ξ_0, ξ_0^* are two distinct zeros of $H^d(\xi)(H^{(k)}(\xi))^n-\xi^m$. We choose a positive number δ small enough such that $D_1\bigcap D_2=\emptyset$ and $H^d(\xi)(H^{(k)}(\xi))^n-\xi^m$ has no other zeros in $D_1 \bigcup D_2$ except for ξ_0 and ξ_0^*, where

$$D_1 = \{\xi \in \mathbf{C} \mid |\xi - \xi_0| < \delta\},$$
$$D_2 = \{\xi \in \mathbf{C} \mid |\xi - \xi_0^*| < \delta\}.$$

By Hurwitz's theorem, there exists a subsequence of $f_j^d(f_j^{(k)})^n - \psi(z_j + \rho_j\xi)$, we still denote it as $f_j^d(f_j^{(k)})^n - \psi(z_j + \rho_j\xi)$, then there exist points $\xi_j^* \to \xi_0^*$ and points $\xi_j \to \xi_0$ such that when j is large enough,

$$f_j^d(\rho_j\xi_j^*)(f_j^{(k)}(\rho_j\xi_j^*))^n - \psi(\rho_j\xi_j^*) = 0,$$
$$f_j^d(\rho_j\xi_j)(f_j^{(k)}(\rho_j\xi_j))^n - \psi(\rho_j\xi_j) = 0.$$

Since, by the assumption in Theorem 1.6, $f_m^d(f_m^{(k)})^n$ and $f_j^d(f_j^{(k)})^n$ share $\psi(z)$, it follows that

$$f_m^d(\rho_j\xi_j^*)(f_m^{(k)}(\rho_j\xi_j^*))^n - \psi(\rho_j\xi_j^*) = 0,$$
$$f_m^d(\rho_j\xi_j)(f_m^{(k)}(\rho_j\xi_j))^n - \psi(\rho_j\xi_j) = 0.$$

Fix m and let $j \to \infty$, note $\rho_j\xi_j \to 0$, $\rho_j\xi_j^* \to 0$, we obtain

$$f_m^d(0)(f_m^{(k)}(0))^n - \psi(0) = 0.$$

Since the zeros of $f_m^d(\xi)(f_m^{(k)}(\xi))^n - \psi(\xi)$ has no accumulation point, for sufficiently large j, we have

$$\rho_j\xi_j = 0, \qquad \rho_j\xi_j^* = 0.$$

Thus, when j is large enough, $\xi_0 = \xi_0^*$. This contradicts with the facts $\xi_n \in D_1$, $\xi_n^* \in D_2$, $D_1 \bigcap D_2 = \emptyset$. Thus $\mathcal{F}_1$ is normal at 0.

Case 1.2. Suppose that $\dfrac{z_j}{\rho_j} \to \infty$. We have

$$\begin{aligned} f_j^{(k)}(z) &= z^{\frac{m}{n+d}} F_j^{(k)}(z) + \sum_{l=1}^{k} C_k^l (z^{\frac{m}{n+d}})^{(l)} F_j^{(k-l)}(z) \\ &= z^{\frac{m}{n+d}} F_j^{(k)}(z) + \sum_{l=1}^{k} c_l z^{\frac{m}{n+d}-l} F_j^{(k-l)}(z), \end{aligned}$$

where

$$c_l = \begin{cases} C_k^l \dfrac{m}{n+d}\Big(\dfrac{m}{n+d} - 1\Big)\cdots\Big(\dfrac{m}{n+d} - l + 1\Big), & l \le \dfrac{m}{n+d}; \\ 0, & l > \dfrac{m}{n+d}. \end{cases}$$

Thus we have

$$\begin{aligned} f_j^d(z)(f_j^{(k)}(z))^n &= \Big(z^{\frac{m}{n+d}} F_j^{(k)}(z) + \sum_{l=1}^{k} c_l z^{\frac{m}{n+d}-l} F_j^{(k-l)}(z)\Big)^n z^{\frac{md}{n+d}} F_j^d(z) \\ &= \Big(z^{\frac{m}{n+d}+\frac{md}{(n+d)n}} F_j^{(k)}(z) F_j^{\frac{d}{n}}(z) \\ &\quad + \sum_{l=1}^{k} c_l z^{\frac{m}{n+d}+\frac{md}{(n+d)n}-l} F_j^{(k-l)}(z) F_j^{\frac{d}{n}}(z)\Big)^n, \end{aligned}$$

$$\frac{f_j^d(z)(f_j^{(k)}(z))^n}{\psi(z)} = \Big(z^{\frac{m}{n+d}+\frac{md}{(n+d)n}-\frac{m}{n}} F_j^{(k)}(z) F_j^{\frac{d}{n}}(z)$$

$$+\sum_{l=1}^{k} c_l z^{\frac{m}{n+d}+\frac{md}{(n+d)n}-\frac{m}{n}-l} F_j^{(k-l)}(z) F_j^{\frac{d}{n}}(z)\Big)^n \frac{1}{h(z)}$$

$$=\left(F_j^{(k)}(z)F_j^{\frac{d}{n}}(z)+\sum_{l=1}^{k} c_l \frac{F_j^{(k-l)}(z)F_j^{\frac{d}{n}}(z)}{z^l}\right)^n \frac{1}{h(z)}.$$

Since

$$F_j^{(k-l)}=\rho_j^{\frac{kn}{n+d}-(k-l)} g_j^{(k-l)},$$

we have

$$\frac{f_j^d(z_j+\rho_j\xi)(f_j^{(k)}(z_j+\rho_j\xi))^n}{\psi(z_j+\rho_j\xi)}$$
$$=\left(g_j^{(k)}(\xi)g_j^{\frac{d}{n}}(\xi)+\sum_{l=1}^{k} c_l \frac{g_j^{(k-l)}(\xi)g_j^{\frac{d}{n}}(\xi)}{\left(\dfrac{z_j}{\rho_j}+\xi\right)^l}\right)^n \frac{1}{h(z_j+\rho_j\xi)}.$$

On the other hand, for $l=1,2,\cdots,k$, we have

$$\lim_{j\to\infty}\frac{c_l}{\left(\dfrac{z_j}{\rho_j}+\xi\right)^l}=0,\qquad \lim_{j\to\infty}\frac{1}{h(z_j+\rho_j\xi)}=1.$$

Thus we have

$$\frac{f_j^d(z_j+\rho_j\xi)(f_j^{(k)}(z_j+\rho_j\xi))^n}{\psi(z_j+\rho_j\xi)}-1\to g^d(\xi)(g^{(k)}(\xi))^n-1$$

spherically locally uniformly in $\mathbf{C}$ disjoint from the poles of g.

If $g^d(\xi)(g^{(k)}(\xi))^n\equiv 1$, then g has no zeros. Of course, g also has no poles. Since g is a non-constant Meromorphic function of order at most 2, there exist constants c_i $(i=1,2)$, $(c_1,\ c_2)\neq(0,\ 0)$, and $g(\xi)=\mathrm{e}^{c_0+c_1\xi+c_2\xi^2}$. Obviously, this is contrary to the case $g^d(\xi)(g^{(k)}(\xi))^n\equiv 1$. Hence

$$g^d(\xi)(g^{(k)}(\xi))^n\not\equiv 1.$$

Since the multiplicity of all zeros of g is at least $p\geq\max\left\{k+\dfrac{m}{d},\ 2m+2\right\}$, by Lemmas 2.2, 2.3 and 2.4, $g^d(\xi)(g^{(k)}(\xi))^n-1$ has at least two distinct zeros.

Suppose that ξ_1, ξ_1^* are two distinct zeros of $g^d(\xi)(g^{(k)}(\xi))^n-1$. We choose a positive number δ small enough such that $D_1\bigcap D_2=\emptyset$ and $g^d(\xi)(g^{(k)}(\xi))^n-1$ has no other zeros in $D_1\bigcup D_2$ except for ξ_1 and ξ_1^*, where

$$D_1=\{\xi\in C\mid |\xi-\xi_1|<\delta\},$$
$$D_2=\{\xi\in C\mid |\xi-\xi_1^*|<\delta\}.$$

By Hurwitz's theorem, there exists a subsequence of $f_j^d(z_j+\rho_j\xi)(f_j^{(k)}(z_j+\rho_j\xi))^n-\psi(z_j+\rho_j\xi)$, we still denote it as $f_j^d(z_j+\rho_j\xi)(f_j^{(k)}(z_j+\rho_j\xi))^n-\psi(z_j+\rho_j\xi)$. Then there exist points $\widehat{\xi_j}\to\xi_1$ and points $\widetilde{\xi_j}\to\xi_1^*$ such that when j is large enough,

$$f_j^d(z_j+\rho_j\widehat{\xi_j})(f_j^{(k)}(z_j+\rho_j\widehat{\xi_j}))^n-\psi(z_j+\rho_j\widehat{\xi_j})=0,$$
$$f_j^d(z_j+\rho_j\widetilde{\xi_j})(f_j^{(k)}(z_j+\rho_j\widetilde{\xi_j}))^n-\psi(z_j+\rho_j\widetilde{\xi_j})=0.$$

Similar to the proof of Case 1.1, we get a contradiction. Then, $\mathcal{F}_1$ is normal at 0.

From Cases 1.1 and 1.2, we know that $\mathcal{F}_1$ is normal at 0, and there exist $\varDelta = \{z : |z| < \rho\}$ and a subsequence of F_j, we still denote it as F_j, such that F_j converges spherically locally uniformly to a meromorphic function $F(z)$ or ∞ in $\varDelta$.

Here we distinguish two cases.

Case (i). When j is large enough, $f_j(0) \neq 0$. Then $F(0) = \infty$. Thus, for each $F_j(z) \in \mathcal{F}_1$, there exists a $\delta > 0$ such that if $F(z) \in \mathcal{F}_1$, then $|F(z)| > 1$ for all $z \in \varDelta_\delta = \{z : |z| < \delta\}$. Thus, for sufficiently large j, $|F_j(z)| \geq 1$, $\dfrac{1}{f_j}$ is holomorphic in $\varDelta_\delta$. Therefore, for all $f_j \in \mathcal{F}$, when $|z| = \delta/2$, we have

$$\Big|\frac{1}{f_j}\Big| = \Big|\frac{1}{F_j(z) z^{\frac{m}{n+d}}}\Big| \leq \Big(\frac{2}{\delta}\Big)^{\frac{m}{n+d}}.$$

By Maximum Principle and Montel's Theorem, $\mathcal{F}$ is normal at $z = 0$.

Case (ii). There exists a subsequence of f_j, we still denote it as f_j, such that $f_j(0) = 0$. Since $f \in \mathcal{F}$, the multiplicity of all zeros of f is at least $p \geq \max\Big\{k + \dfrac{m}{d},\ 2m + 2\Big\}$, then $F(0) = 0$. Thus, there exists $0 < r < \rho$ such that $F(z)$ is holomorphic in $\varDelta_r = \{z : |z| < r\}$ and has a unique zero $z = 0$ in $\varDelta_r$. Then F_j converges spherically locally uniformly to a holomorphic function $F(z)$ in $\varDelta_r$. f_j converges spherically locally uniformly to a holomorphic function $F(z) z^{\frac{m}{n+d}}$ in $\varDelta_r$. Hence $\mathcal{F}$ is normal at $z = 0$.

By Cases (i) and (ii), $\mathcal{F}$ is normal at $z = 0$.

Case 2. $\psi(z_0) \neq 0$.

Suppose that $\mathcal{F}$ is not normal at z_0. By Lemma 2.1 there exist a sequence $\{z_j\}$ of complex numbers with $z_j \to z_0$, a sequence $\{\rho_n\}$ of positive numbers with $\rho_j \to 0$ such that

$$g_j(\xi) = \rho_j^{-\frac{kn}{n+d}} F_j(z_j + \rho_j \xi) \to g(\xi)$$

locally uniformly on compact subsets of $\mathbf{C}$, where $g(\xi)$ is a non-constant meromorphic function in $\mathbf{C}$, all of whose zeros have multiplicity at least $p \geq \max\Big\{k + \dfrac{m}{d},\ 2m + 2\Big\}$. Moreover, $g(\xi)$ has order at most 2.

Hence, by Lemmas 2.2, 2.3 and 2.4, similar to the proof of Case 1.1, we get a contradiction. Thus $\mathcal{F}$ is normal at z_0.

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