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# An Identity with Skew Derivations on Lie Ideals

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**Abstract:** Let $R$ be a 2-torsion free prime ring and $L$ a noncommutative Lie ideal of $R$. Suppose that $(d, \sigma)$ is a skew derivation of $R$ such that $x^s d(x)x^t = 0$ for all $x \in L$, where $s$, $t$ are fixed non-negative integers. Then $d = 0$.

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## 1 Introduction

Throughout this paper, unless specifically stated, $R$ always denotes a prime ring with center $Z(R)$, $Q$ its Martindale quotient ring. Note that $Q$ is also a prime ring and the center $C$ of $Q$, which is called the extended centroid of $R$, is a field (we refer the readers to [1] for the definitions and related properties of these notions). For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$. For subsets $A$, $B$ of $R$, $[A, B]$ is the additive subgroup generated by all $[a, b]$ with $a \in A$ and $b \in B$. An additive subgroup $L$ of $R$ is said to be a Lie ideal of $R$ if $[l, r] \in L$ for all $l \in L$ and $r \in R$. A Lie ideal $L$ is called noncommutative if $[L, L] \neq 0$. Let $L$ be a noncommutative Lie ideal of $R$. It is well known that $[R[L, L]R, R] \subseteq L$ (see the proof of Lemma 1.3 in [2]). Since $[L, L] \neq 0$, we have $0 \neq [I, R] \subseteq L$ for $I = R[L, L]R$ a nonzero ideal of $R$. Recall that a ring $R$ is called prime if for any $x, y \in R$, $xRy = 0$ implies that either $x = 0$ or $y = 0$. An additive mapping $d : R \longrightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. Given any

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automorphism $\sigma$ of $R$, an additive mapping $d : R \to R$ satisfying

$$d(xy) = d(x)y + \sigma(x)d(y), \qquad x, y \in R$$

is called a $\sigma$-derivation of $R$, or a skew derivation of $R$ with respect to $\sigma$, denoted by $(d,\ \sigma)$. It is easy to see that if $\sigma = 1_R$, the identity map of $R$, then a $\sigma$-derivation is merely an ordinary derivation. And if $\sigma \neq 1_R$, then $\sigma - 1_R$ is a skew derivation. Thus the concept of skew derivations can be regarded as a generalization of derivations. When $d(x) = \sigma(x)b - bx$ for some $b \in Q$, then $(d,\ \sigma)$ is called an inner skew derivation, and otherwise it is outer. Any skew derivation $(d,\ \sigma)$ extends uniquely to a skew derivation of $Q$ (see [3]) via extensions of both maps to $Q$. Thus we may assume that any skew derivation of $R$ is the restriction of a skew derivation of $Q$. Recall that $\sigma$ is called an inner automorphism if when acting on $Q$, $\sigma(q) = uqu^{-1}$ for some invertible $u \in Q$. When $\sigma$ is not inner, then it is called an outer automorphism. The skew derivations have been extensively studied by many researchers from various views (see for instance [4]–[7] where further references can be found).

A well-known paper of Herstein[2] states that if $I$ is a right ideal of $R$ such that $x^n = 0$ for all $x \in I$, then $I = 0$. Chang and Lin[8] studied a more general case when $d(x)x^n = 0$ and $x^n d(x) = 0$ for all $x \in I$, where $d$ is a nonzero derivation and $I$ is a nonzero right ideal of a prime ring $R$. Dhara and De Filippis[9] proved the following: Let $R$ be a prime ring, $F$ a generalized derivation of $R$ and $L$ a noncommutative Lie ideal of $R$. Suppose that $x^s F(x)x^t = 0$ for all $x \in L$, where $s \geq 0$, $t \geq 0$ are fixed integers, then $F = 0$ except when $\mathrm{char} R = 2$ and $R$ satisfies $s_4$.

In this paper, we continue to investigation on Lie ideals of prime rings, involving a skew derivation $(d,\ \sigma)$ with a nontrivial associated automorphism $\sigma$. Here we examine what happens replacing the generalized derivation $F$ by a skew derivation $(d,\ \sigma)$ in the result of [9].

# 2 Main Results

**Theorem 2.1** *Let $R$ be a 2-torsion free prime ring and $L$ be a noncommutative Lie ideal of $R$. Suppose that $(d,\ \sigma)$ is a skew derivation of $R$ such that $x^s d(x)x^t = 0$ for all $x \in L$, where $s$, $t$ are fixed non-negative integers. Then $d = 0$.*

*Proof.* Suppose that $d \neq 0$. We divide the proof into two cases.

Case 1. Suppose that $(d,\ \sigma)$ is $X$-outer. Set $I = R[L,\ L]R$. Then $0 \neq [I,\ R] \subseteq L$. By the assumption, we have $[x,\ y]^s(d([x,\ y]))[x,\ y]^t = 0$ for all $x, y \in I$ and also for all $x, y \in Q$ by Theorem 2 in [10]. By Theorem 1 in [11], we get

$$[x,\ y]^s(zy + \sigma(x)w - wx - \sigma(y)z)[x,\ y]^t = 0, \qquad x, y, z, w \in Q. \tag{2.1}$$

Subcase 1.1. If $\sigma$ is $X$-inner, that is, $\sigma(x) = gxg^{-1}$ for some $g \in Q - C$ since $\sigma$ is nontrivial. This implies that

$$[x,\ y]^s(zy + gxg^{-1}w - wx - gyg^{-1}z)[x,\ y]^t = 0, \qquad x, y, z, w \in Q. \tag{2.2}$$

Letting $z = 0$ and replacing $w$ by $gw$ in (2.2), we find that

$$[x,\ y]^s g[x,\ w][x,\ y]^t = 0,$$

and, in particular, when $w = y$, we have

$$[x, y]^s g[x, y][x, y]^t = 0, \qquad x, y \in Q. \tag{2.3}$$

Set $F(x) = gx$ for all $x \in R$. It is easy to see that $F$ is a generalized derivation of $R$. Using Theorem 1 in [9], we find that $F = 0$, that is, $g = 0$, a contradiction.

Subcase 1.2. Suppose that $\sigma$ is $X$-outer. By (2.1), we find that

$$[x, y]^s(zy + mw - wx - nz)[x, y]^t = 0, \qquad x, y, z, m, n, w \in Q. \tag{2.4}$$

Letting $z = 0$ and replacing $m$ by $x$ in (2.4), we get

$$[x, y]^s[x, w][x, y]^t = 0, \qquad x, y, w \in Q.$$

In particular,

$$[x, y]^s[x, y][x, y]^t = 0, \qquad x, y \in Q.$$

Using Theorem 1 in [9] again, we conclude that $1 = 0$, a contradiction.

Case 2. Suppose that $d$ is $X$-inner, that is, $d(x) = \sigma(x)b - bx$ with $0 \neq b \in Q$.

Subcase 2.1. If $\sigma$ is $X$-inner, then there exists an invertible element $q \in Q$ such that $\sigma(x) = qxq^{-1}$, where $q \in Q - C$. So $Q$ satisfies the generalized polynomial identity

$$[x, y]^s(q[x, y]q^{-1}b - b[x, y])[x, y]^t = 0, \qquad x, y \in Q. \tag{2.5}$$

Let $\dim_C V = \infty$ and recall that as Lemma 2 in [12]. The set $[Q, Q]$ is dense in $Q$ and so from

$$[x, y]^s(q[x, y]q^{-1}b - b[x, y])[x, y]^t = 0, \qquad x, y \in [Q, Q], \tag{2.6}$$

we have

$$x^s(qxq^{-1}b - bx)x^t = 0, \qquad x \in Q.$$

Let $v \in V$ such that $\{v, q^{-1}bv\}$ is linearly $C$-independent. Therefore there exist $v_1, \cdots, v_t$, $w \in V$ such that $\{v, q^{-1}bv, v_1, \cdots, v_t, w\}$ is linearly $C$-independent. By the density of $Q$, there exists an $r \in Q$ such that

$$rv_i = v_{i+1}, \quad i = 1, \cdots, t-1, \qquad rv_t = v,$$
$$rv = 0, \qquad r(q^{-1}bv) = q^{-1}w, \qquad rw = w.$$

Thus we get the contradiction

$$0 = r^s(qrq^{-1}b - br)r^t v_1 = w.$$

Hence $\{v, q^{-1}bv\}$ is linearly $C$-dependent for all $v \in V$ and a standard argument shows that $q^{-1}b \in C$, that is, $d = 0$.

Let $\dim_C V = k$ be a finite integer, that is, $Q = M_k(C)$ for $k \geq 2$. Denote $p = q^{-1}b$. Let $i \neq j$ and choose $[x, y] = e_{ii} - e_{jj}$ in (2.5). Both left multiplying by $e_{jj}$ and right multiplying by $e_{ii}$ it follows

$$e_{jj}qe_{ii}pe_{ii} - e_{jj}qe_{jj}pe_{ii} - e_{jj}be_{ii} = 0,$$

in particular,

$$q_{ji}p_{ii} - q_{jj}p_{ji} - b_{ji} = 0. \tag{2.7}$$

Let $\phi$ and $\xi$ be the following automorphisms of $M_k(C)$:

$$\phi(x) = x + e_{ij}x - xe_{ij} - e_{ij}xe_{ij},$$
$$\xi(x) = x - e_{ij}x + xe_{ij} - e_{ij}xe_{ij}.$$

Since $\phi(q)$, $\phi(p)$, $\xi(q)$ and $\xi(p)$ satisfy the same property of $q$ and $p$, it follows that

$$\phi(q)_{ji}\phi(p)_{ii} - \phi(q)_{jj}\phi(p)_{ji} - \phi(b)_{ji} = 0,$$

and also

$$\xi(q)_{ji}\xi(p)_{ii} - \xi(q)_{jj}\xi(p)_{ji} - \xi(b)_{ji} = 0,$$

which means that

$$q_{ji}(p_{ii} + p_{ji}) - (q_{jj} - q_{ji})p_{ji} - b_{ji} = 0,$$

and also

$$q_{ji}(p_{ii} - p_{ji}) - (q_{jj} + q_{ji})p_{ji} - b_{ji} = 0.$$

Comparing these last two relations we get $4q_{ji}p_{ji} = 0$, that is,

$$q_{ji}p_{ji} = 0, \qquad i \neq j. \tag{2.8}$$

If $k \geq 3$, then by Proposition 1 in [13], it follows that either $p \in C$ or $q \in C$. In the first case we get $d = 0$. On the other hand, if $q \in C$ then $d(x) = [x, b]$ and the result follows as an application of main theorem in [14]. Let $k = 2$, that is, $Q \cong M_2(C)$. Assume that neither $\boldsymbol{q}$ nor $\boldsymbol{p}$ is a diagonal matrix in $M_2(C)$. Without loss of generality, we consider $p_{21} \neq 0$. Thus, by (2.8), it follows $q_{21} = 0$, then $q_{12} \neq 0$ and so $p_{12} = 0$. Moreover, since $\boldsymbol{q}$ is invertible, we also have $q_{22} \neq 0$. Notice that, if $\boldsymbol{u} \in [Q, Q]$ is an invertible matrix, then by (2.6) it follows that

$$\boldsymbol{X} = \boldsymbol{qup} - \boldsymbol{bu} = \boldsymbol{0}.$$

For $u = e_{11} - e_{22}$ and by computations it follows that the $(2, 1)$-entry of the matrix $\boldsymbol{X}$ is

$$q_{22}p_{21} + b_{21} = 0. \tag{2.9}$$

On the other hand, for $u = e_{12} + e_{21}$, it follows that the $(2, 2)$-entry of the matrix $\boldsymbol{X}$ is $b_{21} = 0$. Thus, by (2.9) we get the contradiction $q_{22}p_{21} = 0$. The previous contradiction means that either $\boldsymbol{q}$ or $\boldsymbol{p}$ is diagonal. In this case, a standard argument shows that either $\boldsymbol{q}$ or $\boldsymbol{p}$ is central and we are done as above.

Subcase 2.2. If $\sigma$ is $X$-outer, then

$$[x, y]^s([\sigma(x), \sigma(y)]b - b[x, y])[x, y]^t = 0.$$

So by Kharchenko[15] we find that

$$[x, y]^s([m, n]b - b[x, y])[x, y]^t = 0, \qquad x, y, m, n \in Q.$$

Setting $m = 0$, we get

$$[x, y]^s(b[x, y])[x, y]^t = 0, \qquad x, y \in Q.$$

Repeating the same argument already used after (2.3) we get $b = 0$, which is a contradiction.

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