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L^2-harmonic 1-forms on Complete Manifolds

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Abstract: We study the global behavior of complete minimal δ-stable hypersurfaces in $\mathbf{R}^{n+1}$ by using L^2-harmonic 1-forms. We show that a complete minimal δ-stable $\left(\delta > \dfrac{(n-1)^2}{n^2}\right)$ hypersurface in $\mathbf{R}^{n+1}$ has only one end. We also obtain two vanishing theorems of complete noncompact quaternionic manifolds satisfying the weighted Poincaré inequality. These results are improvements of the first author's theorems on hypersurfaces and quaternionic Kähler manifolds.

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1 Introduction

Palmer[1] showed that there is no non-trivial L^2-harmonic 1-form on a complete stable minimal hypersurface in $\mathbf{R}^{n+1}$. Cao *et al.*[2] proved that a complete stable minimal hypersurface in $\mathbf{R}^{n+1}$ ($n \geq 3$) must have only one end. Cheng *et al.*[3] showed that a complete oriented weakly stable minimal hypersurface in $\mathbf{R}^{n+1}$ ($n \geq 3$) must contain no nonconstant bounded harmonic functions with finite Dirichlet integral and have only one end. If the ambient manifold is not the Euclidean space, Cheng[4] gave one end theorem for complete noncompact oriented stable minimal hypersurfaces immersed in an $(n+1)$-dimensional ($n \geq 3$) complete oriented manifold of positive sectional curvature. Recently, by use of the rigidity of complete Riemannian manifolds with weighted Poincaré inequality, Cheng and Zhou[5] showed that: if M is an $\dfrac{n-2}{n}$-stable complete minimal hypersurface in $\mathbf{R}^{n+1}$ ($n \geq 3$) and it has bounded

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norm of second fundamental form, then M either has only one end or is a catenoid. The first author proved that if M^n $(n \geq 2)$ is a complete minimal δ-stable $\left(\delta > \dfrac{(n-1)^2}{n^2}\right)$ hypersurface in $\mathbf{R}^{n+1}$ and it has the bounded norm of the second fundamental form, then the space of L^2 integrable harmonic 1-forms $H^1(L^2(M))$ is trivial (see [6], Corollary 2.5).

In this paper, firstly, we can obtain the following result:

Theorem 1.1 *Suppose that M^n $(n \geq 2)$ is a complete minimal δ-stable $\left(\delta > \dfrac{(n-1)^2}{n^2}\right)$ hypersurface in $\mathbf{R}^{n+1}$. Then the space of L^2 integrable harmonic 1-forms $H^1(L^2(M))$ is trivial and M has only one end.*

Remark 1.1 Theorem 1.1 generalizes Corollary 2.5 in [6] without the restriction of the second fundamental forms.

Secondly, Lam[7] showed that if M^{4n} is a $4n$-dimensional complete noncompact quaternionic Kähler and the Ricci curvature of M satisfies

$$\mathrm{Ric}_M \geq -\frac{4}{3}\lambda_1(M) + \delta$$

for a positive constant δ, where $\lambda_1(M)$ is the lower bound of the spectrum of the Laplacian on M, then

$$H^1(L^2(M)) = \{0\}.$$

Suppose that M is a $4n$-dimensional complete noncompact quaternionic manifold satisfying the weighted Poincaré inequality with a non-negative weight function $\rho(x)$ and the Ricci curvature satisfies

$$\mathrm{Ric}_M(x) \geq -\frac{4}{3}\rho(x) + \sigma(x)$$

for a nonnegative continuous function σ $(\sigma \neq 0)$. If $\rho(x) = O(r_p^{2-\alpha})$, where $r_p(x)$ is the distance function from x to some fixed point p and $0 < \alpha < 2$, then $H^1(L^2(M)) = \{0\}$ (see [6]). It is interesting to see if a similar theorem holds without the restriction of growth rate of the weight function. The following theorems had been established:

Theorem 1.2 *Suppose that M is a $4n$-dimensional complete noncompact quaternionic manifold satisfying the weighted Poincaré inequality with a non-negative continuous weight function $\rho(x)$ ($\rho(x)$ is not identically zero). Assume that the Ricci curvature satisfies*

$$\mathrm{Ric}_M(x) \geq -\alpha\rho(x)$$

for a constant α with $0 < \alpha < \dfrac{4}{3}$. Then $H^1(L^2(M)) = \{0\}$.

Theorem 1.3 *Suppose that M is a $4n$-dimensional complete noncompact quaternionic manifold satisfying the weighted Poincaré inequality with a non-negative continuous weight function $\rho(x)$. Assume that the Ricci curvature satisfies*

$$\mathrm{Ric}_M(x) \geq -\alpha\rho(x) - \beta$$

for constants α with $0 < \alpha < \dfrac{4}{3}$ and $\beta > 0$. If the lower bound of the spectrum $\lambda_1(M)$ of the

Laplacian on M satisfies

$$\lambda_1(M) > \frac{\beta}{\frac{4}{3} - \alpha},$$

then

$$H^1(L^2(M)) = \{0\}.$$

2 One End Theorem on Hypersurfaces in $\mathbf{R}^{n+1}$

In this section, we give the proof of Theorem 1.1.

Let M^n be a minimal hypersurface of $\mathbf{R}^{n+1}$. Let ν denote the unit normal vector field of M and $|A|$ be the norm of the second fundamental form A. A minimal hypersurface $M^n \subset \mathbf{R}^{n+1}$ is called δ-stable if, for each $\phi \in C_0^\infty(M)$,

$$\delta \int_M |A|^2\phi^2 \le \int_M |\nabla\phi|^2.$$

Proof of Theorem 1.1 First, a complete minimal hypersurface in $\mathbf{R}^{n+1}$ is noncompact. For any point $p \in M$ and any unit tangent vector $\boldsymbol{v}$ belonging to tangent space at p, we can choose an orthonormal frame $\{\boldsymbol{e}_1,\ \boldsymbol{e}_2,\ \cdots, \boldsymbol{e}_n\}$ on M at p such that $\boldsymbol{e}_1 = \boldsymbol{v}$. Since M is a minimal hypersurface, there has the following inequality:

$$|A|^2 \ge h_{11}^2 + \frac{\left(\sum_{i=2}^n h_{ii}\right)^2}{n-1} + 2\sum_{i=2}^n h_{1i}^2 \ge \frac{n}{n-1}\sum_{i=1}^n h_{1i}^2. \tag{2.1}$$

The Gauss equation implies that

$$\mathrm{Ric}_M(\boldsymbol{v},\ \boldsymbol{v}) = \sum_{i=2}^n (h_{11}h_{ii} - h_{1i}^2) = -\sum_{i=1}^n h_{1i}^2. \tag{2.2}$$

By (2.1) and (2.2), we have

$$\mathrm{Ric}_M(\boldsymbol{v},\ \boldsymbol{v}) \ge -\frac{n-1}{n}|A|^2. \tag{2.3}$$

Let $\omega \in H^1(L^2(M))$. Then $h = |\omega|$ satisfies a formula (see [8]):

$$\begin{aligned} h\Delta h &\ge \mathrm{Ric}_M(\omega,\ \omega) + \frac{1}{n-1}|\nabla h|^2 \\ &\ge -\frac{n-1}{n}|A|^2h^2 + \frac{1}{n-1}|\nabla h|^2. \end{aligned} \tag{2.4}$$

So, for each $\phi \in C_0^\infty(M)$, we have

$$\phi^2 h\Delta h \ge -\frac{n-1}{n}|A|^2h^2\phi^2 + \frac{1}{n-1}|\nabla h|^2\phi^2. \tag{2.5}$$

Integration by parts implies that

$$\begin{aligned} \left(1 + \frac{1}{n-1}\right)\int_M |\nabla h|^2\phi^2 &\le \frac{n-1}{n}\int_M |A|^2h^2\phi^2 - 2\int_M \phi h\nabla\phi\cdot\nabla h \\ &\le \frac{n-1}{n}\int_M |A|^2h^2\phi^2 + \epsilon_1\int_M |\phi\nabla h|^2 + \frac{1}{\epsilon_1}\int_M |h\nabla\phi|^2 \end{aligned}$$

for each positive constant ϵ_1. That is,

$$\left(1+\frac{1}{n-1}-\epsilon_1\right)\int_M |\nabla h|^2\phi^2 \leq \frac{n-1}{n}\int_M |A|^2h^2\phi^2 + \frac{1}{\epsilon_1}\int_M |h\nabla\phi|^2. \tag{2.6}$$

By the definition of minimal δ-stable hypersurfaces, we have that

$$\begin{aligned}\delta\int_M |A|^2h^2\phi^2 &\leq \int_M |\nabla(h\phi)|^2\\ &\leq \left(1+\frac{1}{\epsilon_2}\right)\int_M h^2|\nabla\phi|^2 + (1+\epsilon_2)\int_M |\nabla h|^2\phi^2\end{aligned} \tag{2.7}$$

for each positive constant ϵ_2. Combining (2.6) with (2.7), we have

$$A_1\int_M |\nabla h|^2\phi^2 \leq A_2\int_M h^2|\nabla\phi|^2, \tag{2.8}$$

where

$$A_1 = 1+\frac{1}{n-1}-\epsilon_1-\frac{(n-1)}{n\delta}(1+\epsilon_2),$$

$$A_2 = \frac{1}{\epsilon_1}+\frac{(n-1)}{n\delta}\left(1+\frac{1}{\epsilon_2}\right).$$

Obviously, A_2 is positive. Since $\delta > \dfrac{(n-1)^2}{n^2}$, we can choose sufficient small constants ϵ_1 and ϵ_2 such that $A_1 > 0$. Choose $\phi \in C_0^\infty(M)$ such that

$$\begin{cases} 0 \leq \phi \leq 1, \\ \phi \equiv 1 \quad \text{on } B\left(\frac{r}{2}\right), \\ \phi \equiv 0 \quad \text{on } M\setminus B(r), \\ |\nabla\phi| \leq \frac{2}{r}. \end{cases} \tag{2.9}$$

Thus, (2.8) implies that

$$A_1\int_{B(\frac{r}{2})} |\nabla h|^2 \leq \frac{4A_2}{r^2}\int_M h^2. \tag{2.10}$$

Note that

$$\int_M h^2 < +\infty. \tag{2.11}$$

Letting $r \to +\infty$, we obtain that h is a constant on M. Since M is a complete noncompact minimal hypersurface in $\mathbf{R}^{n+1}$, it implies that M has infinite volume (see [9]). Thus by (2.11), we have $h = 0$. That is,

$$H^1(L^2(M)) = \{0\}.$$

Since M^n is a minimal hypersurface of $\mathbf{R}^{n+1}$ $(n \geq 3)$, each end of M is non-parabolic (see [2]) and the number of non-parabolic end of M is bounded from above by $\dim H^1(L^2(M))+1$ (see [10]). Therefore, M has only one end.

3 Vanishing Theorems on Quaternionic Manifolds

In this section, we give the proofs of Theorems 1.2 and 1.3, respectively.

If M is a quaternionic manifold and $\omega \in H^1(L^2(M))$, then $h = |\omega|$ satisfies a Bochner type formula (see [11]):

$$h\Delta h \geq \mathrm{Ric}_M(\omega, \omega) + \frac{1}{3}|\nabla h|^2. \tag{3.1}$$

Proof of Theorem 1.2 Note that $\mathrm{Ric}_M(x) \geq -\alpha\rho(x)$. Combining with (3.1), we have

$$h\Delta h \geq -\alpha\rho h^2 + \frac{1}{3}|\nabla h|^2.$$

For each $\phi \in C_0^\infty(M)$, integration by parts implies that

$$-\int_M \nabla(h\phi^2)\cdot\nabla h \geq -\alpha\int_M \rho h^2\phi^2 + \int_M \frac{1}{3}|\nabla h|^2\phi^2. \tag{3.2}$$

That is,

$$\frac{4}{3}\int_M |\nabla h|^2\phi^2 \leq \alpha\int_M \rho h^2\phi^2 - 2\int_M \phi h\nabla\phi\cdot\nabla h. \tag{3.3}$$

Note that

$$-2\int_M \phi h\nabla\phi\cdot\nabla h \leq \frac{1}{\epsilon_1}\int_M h^2|\nabla\phi|^2 + \epsilon_1\int_M \phi^2|\nabla h|^2 \tag{3.4}$$

holds for each positive constant ϵ_1. Since ρ is weight function, we have

$$\int_M \rho h^2\phi^2 \leq \int_M |\nabla(h\phi)|^2 \leq \Big(1+\frac{1}{\epsilon_2}\Big)\int_M h^2|\nabla\phi|^2 + (1+\epsilon_2)\int_M |\nabla h|^2\phi^2 \tag{3.5}$$

for each positive constant ϵ_2. By (3.3), (3.4) and (3.5), we get

$$B_1\int_M |\nabla h|^2\phi^2 \leq B_2\int_M h^2|\nabla\phi|^2, \tag{3.6}$$

where

$$B_1 = \frac{4}{3} - \epsilon_1 - \alpha(1+\epsilon_2), \qquad B_2 = \frac{1}{\epsilon_1} + \alpha\Big(1+\frac{1}{\epsilon_2}\Big) > 0.$$

Choose sufficient small constants ϵ_1 and ϵ_2 such that $B_1 > 0$. Choose $\phi \in C_0^\infty(M)$ satisfying (2.9). Thus, (3.6) implies that

$$B_1\int_{B(\frac{r}{2})} |\nabla h|^2 \leq \frac{4B_2}{r^2}\int_M h^2.$$

Note that (2.11) holds. Letting $r \to +\infty$, we have h is a constant on M. If h is not identically zero, then, by (2.11), the volume of the M is finite. The weighted Poincaré inequality implies that

$$\int_{B(\frac{r}{2})} \rho \leq \int_M \frac{4}{r^2} = \frac{4\mathrm{Vol}(M)}{r^2}.$$

Letting $r \to +\infty$, we have $\displaystyle\int_M \rho \leq 0$ which contradicts the fact that ρ is non-negative continuous weight function and not identically zero. Therefore,

$$H^1(L^2(M)) = \{0\}.$$

Proof of Theorem 1.3 Combining the fact $\mathrm{Ric}_M(x) \geq -\alpha\rho(x) - \beta$ with (3.1), we have

$$h\Delta h \geq (-\alpha\rho - \beta)h^2 + \frac{1}{3}|\nabla h|^2.$$

For each $\phi \in C_0^\infty(M)$, integration by parts implies that

$$-\int_M \nabla(h\phi^2)\cdot\nabla h \geq -\alpha\int_M \rho h^2\phi^2 - \beta\int_M h^2\phi^2 + \int_M \frac{1}{3}|\nabla h|^2\phi^2.$$

That is,

$$\frac{4}{3}\int_M |\nabla h|^2\phi^2 \leq \alpha\int_M \rho h^2\phi^2 + \beta\int_M h^2\phi^2 - 2\int_M \phi h\nabla\phi\cdot\nabla h. \tag{3.7}$$

Note that

$$-2\int_M \phi h\nabla\phi\cdot\nabla h \leq \frac{1}{\epsilon_1}\int_M h^2|\nabla\phi|^2 + \epsilon_1\int_M \phi^2|\nabla h|^2 \tag{3.8}$$

for each positive constant ϵ_1. Since ρ is a weight function, we obtain

$$\int_M \rho h^2\phi^2 \leq \int_M |\nabla(h\phi)|^2 \leq \Big(1+\frac{1}{\epsilon_2}\Big)\int_M h^2|\nabla\phi|^2 + (1+\epsilon_2)\int_M |\nabla h|^2\phi^2 \tag{3.9}$$

for each positive constant ϵ_1. By (3.7), (3.8) and (3.9), we have

$$B_1\int_M |\nabla h|^2\phi^2 \leq B_2\int_M h^2|\nabla\phi|^2 + \beta\int_M h^2\phi^2, \tag{3.10}$$

where

$$B_1 = \frac{4}{3} - \epsilon_1 - \alpha(1+\epsilon_2), \qquad B_2 = \frac{1}{\epsilon_1} + \alpha\Big(1+\frac{1}{\epsilon_2}\Big) > 0.$$

Since $0 < \alpha < \frac{4}{3}$, we can choose sufficient small constants ϵ_1 and ϵ_2 such that $B_1 > 0$. Choose $\phi \in C_0^\infty(M)$ satisfying (2.9). Thus, (3.10) implies that

$$B_1\int_{B(\frac{r}{2})} |\nabla h|^2 \leq \frac{B_2\tilde{C}^2}{r^2}\int_M h^2 + \beta\int_{B(r)} h^2. \tag{3.11}$$

Note that (2.11) holds. Letting $r \to +\infty$, we obtain that

$$B_1\int_M |\nabla h|^2 \leq \beta\int_M h^2. \tag{3.12}$$

Choosing ϵ_1, $\epsilon_2 \to 0$, we get

$$\Big(\frac{4}{3} - \alpha\Big)\int_M |\nabla h|^2 \leq \beta\int_M h^2. \tag{3.13}$$

It is well known that

$$\begin{aligned}\lambda_1(M)\int_M h^2\phi^2 &\leq \int_M |\nabla(h\phi)|^2\\ &\leq (1+\epsilon_3)\int_M |\nabla h|^2\phi^2 + \Big(1+\frac{1}{\epsilon_3}\Big)\int_M h^2|\nabla\phi|^2,\end{aligned} \tag{3.14}$$

for each positive constant ϵ_3. Substituting (2.9) into (3.14), we get

$$\lambda_1(M)\int_{B(\frac{r}{2})} h^2 \leq (1+\epsilon_3)\int_{B(r)} |\nabla h|^2 + \Big(1+\frac{1}{\epsilon_3}\Big)\frac{1}{r^2}\int_M h^2. \tag{3.15}$$

Letting $r \to +\infty$, we have

$$\lambda_1(M)\int_M h^2 \leq (1+\epsilon_3)\int_M |\nabla h|^2. \tag{3.16}$$

Let $\epsilon_3 \to 0$. Then we obtain that

$$\lambda_1(M)\int_M h^2 \leq \int_M |\nabla h|^2. \tag{3.17}$$

Suppose that there exists $\omega \in H^1(L^2(M))$ such that h is not identically constant. Combining (3.13) and (3.17), we have

$$\lambda_1(M)\Big(\frac{4}{3} - \alpha\Big) \leq \beta, \tag{3.18}$$

which is contradiction with the restriction of $\lambda_1(M)$. Thus, h is constant. By (3.17), we obtain that h is identically zero. Therefore,

$$H^1(L^2(M)) = \{0\}.$$

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