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The Value Distribution and Normality Criteria of a Class of Meromorphic Functions

YANG QI
(*School of Mathematics Science, Xinjiang Normal University, Urumqi, 830054*)

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Abstract: In this article, we use Zalcman Lemma to investigate the normal family of meromorphic functions concerning shared values, which improves some earlier related results.

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1 Introduction and Main Results

Let D be a domain of the open complex plane $\mathbf{C}$, $f(z)$ and $g(z)$ be two nonconstant meromorphic functions defined in D, a be a finite complex value. We say that f and g share a CM (or IM) in D provided that $f-a$ and $g-a$ have the same zeros counting (or ignoring) multiplicity in D. When $a=\infty$, the zeros of $f-a$ means the poles of f (see [1]). It is assumed that the reader is familiar with the standard notations and the basic results of Nevanlinna's value-distribution theory (see [2]–[4]).

It is also interesting to find normality criteria from the point of view of shared values. In this area, Schwick[5] first proved an interesting result that a family of meromorphic functions in a domain is normal if in which every function shares three distinct finite complex numbers with its first derivative. And later, more results about shared values' normality criteria related a Hayma conjecture of higher derivative have emerged (see [6]–[13]).

Lately, Chen[14] proved the following theorems.

Theorem 1.1 *Let D be a domain in $\mathbf{C}$ and let $\mathcal{F}$ be a family of meromorphic functions in D. Let $k, n, d \in \mathbf{N}_+$, $n \geq 3$, $d \geq \dfrac{k+1}{n-2}$ and a, b be two finite complex numbers with*

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E-mail address: yangqi_8138@126.com (Yang Q).

$a \neq 0$. Suppose that every $f \in \mathcal{F}$ has all its zeros of multiplicity at least k and all its poles of multiplicity at least d. If $f^{(k)} - af^n$ and $g^{(k)} - ag^n$ share the value b IM for every pair of functions $(f,\ g)$ of $\mathcal{F}$, then $\mathcal{F}$ is a normal family in D.

Theorem 1.2 *Let D be a domain in $\mathbf{C}$ and let $\mathcal{F}$ be a family of meromorphic functions in D. Let $k \in \mathbf{N}_+$ and a, b be two finite complex numbers with $a \neq 0$. Suppose that every $f \in \mathcal{F}$ has all its zeros of multiplicity at least $k+1$ and all its poles of multiplicity at least $k+2$. If $f^{(k)} - af^2$ and $g^{(k)} - ag^2$ share the value b IM for every pair of functions $(f,\ g)$ of $\mathcal{F}$, then $\mathcal{F}$ is a normal family in D.*

A natural problem arises: what can we say if $f^{(k)} - af^n$ in Theorem 1.1 is replaced by the $(f^{(k)})^m - af^n$? In this paper, we prove the following results.

Theorem 1.3 *Let D be a domain in $\mathbf{C}$ and let $\mathcal{F}$ be a family of meromorphic functions in D. Let $k, n, m, d \in \mathbf{N}_+$, $n \geq m+2$, $d \geq \dfrac{mk+1}{n-m-1}$ and a, b be two finite complex numbers with $a \neq 0$. Suppose that every $f \in \mathcal{F}$ has all its zeros of multiplicity at least $k+1$ and all its poles of multiplicity at least d. If $(f^{(k)})^m - af^n$ and $(g^{(k)})^m - ag^n$ share the value b IM for every pair of functions $(f,\ g)$ of $\mathcal{F}$, then $\mathcal{F}$ is a normal family in D.*

Theorem 1.4 *Let D be a domain in $\mathbf{C}$ and let $\mathcal{F}$ be a family of meromorphic functions in D. Let $k, m \in \mathbf{N}_+$ and a, b be two finite complex numbers with $a \neq 0$. Suppose that every $f \in \mathcal{F}$ has all its zeros of multiplicity at least $k+1$ and all its poles of multiplicity at least $mk+2$. If $(f^{(k)})^m - af^{m+1}$ and $(g^{(k)})^m - ag^{m+1}$ share the value b IM for every pair of functions $(f,\ g)$ of $\mathcal{F}$, then $\mathcal{F}$ is a normal family in D.*

2 Some Lemmas

Lemma 2.1[15] *Let $\mathcal{F}$ be a family of meromorphic functions on the unit disc satisfying all zeros of functions in $\mathcal{F}$ have multiplicity $\geq p$ and all poles of functions in $\mathcal{F}$ have multiplicity $\geq q$. Let α be a real number satisfying $-q < \alpha < p$. Then $\mathcal{F}$ is not normal at 0 if and only if there exist*

a) *a number $0 < r < 1$;*
b) *points z_n with $|z_n| < r$;*
c) *functions $f_n \in \mathcal{F}$;*
d) *positive numbers $\rho_n \to 0$*

such that $g_n(\zeta) := \rho_n^{-\alpha} f_n(z_n + \rho_n \zeta)$ converges spherically uniformly on each compact subset of $\mathbf{C}$ to a non-constant meromorphic function $g(\zeta)$, whose all zeros have multiplicity $\geq p$ and all poles have multiplicity $\geq q$ and order is at most 2.

Lemma 2.2 *Let $f(z)$ be a meromorphic function such that $f^{(k)}(z) \not\equiv 0$ and $a \in \mathbf{C}\backslash\{0\}$, $k, m, n, d \in \mathbf{N}_+$ with $n \geq m+2$, $d \geq \dfrac{km+1}{n-m-1}$. If all zeros of f are of multiplicity at least*

$k+1$ and all poles of f are of multiplicity at least d, then

$$T(r,f)\le \frac{1}{k+1}N\Big(r,\ \frac{1}{f}\Big)+\bar{N}\Big(r,\ \frac{1}{(f^{(k)})^m-cf^n}\Big)+S(r,f), \tag{2.1}$$

where

$$S(r,f)=o(T(r,f))\qquad \text{as } r\to\infty,$$

possibly outside a set with finite linear measure.

Proof. Set

$$\Phi(z):=\frac{(f^{(k)}(z))^m}{cf^n(z)}.$$

Since $f^{(k)}(z)\not\equiv 0$, we have $\Phi(z)\not\equiv 0$. Thus

$$f^n(z)=\frac{(f^{(k)}(z))^m}{c\Phi(z)}. \tag{2.2}$$

Hence

$$\begin{aligned} nm(r,f)&=m(r,f^n)\\ &\le m\Big(r,\ \frac{(f^{(k)})^m}{\Phi}\Big)+\log^+\frac{1}{|c|}\\ &\le m\Big(r,\ \frac{1}{\Phi}\Big)+m(r,\ (f^{(k)})^m)+\log^+\frac{1}{|c|}\\ &\le m\Big(r,\ \frac{1}{\Phi}\Big)+mm\Big(r,\ \frac{f^{(k)}}{f}\Big)+mm(r,\ f)+\log^+\frac{1}{|c|}. \end{aligned}$$

So that

$$(n-m)m(r,\ f)\le m\Big(r,\ \frac{1}{\Phi}\Big)+mm\Big(r,\ \frac{f^{(k)}}{f}\Big)+\log^+\frac{1}{|c|}. \tag{2.3}$$

On the other hand, (2.2) gives

$$\begin{aligned} nN(r,\ f)&\le N(r,\ f^n)\\ &=N\Big(r,\ \frac{(f^{(k)})^m}{\Phi}\Big)\\ &\le mN(r,\ f^{(k)})+N\Big(r,\ \frac{1}{\Phi}\Big)-\bar{N}(r,\ \Phi=f^{(k)}=0), \end{aligned} \tag{2.4}$$

where $\bar{N}(r,\ \Phi=f^{(k)}=0)$ denotes the counting function of zeros of both Φ and $f^{(k)}$. We obtain

$$nN(r,\ f)\le mN(r,\ f)+mk\bar{N}(r,\ f)+N\Big(r,\ \frac{1}{\Phi}\Big)-\bar{N}(r,\ \Phi=f^{(k)}=0),$$

$$(n-m)N(r,\ f)\le mk\bar{N}(r,\ f)+N\Big(r,\ \frac{1}{\Phi}\Big)-\bar{N}(r,\ \Phi=f^{(k)}=0). \tag{2.5}$$

By (2.2), we have

$$\bar{N}(r,\ \Phi)+\bar{N}\Big(r,\ \frac{1}{\Phi}\Big)\le \bar{N}\Big(r,\ \frac{1}{f}\Big)+\bar{N}(r,\ f)+\bar{N}(r,\ \Phi=f^{(k)}=0). \tag{2.6}$$

From (2.3)–(2.6), we obtain

$$\begin{aligned} (n-m)T(r,\ f)&\le mk\bar{N}(r,\ f)+T\Big(r,\ \frac{1}{\Phi}\Big)-\bar{N}(r,\ \Phi=f^{(k)}=0)+S(r,f)\\ &\le mk\bar{N}(r,\ f)+T(r,\ \Phi)-\bar{N}(r,\ \Phi=f^{(k)}=0)+S(r,f) \end{aligned}$$

$$\le mk\bar{N}(r,\ f) + \bar{N}\Big(r,\ \frac{1}{\Phi}\Big) + \bar{N}(r,\ \Phi) + \bar{N}\Big(r,\ \frac{1}{\Phi - 1}\Big)$$
$$- \bar{N}(r,\ \Phi = f^{(k)} = 0) + S(r, f)$$
$$\le (mk+1)\bar{N}(r,\ f) + \bar{N}\Big(r,\ \frac{1}{f}\Big) + \bar{N}\Big(r,\ \frac{1}{(f^{(k)})^m - cf^n}\Big) + S(r, f).$$

Since all zeros and poles of f are multiplicities at least k and d respectively, we get

$$\bar{N}(r,\ f) \le \frac{1}{d}N(r,\ f) \le \frac{1}{d}T(r,\ f) \le \frac{n-m-1}{km+1}T(r,\ f),$$
$$\bar{N}\Big(r,\ \frac{1}{f}) \le \frac{1}{k+1}N\Big(r,\ \frac{1}{f}\Big).$$

So that

$$T(r,\ f) \le \frac{1}{k+1}N\Big(r,\ \frac{1}{f}\Big) + \bar{N}\Big(r,\ \frac{1}{(f^{(k)})^m - cf^n}\Big) + S(r,\ f).$$

This completes the proof of Lemma 2.2.

Lemma 2.3 *Let $f(z)$ be a nonconstant rational function such that $f^{(k)}(z) \not\equiv 0$. Let $a \in \mathbf{C}\backslash\{0\}$, and $k, n, m, d \in \mathbf{N}_+$ with $n \ge m+2$ and $d \ge \dfrac{mk+1}{n-m-1}$. If $f \ne 0$ and all poles of f are of multiplicity at least d, then $(f^{(k)})^m - af^n$ has at least two distinct zeros.*

Proof. Suppose to the contrary that $(f^{(k)})^m - af^n$ has at most one zero. Since $f \ne 0$, we get f is a rational but not a polynomial.

Case 1. If $(f^{(k)})^m - af^n$ has only zero z_0 with multiplicity l, then we set

$$f(z) = \frac{A}{(z-z_1)^{\beta_1}(z-z_2)^{\beta_2}\cdots(z-z_t)^{\beta_t}}, \tag{2.7}$$

where A is a nonzero constant and

$$\beta_i \ge \frac{mk+1}{n-m-1}, \qquad i = 1, 2, \cdots, t.$$

For the sake of simplicity, we denote

$$\beta_1 + \beta_2 + \cdots + \beta_t = q.$$

From (2.7), we have

$$f^{(k)} = \frac{g(z)}{(z-z_1)^{\beta_1+k}(z-z_2)^{\beta_2+k}\cdots(z-z_t)^{\beta_t+k}}, \tag{2.8}$$

where $g(z)$ is a polynomial such that $\deg(g(z)) \le k(t-1)$.

From (2.7) and (2.8), we get

$$\begin{aligned}
&(f^{(k)})^m - af^n \\
&= \frac{g^m(z)}{(z-z_1)^{m(\beta_1+k)}(z-z_2)^{m(\beta_2+k)}\cdots(z-z_t)^{m(\beta_t+k)}} \\
&\quad - \frac{aA^n}{(z-z_1)^{n\beta_1}(z-z_2)^{n\beta_2}\cdots(z-z_t)^{n\beta_t}} \\
&= \frac{[g^m(z)(z-z_1)^{(n-m)\beta_1-mk}(z-z_2)^{(n-m)\beta_2-mk}\cdots(z-z_t)^{(n-m)\beta_t-mk} - aA^n]}{(z-z_1)^{n\beta_1}(z-z_2)^{n\beta_2}\cdots(z-z_t)^{n\beta_t}}.
\end{aligned}$$

By the assumption that $(f^{(k)})^m - af^n$ has exactly one zero z_0 with multiply l, we have

$$(f^{(k)})^m - af^n = \frac{C(z-z_0)^l}{(z-z_1)^{n\beta_1}(z-z_2)^{n\beta_2}\cdots(z-z_t)^{n\beta_t}}, \tag{2.9}$$

where C is a nonzero constant. Thus

$$\begin{aligned} &C(z-z_0)^l \\ \equiv\ & g^m(z)(z-z_1)^{(n-m)\beta_1-mk}(z-z_2)^{(n-m)\beta_2-mk}\cdots(z-z_t)^{(n-m)\beta_t-mk} - aA^n. \end{aligned} \tag{2.10}$$

Differentiating (2.10), we obtain

$$\begin{aligned} Cl(z-z_0)^{l-1} \equiv\ & (z-z_1)^{(n-m)\beta_1-mk-1}\cdots(z-z_t)^{(n-m)\beta_t-mk-1} \\ & \cdot\Big[mg^{m-1}g'(z)(z-z_1)\cdots(z-z_t) \\ & + g^m(z)\sum_{i=1}^{t}((n-m)\beta_i - mk)\prod_{j=1,j\neq i}^{t}(z-z_j)\Big]. \end{aligned}$$

For the sake of simplicity, we denote

$$\begin{aligned} g_1(z) &= Cl(z-z_0)^{l-1}, \\ g_2(z) &= (z-z_1)^{(n-m)\beta_1-mk-1}\cdots(z-z_t)^{(n-m)\beta_t-mk-1} \\ &\quad \cdot\Big[mg^{m-1}g'(z)(z-z_1)\cdots(z-z_t) \\ &\quad + g^m(z)\sum_{i=1}^{t}((n-m)\beta_i - mk)\prod_{j=1,j\neq i}^{t}(z-z_j)\Big]. \end{aligned}$$

Hence

$$g_1(z) \equiv g_2(z).$$

Since $(n-m)\beta_i - mk - 1 > 0$, we have

$$g_2(z_i) = 0.$$

But $g_1(z_i) \neq 0$ $(i = 1, 2, \cdots, t)$, a contradiction.

Case 2. If $(f^{(k)})^m - af^n$ has no zeros, then $l = 0$ for (2.9). We have

$$(f^{(k)})^m - af^n = \frac{C}{(z-z_1)^{n\beta_1}(z-z_2)^{n\beta_2}\cdots(z-z_t)^{n\beta_t}},$$

where C is a nonzero constant. Thus

$$C \equiv g^m(z)(z-z_1)^{(n-m)\beta_1-mk}(z-z_2)^{(n-m)\beta_2-mk}\cdots(z-z_t)^{(n-m)\beta_t-mk} - aA^n,$$

i.e.,

$$g^m(z)(z-z_1)^{(n-m)\beta_1-mk}(z-z_2)^{(n-m)\beta_2-mk}\cdots(z-z_t)^{(n-m)\beta_t-mk} \equiv C + aA^n.$$

Obviously, $g^m(z)(z-z_1)^{(n-m)\beta_1-mk}(z-z_2)^{(n-m)\beta_2-mk}\cdots(z-z_t)^{(n-m)\beta_t-mk}$ is not a constant, a contradiction.

This completes the proof of Lemma 2.3.

Lemma 2.4 *Let $f(z)$ be a nonconstant rational function and Let $a \in \mathbf{C}\backslash\{0\}$, and $k, n, m, d \in \mathbf{N}_+$ with $n \geq m+2$ and $d \geq \dfrac{mk+1}{n-m-1}$. If all zeros of f are of multiplicity at least $k+1$ and all poles of f are of multiplicity at least d, then $(f^{(k)})^m - af^n$ has at least two distinct zeros.*

Proof. Suppose to the contrary that $(f^{(k)})^m - af^n$ has at most one zero.

Case I. When f is a non-constant polynomial, noting that all zeros of f have multiplicity at least $k+1$, we know that $(f^{(k)})^m - af^n$ must have zeros. We claim that f has exactly one zero. Otherwise, combing with the conditions of Lemma 2.4, we can get $(f^{(k)})^m - af^n$ has at least two zeros, which contradicts with our assumption.

Set

$$f(z) = B(z-z_0)^s,$$

where $s \geq k+1$, B is a nonzero constant. Then

$$(f^{(k)}(z))^m - af^n(z) = B^m(z-z_0)^{(s-k)m}[s^m(s-1)^m \cdots (s-k+1)^m - aB^{n-m}(z-z_0)^{(n-m)s+mk}]. \tag{2.11}$$

Since $(s-k)m \geq 1$, we obtain that $s^m(s-1)^m \cdots (s-k+1)^m - aB^{n-m}(z-z_0)^{(n-m)s+mk}$ has least one zero which is not z_0 from (2.11). Therefore, $(f^{(k)})^m - af^n$ has at least two distinct zeros, a contradiction.

Case II. When f is rational but not a polynomial, we consider two cases.

Case 1. Suppose that $(f^{(k)})^m - af^n$ has only zero z_0 with multiplicity at least l. If $f \neq 0$, by Lemma 2.3, we get a contradiction. So f has zeros, and then we can deduce that z_0 is the only zero of f. Otherwise, $(f^{(k)})^m - af^n$ has at least two distinct zeros, a contradiction.

We set

$$f(z) = \frac{A(z-z_0)^s}{(z-z_1)^{\beta_1}(z-z_2)^{\beta_2}\cdots(z-z_t)^{\beta_t}}, \tag{2.12}$$

where A is a nonzero constant and $s \geq k+1$, $\beta_i \geq d \geq \dfrac{mk+1}{n-m-1}$ $(i = 1, 2, \cdots, t)$.

For the sake of simplicity, we denote

$$\beta_1 + \beta_2 + \cdots + \beta_t = q.$$

From (2.12), we have

$$f^{(k)} = \frac{A(z-z_0)^{s-k}g(z)}{(z-z_1)^{\beta_1+k}(z-z_2)^{\beta_2+k}\cdots(z-z_t)^{\beta_t+k}}, \tag{2.13}$$

where $g(z)$ is a polynomial with $\deg(g) \leq kt$.

From (2.12) and (2.13), we get

$$\begin{aligned}
&(f^{(k)})^m - af^n \\
= &\frac{A^m(z-z_0)^{m(s-k)}g^m(z)}{(z-z_1)^{m(\beta_1+k)}(z-z_2)^{m(\beta_2+k)}\cdots(z-z_t)^{m(\beta_t+k)}} \\
&- \frac{aA^n(z-z_0)^{ns}}{(z-z_1)^{n\beta_1}(z-z_2)^{n\beta_2}\cdots(z-z_t)^{n\beta_t}} \\
= &\frac{A^m(z-z_0)^{m(s-k)}g^m(z)(z-z_1)^{(n-m)\beta_1-mk}(z-z_2)^{(n-m)\beta_2-mk}\cdots(z-z_t)^{(n-m)\beta_t-mk}}{(z-z_1)^{n\beta_1}(z-z_2)^{n\beta_2}\cdots(z-z_t)^{n\beta_t}} \\
&- \frac{aA^n(z-z_0)^{ns}}{(z-z_1)^{n\beta_1}(z-z_2)^{n\beta_2}\cdots(z-z_t)^{n\beta_t}}.
\end{aligned}$$

By the assumption that $(f^{(k)})^m - af^n$ has exactly one zero z_0 with multiply l, we have

$$(f^{(k)})^m - af^n = \frac{B(z-z_0)^l}{(z-z_1)^{n\beta_1}(z-z_2)^{n\beta_2}\cdots(z-z_t)^{n\beta_t}},$$

where B is a nonzero constant. Thus

$$B(z-z_0)^l \equiv A^m(z-z_0)^{m(s-k)}[g^m(z)(z-z_1)^{(n-m)\beta_1-mk}(z-z_2)^{(n-m)\beta_2-mk}\cdots (z-z_t)^{(n-m)\beta_t-mk} - aA^{n-m}(z-z_0)^{(n-m)s+mk}]. \tag{2.14}$$

Case 1.1. If $l > m(s-k)$, from (2.14), we can deduce that z_0 is a zero of $g^m(z)(z-z_1)^{(n-m)\beta_1-mk}(z-z_2)^{(n-m)\beta_2-mk}\cdots(z-z_t)^{(n-m)\beta_t-mk}$, a contradiction.

Case 1.2. If $l = m(s-k)$, from (2.14), it follows that

$$g^m(z)(z-z_1)^{(n-m)\beta_1-mk}(z-z_2)^{(n-m)\beta_2-mk}\cdots (z-z_t)^{(n-m)\beta_t-mk} - aA^{n-m}(z-z_0)^{(n-m)s+mk} \equiv \frac{B}{A^m}. \tag{2.15}$$

Differentiating (2.15), we have

$$g^{m-1}(z)(z-z_1)^{(n-m)\beta_1-mk-1}\cdots(z-z_t)^{(n-m)\beta_t-mk-1} \cdot\left[mg'(z)(z-z_1)\cdots(z-z_t)+g(z)\sum_{i=1}^{t}((n-m)\beta_i-mk)\prod_{j=1,j\neq i}^{t}(z-z_j)\right] \equiv a((n-m)s+mk)A^{n-m}(z-z_0)^{(n-m)s+mk-1}.$$

For the sake of simplicity, we denote

$$g_1(z) = g^{m-1}(z)(z-z_1)^{(n-m)\beta_1-mk-1}\cdots(z-z_t)^{(n-m)\beta_t-mk-1} \cdot\left[mg'(z)(z-z_1)\cdots(z-z_t)+g(z)\sum_{i=1}^{t}((n-m)\beta_i-mk)\prod_{j=1,j\neq i}^{t}(z-z_j)\right],$$

$$g_2(z) = a((n-m)s+mk)A^{n-m}(z-z_0)^{(n-m)s+mk-1}.$$

Thus

$$g_1(z)\equiv g_2(z).$$

Since $(n-m)\beta_i - mk - 1 > 0$, we get

$$g_1(z_i) = 0.$$

But $g_2(z_i)\neq 0$ $(i=1,2,\cdots,t)$, a contradiction.

Case 2. If $(f^{(k)})^m - af^n$ has no zeros, then f has no zeros. It is a contradiction with Lemma 2.3.

This completes the proof of Lemma 2.4.

Lemma 2.5 *Let $f(z)$ be a transcendental meromorphic function, and let $k,m\in\mathbf{N}_+$ and $c\in\mathbf{C}\backslash\{0\}$. If all zeros of f are of multiplicity at least $k+1$ and all poles of f are of multiplicity at least $mk+2$, then $(f^{(k)})^m - cf^{m+1}$ has infinitely many zeros.*

Proof. Suppose that $(f^{(k)})^m - cf^{m+1}$ has only finitely many zeros. Then

$$\bar{N}\Big(r,\ \frac{1}{(f^{(k)})^m - cf^{m+1}}\Big) = S(r,f).$$

Clearly, an arbitrary zero of f is a zero of $(f^{(k)})^m - cf^{m+1}$. Since all zeros of f are of multiplicity at least $k+1$, we can deduce that f has only finitely zeros, and so

$$\bar{N}\Big(r,\ \frac{1}{f}\Big) = O(\log r) = S(r,f).$$

Set

$$\varPhi(z) := \frac{(f^{(k)}(z))^m}{cf^{m+1}(z)}.$$

Similarly with the proof of Lemma 2.2, we can get

$$T(r,f) \le (mk+1)\bar{N}(r,\ f) + \bar{N}\Big(r,\ \frac{1}{f}\Big) + \bar{N}\Big(r,\ \frac{1}{(f^{(k)})^m - cf^{m+1}}\Big) + S(r,f).$$

Since all poles of f are multiplicities at least $mk+2$, we obtain

$$\bar{N}(r,\ f) \le \frac{1}{mk+2}N(r,f) \le \frac{1}{mk+2}T(r,f).$$

So that

$$T(r,\ f) \le (mk+2)\bar{N}\Big(r,\ \frac{1}{f}\Big) + (mk+2)\bar{N}\Big(r,\ \frac{1}{(f^{(k)})^m - cf^{m+1}}\Big) + S(r,f) = S(r,f).$$

This contradicts with f is transcendental.

This completes the proof of Lemma 2.5.

Similarly to the proofs of Lemmas 2.3 and 2.4, we can get the following Lemmas.

Lemma 2.6 *Let $f(z)$ be a nonconstant rational function such that $f^{(k)}(z) \not\equiv 0$, and $a \in \mathbf{C}\backslash\{0\}$, and $k, n, m, d \in \mathbf{N}_+$ with $n \ge m+1$ and $d \ge \dfrac{mk+2}{n-m}$. If $f \ne 0$ and all poles of f are of multiplicity at least d, then $(f^{(k)})^m - af^n$ has at least two distinct zeros.*

Lemma 2.7 *Let $f(z)$ be a nonconstant rational function, and $a \in \mathbf{C}\backslash\{0\}$, and $k, n, m, d \in \mathbf{N}_+$ with $n \ge m+1$ and $d \ge \dfrac{mk+2}{n-m}$. If all zeros of f are of multiplicity at least $k+1$ and all poles of f are of multiplicity at least d, then $(f^{(k)})^m - af^n$ has at least two distinct zeros.*

3 Proofs of Theorems

Proof Theorem 1.3 Suppose that $\mathcal{F}$ is not normal in D. Then there exists at least one point z_0 such that $\mathcal{F}$ is not normal at the point z_0. Without loss of generality we assume that $z_0 = 0$. By Lemma 2.1, there exist points $z_j \to 0$, positive numbers $\rho_j \to 0$ and functions $f_j \in \mathcal{F}$ such that

$$g_j(\xi) = \rho_j^{\frac{mk}{n-m}} f_j(z_j + \rho_j\xi) \to g(\xi) \tag{3.1}$$

locally uniformly with respect to the spherical metric, where g is a non-constant meromorphic function in $\mathbf{C}$ and whose poles and zeros are of multiplicity at least d and $k+1$, respectively. Moreover, the order of g is at most 2.

From (3.1) we know that

$$(g_j^{(k)}(\xi))^m = \rho_j^{\frac{mnk}{n-m}}(f_j^{(k)}(z_j + \rho_j\xi))^m \to (g^{(k)}(\xi))^m$$

and

$$\begin{aligned}(g_j^{(k)}(\xi))^m - ag_j^n(\xi) - \rho_j^{\frac{mnk}{n-m}}b = \rho_j^{\frac{mnk}{n-m}}[(f_j^{(k)}(z_j + \rho_j\xi))^m - af_j^n(z_j + \rho_j\xi) - b]\\ \to (g^{(k)}(\xi))^m - ag^n(\xi)\end{aligned} \tag{3.2}$$

also locally uniformly with respect to the spherical metric.

If $(g^{(k)}(\xi))^m - ag^n(\xi) \equiv 0$, since all poles of g have multiplicity at least d, we have

$$\begin{aligned} nT(r,\ g) &= T(r,\ g^n) \\ &= T(r,\ (g^{(k)})^m) + O(1) \\ &= mm(r,\ g^{(k)}) + mN(r,\ g^{(k)}) + O(1) \\ &\le mm(r,\ g) + mN(r,\ g) + mk\bar{N}(r,\ g) + S(r,\ g) \\ &\le mT(r,\ g) + \frac{mk(n-m-1)}{mk+1}T(r,\ g) + S(r,\ g). \end{aligned}$$

Because $n - m > \dfrac{mk(n-m-1)}{mk+1}$, we know that $g(\xi)$ is a constant, a contradiction. So

$$(g^{(k)}(\xi))^m - ag^n(\xi) \not\equiv 0.$$

By Lemma 2.2, we have

$$\begin{aligned} T(r,\ g) &\le \frac{1}{k+1}N\Big(r,\ \frac{1}{g}\Big) + \bar{N}\Big(r,\ \frac{1}{(g^{(k)})^m - ag^n}\Big) + S(r,\ g) \\ &\le \frac{1}{k+1}T\Big(r,\ \frac{1}{g}\Big) + \bar{N}\Big(r,\ \frac{1}{(g^{(k)})^m - ag^n}\Big) + S(r,\ g). \end{aligned}$$

Then

$$\Big(1 - \frac{1}{k+1}\Big)T(r,\ g) \le \bar{N}\Big(r,\ \frac{1}{(g^{(k)})^m - ag^n}\Big) + S(r,\ g),$$

i.e.,

$$T(r,\ g) \le \Big(1 + \frac{1}{k}\Big)\bar{N}\Big(r,\ \frac{1}{(g^{(k)})^m - ag^n}\Big) + S(r,g). \tag{3.3}$$

If $(g^{(k)}(\xi))^m - ag^n(\xi) \neq 0$, then (3.3) gives that $g(\xi)$ is also a constant. Hence, $(g^{(k)}(\xi))^m - ag^n(\xi)$ is a non-constant meromorphic function and has at least one zero.

Next we prove that $(g^{(k)}(\xi))^m - ag^n(\xi)$ has just a unique zero. Suppose to the contrary, let ξ_0 and ξ_0^* be two distinct zeros of $(g^{(k)}(\xi))^m - ag^n(\xi)$, and choose $\delta(>0)$ small enough such that

$$D(\xi_0,\ \delta) \bigcap D(\xi_0^*,\ \delta) = \emptyset,$$

where

$$D(\xi_0,\ \delta) = \{\xi : |\xi - \xi_0| < \delta\}, \qquad D(\xi_0^*,\ \delta) = \{\xi : |\xi - \xi_0^*| < \delta\}.$$

From (3.2) and by Hurwitz's theorem, there exist points $\xi_j \in D(\xi_0,\ \delta)$, $\xi_j^* \in D(\xi_0^*,\ \delta)$ such that for sufficiently large j,

$$\begin{aligned} &(f_j^{(k)}(z_j + \rho_j\xi_j))^m - af_j^n(z_j + \rho_j\xi_j) - b = 0, \\ &(f_j^{(k)}(z_j + \rho_j\xi_j^*))^m - af_j^n(z_j + \rho_j\xi_j^*) - b = 0. \end{aligned}$$

By the hypothesis that for each pair of functions f and g in $\mathcal{F}$, $(f^{(k)}(\xi))^m - af^n(\xi)$ and $(g^{(k)}(\xi))^m - ag^n(\xi)$ share b in D, we know that for any positive integer t,

$$\begin{aligned} &(f_t^{(k)}(z_j + \rho_j\xi_j))^m - af_t^n(z_j + \rho_j\xi_j) - b = 0, \\ &(f_t^{(k)}(z_j + \rho_j\xi_j^*))^m - af_t^n(z_j + \rho_j\xi_j^*) - b = 0. \end{aligned}$$

Fix t, take $j \to \infty$, and note $z_j + \rho_j \xi_j \to 0$, $z_j + \rho_j \xi_j^* \to 0$, then

$$(f_t^{(k)}(0))^m - af_t^n(0) - b = 0.$$

Since the zeros of $(f_t^{(k)})^m - af_t^n - b$ has no accumulation point, one has

$$z_j + \rho_j \xi_j = 0, \qquad z_j + \rho_j \xi_j^* = 0.$$

Hence

$$\xi_j = -\frac{z_j}{\rho_j}, \qquad \xi_j^* = -\frac{z_j}{\rho_j}.$$

This contradicts with $\xi_j \in D(\xi_0,\ \delta)$, $\xi_j^* \in D(\xi_0^*,\ \delta)$ and $D(\xi_0,\ \delta) \bigcap D(\xi_0^*,\ \delta) = \emptyset$. So $(g^{(k)}(\xi))^m - ag^n(\xi)$ has just a unique zero, which can be denoted by ξ_0.

Noting that g has poles and zeros of multiplicities at least d and $k+1$, respectively, (3.3) deduces that $g(\xi)$ is a rational function with degree at most 2. By Lemmas 2.3 and 2.4, this is a contradiction.

This completes the proof of Theorem 1.3.

Proof Theorem 1.4 Suppose that $\mathcal{F}$ is not normal in D. Then there exists at least one point z_0 such that $\mathcal{F}$ is not normal at the point z_0. Without loss of generality we assume that $z_0 = 0$. By Lemma 2.1, there exist points $z_j \to 0$, positive numbers $\rho_j \to 0$ and functions $f_j \in \mathcal{F}$ such that

$$g_j(\xi) = \rho_j^{km} f_j(z_j + \rho_j \xi) \to g(\xi) \tag{3.4}$$

locally uniformly with respect to the spherical metric, where g is a non-constant meromorphic function in $\mathbf{C}$ and whose poles and zeros are of multiplicity at least $mk + 2$ and $k + 1$, respectively. Moreover, the order of g is at most 2.

From (3.4) we know that

$$g_j^{(k)}(\xi) = \rho_j^{k(m+1)} f_j^{(k)}(z_j + \rho_j \xi) \to g^{(k)}(\xi)$$

and

$$\begin{aligned} &(g_j^{(k)}(\xi))^m - ag_j^{m+1}(\xi) - \rho_j^{km(m+1)} b \\ =\ &\rho_j^{km(m+1)}((f_j^{(k)}(z_j + \rho_j \xi))^m - af_j^{m+1}(z_j + \rho_j \xi) - b) \\ \to\ &(g^{(k)}(\xi))^m - ag^{m+1}(\xi) \end{aligned}$$

also locally uniformly with respect to the spherical metric.

If $(g^{(k)}(\xi))^m - ag^{m+1}(\xi) \equiv 0$, since all poles of g have multiplicity at least $mk + 2$, we can deduce that $g(\xi)$ is an entire function easily. Thus

$$\begin{aligned} (m+1)T(r,\ g) &= T(r,\ g^{m+1}) \\ &= T(r,\ (g^{(k)})^m) + O(1) \\ &= mm(r,\ g^{(k)}) + mN(r,\ g^{(k)}) + O(1) \\ &\le mm(r,\ g) + mN(r,\ g) + mk\bar{N}(r,\ g) + S(r,\ g) \\ &\le mT(r,\ g) + S(r,\ g). \end{aligned}$$

Therefore, $g(\xi)$ is a constant, a contradiction. So

$$(g^{(k)}(\xi))^m - ag^{m+1}(\xi) \not\equiv 0.$$

By Lemmas 2.5, 2.6 and 2.7, $(g^{(k)}(\xi))^m - ag^{m+1}(\xi)$ has at least two distinct zeros. Proceeding as in the later proof of Theorem 1.3, we will get a contradiction. The proof is completed.

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