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Multiplicative Jordan Decomposition in Integral Group Ring of Group $K_8 \times C_5$

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Abstract: In this article, we present the multiplicative Jordan decomposition in integral group ring of group $K_8 \times C_5$, where K_8 is the quaternion group of order 8. Thus, we give a positive answer to the question raised by Hales A W, Passi I B S and Wilson L E in the paper "The multiplicative Jordan decomposition in group rings II. *J. Algebra*, 2007, **316**: 109–132".

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1 Introduction

Let G be a finite group and $\mathbf{Q}$ the field of rational numbers. Then every element α in the group algebra $\mathbf{Q}G$ has a unique additive Jordan decomposition $\alpha = \alpha_s + \alpha_n$ with $\alpha_s, \alpha_n \in \mathbf{Q}G$, α_s is semisimple, α_n is nilpotent and $\alpha_s\alpha_n = \alpha_n\alpha_s$. Recall that an element $\alpha \in FG$ is said to be semisimple if the minimal polynomial $m(X)$ of α over F does not have repeated roots in the algebraic closure $\overline{F}$ of F with F a field of characteristic 0. Furthermore, if α is a unit in $\mathbf{Q}G$, then so is α_s, and α has a unique multiplicative Jordan decomposition $\alpha = \alpha_s\alpha_u$ with $\alpha_u = 1+\alpha_s^{-1}\alpha_n$ unipotent and $\alpha_s\alpha_u = \alpha_u\alpha_s$. But, when $\alpha \in \mathbf{Z}G$, the integral group ring over G, the semisimple component α_s does not always lie in $\mathbf{Z}G$. The integral group ring $\mathbf{Z}G$ is said to have the additive Jordan decomposition (or AJD for short) property if $\alpha_s \in \mathbf{Z}G$ (and hence $\alpha_n \in \mathbf{Z}G$) for every $\alpha \in \mathbf{Z}G$, and to have the multiplicative Jordan decomposition (or MJD for short) property if α_s and $\alpha_u \in \mathbf{Z}G$ for every unit $\alpha \in \mathbf{Z}G$. If $\mathbf{Z}G$ has the AJD property, then in fact it also has the MJD property. Therefore, to consider

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the groups G such that $\mathbf{Z}G$ has the MJD property, we need to consider the case that if $\mathbf{Z}G$ has the AJD property.

The finite groups G whose integral group ring $\mathbf{Z}G$ has the AJD property are completely classified in [1] and [2]. If G is abelian or a Hamiltonian 2-group, then every element of $\mathbf{Q}G$ is semisimple and consequently $\mathbf{Z}G$ trivially has the MJD property. In [3], the necessary conditions for a finite group G whose integral group ring $\mathbf{Z}G$ has the MJD property are given as follows:

Theorem 1.1 ([3], Theorem 29) *Let G be a finite group such that $\mathbf{Z}G$ has MJD. Then one of the following holds*:

(1) *G is either abelian or of the form $K_8 \times E \times H$, where E is an elementary abelian 2-group and H is abelian of odd order so that 2 has odd multiplicative order mod $|H|$. (Such G have AJD and hence MJD for trivial reasons, since $\mathbf{Q}G$ contains no nilpotent.)*

(2) *G has order $2^a 3^b$ for some nonnegative integers a, b.*

(3) *$G = K_8 \times C_p$ for some prime $p \geq 5$ so that 2 has even multiplicative order mod p.*

(4) *G is the split extension of C_p ($p \geq 5$) by a cyclic group $\langle g \rangle$ of order 2^k or 3^k for some $k \geq 1$, and g^2 or g^3 acts trivially on C_p.*

To completely classify the finite groups G such that $\mathbf{Z}G$ has the MJD property, we need only to investigate the four cases listed in Theorem 1.1. It has been shown that the integral group rings of abelian groups have the AJD property in [2], and so the MJD property. For the finite non-abelian 2-groups whose integral group rings possess the MJD property, there are two groups of order 8 (see [4]), five groups of order 16 (see [5]), four groups of order 32 and only the Hamiltonian groups of larger order (see [3] for details). Liu and Passman[6] showed that for the finite non-abelian 3-groups whose integral group rings have the MJD property, there are two groups of order 27 and at most three other groups (all of order 81) for larger order. Liu and Passman[7] also proved that there are precisely three non-abelian 2,3-groups of order divisible by 6, with $\mathbf{Z}G$ satisfying MJD.

Since $\mathbf{Q}(K_8 \times C_p)$ has no nilpotent elements for p a prime such that 2 has odd multiplicative order mod p, the integral group ring $\mathbf{Z}(K_8 \times C_p)$ trivially has the MJD property. When p is some odd prime such that 2 has even multiplicative order mod p, Hales *et al.*[3] claimed that "We do not know if these groups ever have MJD for their integral group rings. The first example to investigate would be $\mathbf{Z}(K_8 \times C_5)$."

In this article, we present the multiplicative Jordan decomposition in integral group ring of group $K_8 \times C_5$, where K_8 is the quaternion group of order 8. Thus, we give a positive answer to the question raised by Hales *et al.* in [3].

2 The AJD for the Units of $\mathbf{Z}(K_8 \times C_p)$ in $\mathbf{Q}(K_8 \times C_p)$

Lemma 2.1[1] *If α in $\mathbf{Q}G$ is central and β in $\mathbf{Q}G$ is semisimple, then $\alpha+\beta$ is semisimple.*

Let $C_p = \langle t \rangle$ be a cyclic group of order p and ζ a primitive pth root of unity. Let $U_1(\mathbf{Z}C_p)$ and $U_1(\mathbf{Z}[\zeta])$ denote the sets of 1-units for $\mathbf{Z}C_p$ and $\mathbf{Z}[\zeta]$, separately. We consider the map

$$\lambda : \mathbf{Z}C_p \to \mathbf{Z} \oplus \mathbf{Z}[\zeta], \qquad u = \sum c_i t^i \mapsto \big(\sum c_i,\ \sum c_i \zeta^i\big).$$

If $u = \sum c_i t^i \in U_1(\mathbf{Z}C_p)$, then $u \mapsto \big(1,\ \sum c_i\zeta^i\big)$ with $\sum c_i = 1$ and $\sum c_i\zeta^i \in U_1(\mathbf{Z}[\zeta])$. Conversely, given any $\sum b_i\zeta^i \in U_1(\mathbf{Z}[\zeta])$ with $\sum b_i = 1$, it is easy to see that $u = \sum b_i t^i$ lies in $U_1(\mathbf{Z}C_p)$ and $u \mapsto \big(1,\ \sum b_i\zeta^i\big)$. Thus, $U_1(\mathbf{Z}C_p) \cong U_1(\mathbf{Z}[\zeta])$.

Theorem 2.1 *Every non-semisimple unit of $\mathbf{Z}(K_8 \times C_p)$ can be written as a sum of a semisimple unit and a nilpotent element in $\mathbf{Q}(K_8 \times C_p)$, where*

$$K_8 \times C_p = \langle x, y \mid x^4 = 1,\ y^2 = x^2,\ yxy^{-1} = x^{-1}\rangle \times \langle t \mid t^p = 1\rangle,$$

and p is some prime such that 2 has even multiplicative order mod p.

Proof. It is well known that $K_8 \times C_p$ has $5p$ conjugacy classes, then there are $5p$ irreducible complex representations, in which $4p$ have degree 1 and p have degree 2. The $5p$ irreducible complex representations of $K_8 \times C_p$ are given by

$$R_{4k+1} : x \to 1,\ y \to 1,\ t \to \zeta^k,$$
$$R_{4k+2} : x \to 1, y \to -1, t \to \zeta^k,$$
$$R_{4k+3} : x \to -1, y \to 1, t \to \zeta^k,$$
$$R_{4k+4} : x \to -1, y \to -1, t \to \zeta^k,$$
$$R_{4p+l} : x \to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ y \to \begin{pmatrix} f(\zeta) & g(\zeta) \\ g(\zeta) & -f(\zeta) \end{pmatrix}, t \to \begin{pmatrix} \zeta^l & 0 \\ 0 & \zeta^l \end{pmatrix},$$
$$R_{5p} : x \to i,\ y \to j,\ t \to 1,$$

where $0 \le k \le p-1$, $1 \le l \le p-1$, and the polynomials f and g satisfy

$$f(\zeta)^2 + g(\zeta)^2 + 1 = 0$$

since p is some prime such that 2 has even multiplicative order mod p.

We can see that R_i $(1 \le i \le 8)$, R_{4p+1} and R_{5p} are the irreducible rational representations and the Wedderburn decomposition of the rational group algebra $\mathbf{Q}(K_8 \times C_p)$ is given as:

$$\mathbf{Q}(K_8 \times C_p) \cong \mathbf{Q}^4 \oplus \mathbf{Q}(\zeta)^4 \oplus \mathbb{H} \oplus \mathbb{M}_2(\mathbf{Q}(\zeta)),$$

where $\mathbb{H}$ is the rational quaternion algebra and $\mathbb{M}_2(\mathbf{Q}(\zeta))$ is the set of 2×2 matrices with entries in $\mathbf{Q}(\zeta)$.

Suppose that

$$U = \sum_{n=0}^{p-1}\left[\sum_{s=0}^{3}(a_{sn}x^s + a'_{sn}x^s y)\right]t^n \in U_1(\mathbf{Z}(K_8 \times C_p))$$

is not semisimple. Write

$$\beta = \sum_{n=0}^{p-1}\left[\frac{a_{1n} - a_{3n}}{2}(x - x^3) + \frac{a'_{0n} - a'_{2n}}{2}(1 - x^2)y + \frac{a'_{1n} - a'_{3n}}{2}(x - x^3)y\right]t^n.$$

Then

$$\begin{aligned}\alpha &= U - \beta \\ &= \sum_{n=0}^{p-1}\left[a_{0n} + a_{2n}x^2 + \frac{a_{1n} + a_{3n}}{2}(x + x^3) + \frac{a'_{0n} + a'_{2n}}{2}(1 + x^2)y + \frac{a'_{1n} + a'_{3n}}{2}(x + x^3)y\right]t^n\end{aligned}$$

is central, and hence, is semisimple. In order to show that $U = \alpha + \beta$ is the decomposition with α semisimple and β nilpotent in $\mathbf{Q}(K_8 \times C_p)$, it is sufficient to show that the image of β in every Wedderburn component is nilpotent.

Write

$$h(t) = \sum_{n=0}^{p-1} \left[\sum_{s=0}^{3} (a_{sn} + a'_{sn}) \right] t^n.$$

Since

$$R_1(U) = \sum_{n=0}^{p-1} \left[\sum_{s=0}^{3} (a_{sn} + a'_{sn}) \right] = 1,$$

$$R_5(U) = \sum_{n=0}^{p-1} \left[\sum_{s=0}^{3} (a_{sn} + a'_{sn}) \right] \zeta^n \in U(\mathbf{Z}[\zeta]),$$

it is easy to see that $h(t) \in U_1(\mathbf{Z}C_p)$, and hence

$$R_1(Uh(t)^{-1}) = 1, \qquad R_5(Uh(t)^{-1}) = 1.$$

Replacing U by $Uh(t)^{-1}$, we can assume

$$R_1(U) = 1, \qquad R_5(U) = 1,$$

and then

$$\begin{cases} (a_{00} + a_{10} + a_{20} + a_{30}) + (a'_{00} + a'_{10} + a'_{20} + a'_{30}) = 1, \\ (a_{01} + a_{11} + a_{21} + a_{31}) + (a'_{01} + a'_{11} + a'_{21} + a'_{31}) = 0, \\ \vdots \\ (a_{0p-1} + a_{1p-1} + a_{2p-1} + a_{3p-1}) + (a'_{0p-1} + a'_{1p-1} + a'_{2p-1} + a'_{3p-1}) = 0. \end{cases} \tag{2.1}$$

Furthermore,

$$R_2(U) = \sum_{n=0}^{p-1} \left[\sum_{s=0}^{3} (a_{sn} - a'_{sn}) \right] = \pm 1,$$

$$R_6(U) = \sum_{n=0}^{p-1} \left[\sum_{s=0}^{3} (a_{sn} - a'_{sn}) \right] \zeta^n \in U(\mathbf{Z}[\zeta]),$$

$$R_3(U) = \sum_{n=0}^{p-1} [(a_{0n} - a_{1n} + a_{2n} - a_{3n}) + (a'_{0n} - a'_{1n} + a'_{2n} - a'_{3n})] = \pm 1,$$

$$R_7(U) = \sum_{n=0}^{p-1} [(a_{0n} - a_{1n} + a_{2n} - a_{3n}) + (a'_{0n} - a'_{1n} + a'_{2n} - a'_{3n})] \zeta^n \in U(\mathbf{Z}[\zeta]),$$

$$R_4(U) = \sum_{n=0}^{p-1} [(a_{0n} - a_{1n} + a_{2n} - a_{3n}) - (a'_{0n} - a'_{1n} + a'_{2n} - a'_{3n})] = \pm 1,$$

$$R_8(U) = \sum_{n=0}^{p-1} [(a_{0n} - a_{1n} + a_{2n} - a_{3n}) - (a'_{0n} - a'_{1n} + a'_{2n} - a'_{3n})] \zeta^n \in U(\mathbf{Z}[\zeta]).$$

By (2.1), all the coefficients of $\zeta, \zeta^2, \cdots, \zeta^{p-1}$ in $R_6(U)$, $R_7(U)$ and $R_8(U)$ are even. Because of $U \in U_1(\mathbf{Z}(K_8 \times C_p))$, we also have

$$R_{5p}(U) = \sum_{n=0}^{p-1} (a_{0n} - a_{2n}) + \sum_{n=0}^{p-1} (a_{1n} - a_{3n}) i + \sum_{n=0}^{p-1} (a'_{0n} - a'_{2n}) j + \sum_{n=0}^{p-1} (a'_{1n} - a'_{3n}) ij$$
$$\in \{\pm 1, \ \pm i, \ \pm j, \ \pm ij\}.$$

It follows that either

(i) $\sum_{n=0}^{p-1}(a_{0n}-a_{2n})=0$; or

(ii) $\sum_{n=0}^{p-1}(a_{0n}-a_{2n})=\pm 1,\ \sum_{n=0}^{p-1}(a_{1n}-a_{3n})=0,\ \sum_{n=0}^{p-1}(a'_{0n}-a'_{2n})=0,\ \sum_{n=0}^{p-1}(a'_{1n}-a'_{3n})=0.$

Then, by (i) and (ii),

$$R_{5p}(\beta)=\sum_{n=0}^{p-1}(a_{1n}-a_{3n})i+\sum_{n=0}^{p-1}(a'_{0n}-a'_{2n})j+\sum_{n=0}^{p-1}(a'_{1n}-a'_{3n})ij\in\{0,\ \pm i,\ \pm j,\ \pm ij\}.$$

Moreover,

$$\boldsymbol{R}_{4p+1}(\beta)=\begin{pmatrix} Yf(\zeta)+Sg(\zeta) & X+Yg(\zeta)-Sf(\zeta) \\ -X+Yg(\zeta)-Sf(\zeta) & -Yf(\zeta)-Sg(\zeta)\end{pmatrix},$$

where

$$X=\sum_{n=0}^{p-1}(a_{1n}-a_{3n})\zeta^n,\qquad Y=\sum_{n=0}^{p-1}(a'_{0n}-a'_{2n})\zeta^n,\qquad S=\sum_{n=0}^{p-1}(a'_{1n}-a'_{3n})\zeta^n.$$

Note that

$$\mathrm{Tr}\boldsymbol{R}_{4p+1}(\beta)=0,\qquad \det\boldsymbol{R}_{4p+1}(\beta)=X^2+Y^2+S^2,$$

where "Tr" and "det" denote trace and determinant of the matrix $\boldsymbol{R}_{4p+1}(\beta)$, separately.

Case 1. If $\det\boldsymbol{R}_{4p+1}(\beta)\neq 0$, associate with $\mathrm{Tr}\boldsymbol{R}_{4p+1}(\beta)=0$, then $\boldsymbol{R}_{4p+1}(\beta)$ is semisimple. Moreover, $R_i(\beta)=0$ $(1\le i\le 8)$ and $R_{5p}(\beta)\in\{0,\ \pm i,\ \pm j,\ \pm ij\}$ are also semisimple. It follows that β is a semisimple element, and then $U=\alpha+\beta$ is also semisimple by Lemma 2.1, which is a contradiction to the assumption that U is not semisimple.

Case 2. If $\det\boldsymbol{R}_{4p+1}(\beta)=0$, associate with $\mathrm{Tr}\boldsymbol{R}_{4p+1}(\beta)=0$, then $\boldsymbol{R}_{4p+1}(\beta)$ is nilpotent.

Immediately, we have

$$\det\boldsymbol{R}_{4p+1}(\beta)=X^2+Y^2+S^2=T_0+T_1\zeta^2+T_2\zeta^4+\cdots+T_{p-1}\zeta^{2(p-1)}=0.$$

If (i) holds, then

$$pT_0=\sum_{i=0}^{p-1}T_i=\left[\sum_{n=0}^{p-1}(a_{1n}-a_{3n})\right]^2+\left[\sum_{n=0}^{p-1}(a'_{0n}-a'_{2n})\right]^2+\left[\sum_{n=0}^{p-1}(a'_{1n}-a'_{3n})\right]^2=1,$$

which is impossible since $T_0\in\mathbf{Z}$. Thus, (ii) holds and

$$\begin{aligned}T_i&=[(a_{1i}-a_{3i})^2+(a'_{0i}-a'_{2i})^2+(a'_{1i}-a'_{3i})^2]+2\sum_{k+m\equiv 2i(\mathrm{mod}\ p)}[(a_{1k}-a_{3k})(a_{1m}-a_{3m})\\&\quad+(a'_{0k}-a'_{2k})(a'_{0m}-a'_{2m})+(a'_{1k}-a'_{3k})(a'_{1m}-a'_{3m})]\\&=0,\qquad 0\le i\le p-1.\end{aligned}$$

This shows that $R_{5p}(\beta)=0$. Then we obtain that $R_i(\beta)$ is nilpotent for $1\le i\le 8$, $4p+1$, $5p$, and hence β is nilpotent in $\mathbf{Q}(K_8\times C_p)$. Thus, it is proved that the decomposition $U=\alpha+\beta$ in $\mathbf{Q}(K_8\times C_p)$ is the desired.

Remark 2.1 Recall that $T_i=0$ for $0\le i\le p-1$. Then we have that

$$(a_{1i}-a_{3i})^2+(a'_{0i}-a'_{2i})^2+(a'_{1i}-a'_{3i})^2\equiv 0\,(\mathrm{mod}\ 2),$$

and $(a_{1i}-a_{3i})$, $(a'_{0i}-a'_{2i})$ and $(a'_{1i}-a'_{3i})$ either all are even, or two are odd and the other one is even.

Lemma 2.2([8], Lemma 8.3.5) *Let A be a finite abelian group. Then*

$$U_1(\mathbf{Z}A) = A \times U_2(\mathbf{Z}A),$$

where $U_2(\mathbf{Z}A) = \{u \in U(\mathbf{Z}A) : u \equiv 1 \bmod (\Delta A)^2\}$. Moreover,

$$U_2(\mathbf{Z}A) \subseteq U_*(\mathbf{Z}A) = \{u \in U(\mathbf{Z}A) : u^* = u\},$$

where $u^* = \sum c_i g_i^{-1}$ if $u = \sum c_i g_i$.

Lemma 2.3([9], Lemma 8.1) *Let p be prime and $m \geq 1$, ζ_{p^m} a primitive p^mth root of unity.*

(a) *The cyclotomic units of $\mathbf{Q}(\zeta_{p^m})^+$ are generated by -1 and the units*

$$\zeta_a = \zeta_{p^m}^{\frac{1-a}{2}} \frac{1-\zeta_{p^m}^a}{1-\zeta_{p^m}}, \qquad 1 < a < \frac{1}{2}p^m,\ (a,\ p) = 1.$$

(b) *The cyclotomic units of $\mathbf{Q}(\zeta_{p^m})$ are generated by ζ_{p^m} and the cyclotomic units of $\mathbf{Q}(\zeta_{p^m})^+$.*

Lemma 2.4 *Let ζ be a primitive pth root of unity. Then $U_1(\mathbf{Z}[\zeta])$ is generated by ζ, $(1+\zeta), \cdots, (1+\zeta+\cdots+\zeta^{\frac{p-3}{2}})$.*

Proof. Let $C_p = \langle t \rangle$. Then $U_1(\mathbf{Z}C_p) = \langle t \rangle \times F$, where F is free and ran$F = \dfrac{p-3}{2}$ (see [8], 8.3, Exercise 1). By Lemma 2.3 and $U_1(\mathbf{Z}[\zeta]) \cong U_1(\mathbf{Z}C_p)$, $U_1(\mathbf{Z}[\zeta])$ is generated by ζ, $(1+\zeta), \cdots, (1+\zeta+\cdots+\zeta^{\frac{p-3}{2}})$.

Let $\mathcal{U} = \{u \mid u = u_0 + u_1\zeta + u_2\zeta^2 + \cdots + u_{p-1}\zeta^{p-1} \in U(\mathbf{Z}[\zeta]),\ u_0 \equiv 1 (\bmod\ 2),\ u_i \equiv 0 (\bmod\ 2)\ (1 \leq i \leq p-1)\}$ and $\mathcal{U}_1 = \big\{u \mid u \in \mathcal{U},\ \sum u_i = 1\big\}$.

Proposition 2.1 *Let $u = a_0 + a_1\zeta + a_2\zeta^2 + \cdots + a_{p-1}\zeta^{p-1} \in \mathcal{U}_1$ and $u^2 = b_0 + b_1\zeta + b_2\zeta^2 + \cdots + b_{p-1}\zeta^{p-1}$. Then $b_0 \equiv 1 (\bmod\ 4)$, $b_0 \equiv 1 (\bmod\ 8)$, $b_i \equiv 0 (\bmod\ 4)$ and $b_i \equiv a_{\frac{i}{2}}^2 + 2a_0a_i (\bmod\ 8)$ if i is even or $b_i \equiv a_{\frac{p+i}{2}}^2 + 2a_0a_i (\bmod\ 8)$ if i is odd for $1 \leq i \leq p-1$. Moreover, if $b_i \equiv 0 (\bmod\ 8)$ for all $1 \leq i \leq p-1$, then $u^2 \equiv 1 (\bmod\ 8)$ and if there is a $b_i \equiv 4 (\bmod\ 8)$ for some i, then $u^4 \equiv 1 (\bmod\ 8)$.*

Proof. Since Lemma 2.2 and $u \in \mathcal{U}_1$, $a_i = a_{p-i} \equiv 0 (\bmod\ 2)$ for $1 \leq i \leq p-1$. Then

$$1 = \sum_{i=0}^{p-1} a_i = a_0 + 2\sum_{i=1}^{\frac{p-1}{2}} a_i, \qquad a_0 \equiv 1 (\bmod\ 4).$$

It is easy to calculate that

$$b_0 = a_0^2 + 2a_1a_{p-1} + \cdots + 2a_ia_{p-i} + \cdots + 2a_{\frac{p-1}{2}}a_{\frac{p+1}{2}} \equiv a_0^2 \equiv 1 (\bmod\ 8),$$

and for $1 \leq i \leq p-1$,

$$b_i = \begin{cases} a_{\frac{i}{2}}^2 + 2\sum\limits_{l=0}^{\frac{i}{2}-1} a_la_{i-l} + 2\sum\limits_{l=i+1}^{\frac{p+i-1}{2}} a_la_{p+i-l} \equiv a_{\frac{i}{2}}^2 + 2a_0a_i (\bmod\ 8) & \text{if } i \text{ is even;} \\ a_{\frac{p+i}{2}}^2 + 2\sum\limits_{l=0}^{\frac{i}{2}-1} a_la_{i-l} + 2\sum\limits_{l=i+1}^{\frac{p+i}{2}-1} a_la_{p+i-l} \equiv a_{\frac{p+i}{2}}^2 + 2a_0a_i (\bmod\ 8) & \text{if } i \text{ is odd.} \end{cases}$$

Similarly, the rest of the assertions is obvious.

Let $(a,\ b,\ c,\ d)$ be an array, $\alpha = \dfrac{a+b+c+d}{4}$, $\beta = \dfrac{a+b-c-d}{4}$, $\gamma = \dfrac{a-b+c-d}{4}$ and $\delta = \dfrac{a-b-c+d}{4}$.

Lemma 2.5 *Let* b, c, d *be even and* $a = 0$. *If* α, β, γ, δ *are all in* $\mathbf{Z}$, *then* $(a,\ b,\ c,\ d) \pmod 8$ *just be* $(0,\ 0,\ 0,\ 0)$, $(0,\ 0,\ 4,\ 4)$, $(0,\ 4,\ 0,\ 4)$, $(0,\ 4,\ 4,\ 0)$, $(0,\ 0,\ 2,\ 6)$, $(0,\ 2,\ 0,\ 6)$, $(0,\ 2,\ 6,\ 0)$, $(0,\ 0,\ 6,\ 2)$, $(0,\ 6,\ 0,\ 2)$, $(0,\ 6,\ 2,\ 0)$, $(0,\ 2,\ 2,\ 4)$, $(0,\ 2,\ 4,\ 2)$, $(0,\ 4,\ 2,\ 2)$, $(0,\ 6,\ 6,\ 4)$, $(0,\ 6,\ 4,\ 6)$ *or* $(0,\ 4,\ 6,\ 6)$ *such that* β, γ, δ *either all are even, or one is even and the other two are odd.*

Lemma 2.6 *Let* b, c, d *be odd and* $a = 1$. *If* α, β, γ, δ *are all in* $\mathbf{Z}$, *then* $(a,\ b,\ c,\ d) \pmod 8$ *just be* $(1,\ 1,\ 1,\ 1)$, $(1,\ 1,\ 5,\ 5)$, $(1,\ 5,\ 1,\ 5)$, $(1,\ 5,\ 5,\ 1)$, $(1,\ 1,\ 3,\ 7)$, $(1,\ 3,\ 1,\ 7)$, $(1,\ 3,\ 7,\ 1)$, $(1,\ 1,\ 7,\ 3)$, $(1,\ 7,\ 1,\ 3)$, $(1,\ 7,\ 3,\ 1)$, $(1,\ 3,\ 3,\ 5)$, $(1,\ 3,\ 5,\ 3)$, $(1,\ 5,\ 3,\ 3)$, $(1,\ 7,\ 7,\ 5)$, $(1,\ 7,\ 5,\ 7)$ *or* $(1,\ 5,\ 7,\ 7)$ *such that* β, γ, δ *either all are even, or one is even and the other two are odd.*

Remark 2.2 Let $\hat{\zeta} = 1 + \zeta + \cdots + \zeta^{p-1}$. Then

$$u = \pm\zeta^{i_0}(1+\zeta)^{i_1}(1+\zeta+\zeta^2)^{i_2}\cdots(1+\zeta+\cdots+\zeta^{\frac{p-3}{2}})^{i_{\frac{p-3}{2}}} + \lambda\hat{\zeta} \in \mathcal{U}_1.$$

If there is no the item like $(1+\zeta+\cdots+\zeta^l)^{i_l}$ (l is odd), then the sum of the coefficients of $u - \lambda\hat{\zeta}$ is odd, thus λ is even, and there exists an integer j such that

$$\zeta^j(u - \lambda\hat{\zeta}) \equiv 1 + 0\zeta + 0\zeta^2 + \cdots + 0\zeta^{p-1} \pmod 2.$$

If there is the item like $(1+\zeta+\cdots+\zeta^l)^{i_l}$ (l is odd), then the sum of the coefficients of $u - \lambda\hat{\zeta}$ is even, thus λ is odd, and there exists an integer k such that

$$\zeta^k(u - \lambda\hat{\zeta}) \equiv 0 + 1\zeta + 1\zeta^2 + \cdots + 1\zeta^{p-1} \pmod 2.$$

Let $p = 5$. We obtain the main results of the article in Section 3.

3 $\mathbf{Z}(K_8 \times C_5)$ Has the MJD Property

Proposition 3.1 *The* 1*-units of* $\mathbf{Z}[\zeta]$ *are generated by* $[-(1+\zeta)^2 + \hat{\zeta}]$ *and* ζ.

Proof. By Lemma 2.4, $U_1(\mathbf{Z}[\zeta])$ is generated by $1+\zeta$ and ζ, then the 1-units of $\mathbf{Z}[\zeta]$ have the form $\pm\zeta^k(1+\zeta)^n + \lambda\hat{\zeta}$ for some integer k, n and λ. Obviously, $u = -(1+\zeta)^2 + \hat{\zeta}$ is a 1-unit of $\mathbf{Z}[\zeta]$, and $n = 2$, $\lambda = 1$ are the smallest positive integers such that

$$\pm\zeta^k(1+\zeta)^n + \lambda\hat{\zeta} \in U_1(\mathbf{Z}[\zeta]).$$

Therefore, the 1-units are generated by $[-(1+\zeta)^2 + \hat{\zeta}]$ and ζ.

Proposition 3.2 *Let* u *be the generator of* $\mathcal{U}_1$. *Then* $u \equiv 1 + 6\zeta + 6\zeta^2 + 6\zeta^3 + 6\zeta^4 \pmod 8$.

Proof. We observe that

$$(1+\zeta)^3 = 1 + 3\zeta + 3\zeta^2 + \zeta^3 + 0\zeta^4 \equiv 1 + \zeta + \zeta^2 + \zeta^3 + 0\zeta^4 \pmod 2,$$

and $n = 3$ is the smallest positive integer such that

$$\pm\zeta^k(1+\zeta)^n \equiv 0 + \zeta + \zeta^2 + \zeta^3 + \zeta^4 \pmod 2$$

for some positive integer k. By Proposition 3.1 and the definition of $\mathcal{U}_1$, $n = 6$ is the smallest positive integer such that $\pm\zeta^k(1+\zeta)^n + \lambda\hat{\zeta} \in \mathcal{U}_1$ for some integer k and λ. Then the unique generator of $\mathcal{U}_1$ should be

$$u = \pm\zeta^k(1+\zeta)^6 + \lambda\hat{\zeta}$$

for $k = 2$ and $\lambda = 5$, i.e.,

$$\begin{aligned} u = & -\zeta^2(1+\zeta)^6 + 5\hat{\zeta} \\ = & -(20 + 15\zeta + 7\zeta^2 + 7\zeta^3 + 15\zeta^4) + 5\hat{\zeta} \\ \equiv & \, 1 + 6\zeta + 6\zeta^2 + 6\zeta^3 + 6\zeta^4 (\text{mod } 8). \end{aligned}$$

Theorem 3.1 *Integral group ring* $\mathbf{Z}(K_8 \times C_5)$ *has the MJD property.*

Proof. For any $U \in U_1(\mathbf{Z}(K_8 \times C_5))$, by Theorem 2.1, $U = \alpha + \beta$ with α semisimple and β nilpotent in $\mathbf{Q}(K_8 \times C_5)$. Then it remains to show that $\beta \in \mathbf{Z}(K_8 \times C_5)$. Following Theorem 2.1, we have that

$$R_5(U) = \sum_{n=0}^{p-1} A_n\zeta^n = 1$$

with $A_0 = 1$ and $A_n = 0$ for $1 \le n \le p-1$,

$$R_6(U) \equiv \sum_{n=0}^{p-1} B_n\zeta^n (\text{mod } 8),$$

$$R_7(U) \equiv \sum_{n=0}^{p-1} C_n\zeta^n (\text{mod } 8),$$

$$R_8(U) \equiv \sum_{n=0}^{p-1} D_n\zeta^n (\text{mod } 8),$$

where $0 \le B_n, C_n, D_n \le 7$.

Let

$$\alpha_n = \frac{A_n + B_n + C_n + D_n}{4},$$
$$\beta_n = \frac{A_n + B_n - C_n - D_n}{4},$$
$$\gamma_n = \frac{A_n - B_n + C_n - D_n}{4},$$
$$\delta_n = \frac{A_n - B_n - C_n + D_n}{4}$$

for $0 \le n \le p-1$. It is not difficult to see that

$$\beta_n \equiv a_{1n} + a_{3n} \equiv a_{1n} - a_{3n} (\text{mod } 2),$$
$$\gamma_n \equiv a'_{0n} + a'_{2n} \equiv a'_{0n} - a'_{2n} (\text{mod } 2),$$
$$\delta_n \equiv a'_{1n} + a'_{3n} \equiv a'_{1n} - a'_{3n} (\text{mod } 2).$$

Recall Theorem 2.1 again,

$$T_n = \beta_n^2 + \gamma_n^2 + \delta_n^2 + 2\sum_{k+m\equiv 2n (\text{mod } p)} (\beta_k\beta_m + \gamma_k\gamma_m + \delta_k\delta_m) = 0,$$

and hence,

$$0 = T_n \equiv \beta_n^2 + \gamma_n^2 + \delta_n^2 (\text{mod } 2).$$

Thus, β_n, γ_n, δ_n either all are even or one is even and the other two are odd.

Theorem 2.1 provides that all the coefficients of ζ, ζ^2, $\cdots$, ζ^{p-1} in $R_6(U)$, $R_7(U)$ and $R_8(U)$ are even, thus, $R_6(U)$, $R_7(U)$ and $R_8(U)$ are all in $\pm\mathcal{U}_1$. By Propositions 2.1 and 3.2, the generator $u \equiv 1+6\zeta+6\zeta^2+6\zeta^3+6\zeta^4 \pmod 8$ of $\mathcal{U}_1$ is an element of order 2. Thus,

$$R_6(U) \pmod 8, \quad R_7(U) \pmod 8, \quad R_8(U) \pmod 8 \in \{u,\ -u,\ 1,\ -1\} \pmod 8.$$

By using Lemma 2.6, B_0, C_0, D_0 must all be $1 \pmod 8$ since $A_0 = 1$. Then

$$R_6(U) \pmod 8, \quad R_7(U) \pmod 8, \quad R_8(U) \pmod 8 \in \{u,\ 1\} \pmod 8,$$

and β_0, γ_0 and δ_0 are all even. Since $A_n = 0$, by Lemma 2.5, we obtain that

$$R_6(U) \equiv 1 \pmod 8, \quad R_7(U) \equiv 1 \pmod 8, \quad R_8(U) \equiv 1 \pmod 8,$$

and β_n, γ_n and δ_n all are even for all $1 \le n \le p-1$, which deduces that $\beta \in \mathbf{Z}(K_8 \times C_5)$.

References

[1] Hales A W, Luthar I S, Passi I B S. Partial augmentations and Jordan decomposition in group rings. *Comm. Algebra*, 1990, **18**: 2327–2341.

[2] Hales A W, Passi I B S. Integral group rings with Jordan decomposition. *Arch. Math.*, 1991, **57**: 21–27.

[3] Hales A W, Passi I B S, Wilson L E. The multiplicative Jordan decomposition in group rings II. *J. Algebra*, 2007, **316**: 109–132.

[4] Arora S R, Hales A W, Passi I B S. Jordan decomposition and hypercentral units in integral group rings. *Comm. Algebra*, 1993, **21**: 25–35.

[5] Parmenter M M. Multiplicative Jordan decomposition in integral group rings of groups of order 16. *Comm. Algebra*, 2002, **30**: 4789–4797.

[6] Liu C H, Passman D S. Multiplicative Jordan decomposition in group rings of 3-groups. *J. Pure Appl. Algebra*, 2009, **8**: 505–519.

[7] Liu C H, Passman D S. Multiplicative Jordan decomposition in group rings of 2,3-groups. *J. Pure Appl. Algebra*, 2010, **9**: 483–492.

[8] Milies C P, Sehgal S K. An Introduction to Group Rings. Dordrecht: Kluwer, 2002: 237–258.

[9] Washington L C. Introduction to Cyclotomic Fields. Second Edition. Berlin: Springer-Verlag, 1996: 278–286.