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Endpoint Estimates for Commutators of Fractional Integrals Associated to Operators with Heat Kernel Bounds

Liu Xian-jun, Li Wen-ming and Yan Xue-fang*
(*College of Mathematics and Information Science, Hebei Normal University, Shijiazhuang, 050024*)

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Abstract: Let L be the infinitesimal generator of an analytic semigroup on $L^2(\mathbf{R}^n)$ with pointwise upper bounds on heat kernel, and denote by $L^{-\alpha/2}$ the fractional integrals of L. For a BMO function $b(x)$, we show a weak type $L\log L$ estimate of the commutators $[b,\ L^{-\alpha/2}](f)(x) = b(x)L^{-\alpha/2}(f)(x) - L^{-\alpha/2}(bf)(x)$. We give applications to large classes of differential operators such as the Schrödinger operators and second-order elliptic operators of divergence form.

Key words: fractional integral, commutator, $L\log L$ estimate, semigroup, sharp maximal function

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1 Introduction and Main Results

Suppose that L is a linear operator on $L^2(\mathbf{R}^n)$ which generates an analytic semigroup e^{-tL} with a kernel $a_t(x,\ y)$ satisfying an upper bound of the form

$$|a_t(x,\ y)| \leq t^{-\frac{n}{m}} g\left(\frac{|x-y|^m}{t}\right), \tag{1.1}$$

where m is a positive fixed constant and g is a positive, bounded, decreasing function satisfying

$$\lim_{r\to\infty} r^{n+\varepsilon} g(r^m) = 0 \tag{1.2}$$

for some $\varepsilon > 0$.

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* **Corresponding author**.
E-mail address: liuxianjun@126.com (Liu X J), yanxuefang2008@163.com (Yan X F).

For $0<\alpha<\dfrac{2n}{m}$, the fractional integrals $L^{-\alpha/2}$ of the operator L is defined by

$$L^{-\frac{\alpha}{2}}f(x)=\frac{1}{\Gamma\left(\dfrac{\alpha}{2}\right)}\int_0^{\infty}\mathrm{e}^{-tL}(f)\frac{\mathrm{d}t}{t^{-\frac{\alpha}{2}+1}}(x). \tag{1.3}$$

Note that if $L=-\Delta$ is the Laplacian on $\mathbf{R}^n$, then $L^{-\frac{\alpha}{2}}$ is the classical fractional integrals $\mathcal{I}_\alpha$ (see, for example, Chapter 5 in [1]),

$$\mathcal{I}_\alpha(f)(x)=\int_{\mathbf{R}^n}\frac{f(y)}{|x-y|^{n-\frac{m\alpha}{2}}}\mathrm{d}y,\qquad 0<\alpha<\frac{2n}{m}.$$

Let b be a BMO function on $\mathbf{R}^n$. The commutator of b and $L^{-\frac{\alpha}{2}}$ is defined by

$$[b,\ L^{-\frac{\alpha}{2}}](f)(x)=b(x)L^{-\frac{\alpha}{2}}(f)(x)-L^{-\frac{\alpha}{2}}(bf)(x).$$

It is well known that when $b\in\mathrm{BMO}(\mathbf{R}^n)$, the commutator $[b,\ \mathcal{I}_\alpha]$ is bounded from $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$, $1<p<\dfrac{n}{\alpha}$, $\dfrac{1}{q}=\dfrac{1}{p}-\dfrac{\alpha}{n}$ (see [2]), and of weak type $L\log L$ estimate for $p=1$ (see [3] and [4]). For commutators of fractional integrals on homogeneous spaces, we refer the reader to [5], also to [6] for commutators of fractional integrals on non-homogeneous spaces.

The aim of this paper is to prove the following estimate.

Theorem 1.1 *Let $b\in BMO$, $\Phi(t)=t(1+\log^+t)$. Then for every $0<\alpha<\dfrac{2n}{m}$, and $\dfrac{1}{q}=\dfrac{1}{p}-\dfrac{m\alpha}{2n}$,*

(i) $\|[b,\ L^{-\frac{\alpha}{2}}]f\|_q\le c\|b\|_*\|f\|_p,\ 1<p<\dfrac{2n}{m\alpha}$;

(ii) *When $p=1$, $[b,\ L^{-\frac{\alpha}{2}}]$ is of weak type $L\log L$, that is,*

$$\begin{aligned}&|\{x\in\mathbf{R}^n:|[b,\ L^{-\frac{\alpha}{2}}](f)(x)|>\lambda\}|^{\frac{1}{q}}\\&\le C\left[\int_{\mathbf{R}^n}\Phi\left(\frac{\|b\|_*|f(x)|}{\lambda}\right)\mathrm{d}x\right]\left[1+\frac{m\alpha}{2n}\log^+\int_{\mathbf{R}^n}\Phi\left(\frac{\|b\|_*|f(x)|}{\lambda}\right)\mathrm{d}x\right],\end{aligned}\tag{1.4}$$

where $\|b\|_*$ denotes the BMO norm of $b(x)$.

Our result extends the results of [3] and [4] from $(-\Delta)$ to a general operator L, while we only assumes pointwise upper bounds on kernel $a_t(x,\ y)$ of e^{-tL} and no regularity on its space variables. Under our assumptions, the kernel of the operator $L^{-\frac{\alpha}{2}}$ does not have any regularity on space variables x and y. This allows flexibility on the choice of operator L in applications.

The paper is organized as follows. In Section 2, we recall some important estimates on BMO functions, maximal functions and fractional integrals. In Section 3, we prove some estimates on fractional integrals, which play a key role in the proof of the main result Theorem 1.1, which will be shown in Section 4 by using the approach of [4] and [7], combining with some estimates on the sharp maximal function $\mathcal{M}_L^{\#}f$. We conclude this paper by giving applications to large classes of differential operators which include the Schrödinger operators and second-order elliptic operators of divergence form.

Throughout, the letter "C" denote (possibly different) constants that are independent of the essential variables.

2 Definitions and Preliminary Results

Denote the Hardy-Littlewood maximal function Mf and its variant $M_\alpha f$ by

$$Mf(x)=\sup_{x\in B}\frac{1}{|B|}\int_B|f(y)|\mathrm{d}y,$$

and

$$M_\alpha f(x)=\sup_{x\in B}\frac{1}{|B|^{1-\frac{m\alpha}{2n}}}\int_B|f(y)|\mathrm{d}y.$$

For any $f\in L^p(\mathbf{R}^n)$, $p\ge 1$, the sharp maximal function $M_L^{\#}f$ associated with "generalized approximations to the identity" $\{\mathrm{e}^{-tL},\ t>0\}$, is given by

$$M_L^{\#}f(x)=\sup_{x\in B}\frac{1}{|B|}\int_B|f(y)-\mathrm{e}^{-t_BL}f(y)|\mathrm{d}y, \tag{2.1}$$

where $t_B=r_B^m$ and r_B is the radius of the ball B (see [8]).

A function $\mathcal{A}:[0,\ \infty)\to[0,\ \infty)$ is said to be a Young function if it is continuous, convex, and increasing satisfying $\mathcal{A}(0)=0$, $\mathcal{A}(t)\mapsto+\infty$ as $t\mapsto+\infty$. We define the $\mathcal{A}$-average of a function f over a ball B by means of the following Luxemburg norm:

$$\|f\|_{\mathcal{A},B}=\inf\Big\{\lambda>0:\frac{1}{|B|}\int_B\mathcal{A}\Big(\frac{|f(y)|}{\lambda}\Big)\mathrm{d}y\le 1\Big\}.$$

For the mean Luxemburg norm, the following generalized Hölder inequality holds (see [9]):

$$\frac{1}{|B|}\int_B|f(y)g(y)|\mathrm{d}y\le\|f\|_{\mathcal{A},B}\|g\|_{\bar{\mathcal{A}},B}, \tag{2.2}$$

where $\bar{\mathcal{A}}$ is the complementary Young function associated to $\mathcal{A}$.

We use a Young function $\varPhi(t)=t(1+\log^+t)$ with the corresponding average denoted by $\|f\|_{\varPhi,B}=\|f\|_{L\log L,B}$. Its complementary Young function is $\bar{\varPhi}(t)\approx\mathrm{e}^t$ with the corresponding average denoted by $\|f\|_{\bar{\varPhi},B}=\|f\|_{\exp L,B}$. We also introduce the maximal operator of the fractional order associated with $\|\cdot\|_{L\log L,B}$, which is defined by

$$M_{L\log L,\alpha}f(x)=\sup_{x\in B}|B|^{\frac{m\alpha}{2n}}\|f\|_{L\log L,B}.$$

A function $b\in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ is said to be in $\mathrm{BMO}(\mathbf{R}^n)$ if and only if

$$\sup_{x\in B}\frac{1}{|B|}\int_B|b(y)-b_B|\mathrm{d}y<\infty,$$

where $b_B=\dfrac{1}{|B|}\displaystyle\int_B b(y)\mathrm{d}y$. The BMO norm of b is defined by

$$\|b\|_*=\sup_B\frac{1}{|B|}\int_B|b(y)-b_B|\mathrm{d}y.$$

Lemma 2.1 (i) *Assume that* $b\in BMO$ *and* $N>1$. *Then for every ball* B, *we have*

$$|b_B-b_{NB}|\le C\|b\|_*\log N.$$

(ii) (John-Nirenberg Lemma) *Let* $1\le p<\infty$. *Then* $b\in BMO$ *if and only if*

$$\frac{1}{|B|}\int_Q|b-b_B|^p\mathrm{d}x\le C\|b\|_*^p.$$

(iii) *If* $b \in BMO$, *then there exists a constant* C *such that for every ball* B,

$$\frac{1}{|B|}\int_B \exp\left\{\frac{|b(x)-b_B|}{C\|b\|_*}\right\}\mathrm{d}x \le \infty.$$

(iv) *For every* $p \in [1,\infty]$, *there exists a constant* C *such that for every* $f \in L^p$,

$$|\mathrm{e}^{-tL}f(x)| \le CMf(x).$$

Proof. For the proofs of (i) and (ii), see Lemma 2.1 of [7]. For (iii), see Chapter 6 of [10], and for (iv), see Proposition 2.4 of [11].

Lemma 2.2 *Given* α, $0<\alpha<\dfrac{2n}{m}$, *and a non-negative* f, *the following statements are true*:

(i) *There exists a constant* C *such that for any ball* B,

$$\int_B \mathcal{I}_\alpha(f)(x)\mathrm{d}x \le C|B|^{\frac{m\alpha}{2n}}\int_{\mathbf{R}^n} f(x)\mathrm{d}x.$$

(ii) $\mathcal{I}_\alpha f \in A_1$; *in particular, it satisfies the reverse Hölder inequality for some exponent* $r>1$.

(iii) $\dfrac{1}{q}=1-\dfrac{m\alpha}{2n}$, $\mathcal{I}_\alpha$ *is weak* $(1,\ q)$: *for all* $\lambda>0$,

$$\big|\{x\in\mathbf{R}^n : |\mathcal{I}_\alpha(f)(x)|>\lambda\}\big|^{1/q} \le \frac{C}{\lambda}\int_{\mathbf{R}^n}|f(x)|\mathrm{d}x.$$

(iv) *If* Mf *is locally integrable, then there exists a constant* C *independent of* f *and* x *such that*

$$CM_\alpha Mf(x) \le M_{L\log L,\alpha}f(x) \le C^{-1}M_\alpha Mf(x).$$

Proof. For the proofs of (i)–(iii), see Lemma 5.2 of [4]. For (iv), see Lemma 2.3 of [3].

Lemma 2.3 *Let* $\varPhi(t)=t(1+\log^+ t)$. *Then for* $0<\alpha<\dfrac{2n}{m}$, *there exist a constant* C *such that for any bounded function* f *with bounded support and for all* $\lambda>0$,

$$\begin{aligned}&|\{x\in\mathbf{R}^n : M_{L\log L,\alpha}f(x)>\lambda\}|^{\frac{1}{q}}\\ &\le C\bigg[\int_{\mathbf{R}^n}\varPhi\Big(\frac{|f(x)|}{\lambda}\Big)\mathrm{d}x\bigg]\bigg[1+\frac{m\alpha}{2n}\log^+\int_{\mathbf{R}^n}\varPhi\Big(\frac{|f(x)|}{\lambda}\Big)\mathrm{d}x\bigg].\end{aligned} \tag{2.3}$$

Proof. For the proof of this lemma, see Lemma 2.7 of [3].

In the end of this section, we state the following analogue of the Fefferman-Stein inequality on the sharp maximal function $M_L^{\#}f$.

Lemma 2.4 *Let* $\lambda>0$ *and* $f\in L^p(\mathbf{R}^n)$ *for some* $1<p<\infty$. *Then for every* $0<\eta<1$, *we can find* $\gamma>0$ *independent of* λ, f *in such a way that*

$$|\{x\in\mathbf{R}^n : Mf(x)>A\lambda,\ M_L^{\#}f(x)\le\gamma\lambda\}| \le \eta|\{x\in\mathbf{R}^n : Mf(x)>\lambda\}|, \tag{2.4}$$

where $A>1$ *is a fixed constant which depends only on* n.

As a consequence, we have the following estimate:

(i) $\|f\|_p \le \|Mf\|_p \le c\|M_L^{\#}f\|_p$ *for every* $f\in L^p(\mathbf{R}^n)$, $1<p<\infty$.

(ii) *Let $\varphi : (0,\infty) \to (0,\infty)$ be a doubling function. Then, for any positive constant q, there exists a positive constant $c = c(q)$ such that*

$$\sup_{\lambda>0} \varphi(\lambda)|\{x : Mf(x) > \lambda\}|^{\frac{1}{q}} \le c \sup_{\lambda>0} \varphi(\lambda)|\{x : M_L^{\#} f(x) > \lambda\}|^{\frac{1}{q}}$$

for all functions f such that the left side is finite.

Proof. For the proof of (2.4), we refer to Proposition 4.1 of [8].

3 Some Estimates on Fractional Integrals

In this section, we prove several lemmas on fractional integrals $L^{-\frac{\alpha}{2}}$ which will play a key role in the proof of Theorem 1.1.

Lemma 3.1 *Let ε be the constants in (1.2), and let $0 < \alpha < \dfrac{2n}{m}$. Then, the difference operator $L^{-\frac{\alpha}{2}} - \mathrm{e}^{-tL}L^{-\frac{\alpha}{2}}$ has an associated kernel $K_{\alpha,t}(x,\ y)$ which satisfies*

$$|K_{\alpha,t}(x,\ y)| \le C\frac{1}{|x-y|^{n-\frac{m\alpha}{2}}}\left(\frac{t}{|x-y|^m}\right)^{\varepsilon_0} \tag{3.1}$$

for $0 < \varepsilon_0 \le \min\left\{1,\ \dfrac{\varepsilon}{m}\right\}$.

Proof. Note that

$$I - \mathrm{e}^{-tL} = \int_0^t \frac{\mathrm{d}}{\mathrm{d}r}\mathrm{e}^{-rL}\mathrm{d}r = -\int_0^t L\mathrm{e}^{-rL}\mathrm{d}r.$$

Hence, by (1.3),

$$(I - \mathrm{e}^{-tL})L^{-\frac{\alpha}{2}} = \frac{1}{\Gamma\left(\frac{\alpha}{2}\right)}\int_0^t\int_0^\infty \left(v\frac{\mathrm{d}}{\mathrm{d}v}\mathrm{e}^{-vL}\right)\Big|_{v=r+s}\frac{1}{r+s}\cdot\frac{\mathrm{d}s\mathrm{d}r}{s^{-\frac{\alpha}{2}+1}}.$$

The kernel of $v\dfrac{\mathrm{d}}{\mathrm{d}v}\mathrm{e}^{-vL}$ also satisfies (1.1) (see [12]). Hence, the operator $(I-\mathrm{e}^{-tL})L^{-\frac{\alpha}{2}}$ has an associated kernel $K_{\alpha,t}(x,\ y)$ which satisfies

$$\begin{aligned}
|K_{\alpha,t}(x,y)| &\le C\frac{1}{\Gamma\left(\frac{\alpha}{2}\right)}\int_0^t\int_0^\infty (r+s)^{-\frac{n}{m}} g\left(\frac{|x-y|^m}{r+s}\right)\frac{1}{r+s}\cdot\frac{\mathrm{d}s\mathrm{d}r}{s^{-\frac{\alpha}{2}+1}}\\
&\le C\int_0^t\int_0^r (r+s)^{-\frac{n}{m}} g\left(\frac{|x-y|^m}{r+s}\right)\frac{1}{r+s}\cdot\frac{\mathrm{d}s\mathrm{d}r}{s^{-\frac{\alpha}{2}+1}}\\
&\quad + C\int_0^t\int_r^\infty (r+s)^{-\frac{n}{m}} g\left(\frac{|x-y|^m}{r+s}\right)\frac{1}{r+s}\cdot\frac{\mathrm{d}s\mathrm{d}r}{s^{-\frac{\alpha}{2}+1}}\\
&= \mathrm{I} + \mathrm{II}.
\end{aligned}$$

It follows from (1.2) that one has $\lim\limits_{r\to\infty} r^a g(r^m) = 0$ for any a, $0 < a < n+\varepsilon$. Choose an ε_0 such that $0 < \varepsilon_0 \le \min\left\{1,\ \dfrac{\varepsilon}{m}\right\}$, then since $\alpha < \dfrac{2n}{m}$, we get $0 < n + m\varepsilon_0 - \dfrac{m\alpha}{2} < n+\varepsilon$, this implies that $\lim\limits_{r\to\infty} r^{n+m\varepsilon_0-\frac{m\alpha}{2}} g(r^m) = 0$. Furthermore, we have that $g\left(\dfrac{1}{s}\right) \le Cs^{(\frac{n}{m}+\varepsilon_0-\frac{\alpha}{2})}$.

Let us estimate term I. Note that $0 < s < r$, one has

$$\begin{aligned}
\mathrm{I} &\le C\int_0^t\int_0^r r^{-\frac{n}{m}} g\Big(\frac{|x-y|^m}{2r}\Big)\frac{1}{r}\cdot\frac{\mathrm{d}s\mathrm{d}r}{s^{-\frac{\alpha}{2}+1}}\\
&= C\frac{1}{|x-y|^{n-\frac{m\alpha}{2}}}\int_0^{\frac{t}{|x-y|^m}} u^{\frac{\alpha}{2}-\frac{n}{m}-1} g\Big(\frac{1}{u}\Big)\mathrm{d}u\\
&\le C\frac{1}{|x-y|^{n-\frac{m\alpha}{2}}}\int_0^{\frac{t}{|x-y|^m}} u^{\frac{\alpha}{2}-\frac{n}{m}-1} u^{\frac{n}{m}+\varepsilon_0-\frac{\alpha}{2}}\mathrm{d}u\\
&\le C\frac{1}{|x-y|^{n-\frac{m\alpha}{2}}}\Big(\frac{t}{|x-y|^m}\Big)^{\varepsilon_0}.
\end{aligned}$$

For II, note that $0<r<s$, one has

$$\begin{aligned}
\mathrm{II} &\le C\int_0^t\int_r^\infty s^{-\frac{n}{m}} g\Big(\frac{|x-y|^m}{2s}\Big)\frac{1}{s}\cdot\frac{\mathrm{d}s\mathrm{d}r}{s^{-\frac{\alpha}{2}+1}}\\
&\le C\int_0^t\int_0^s s^{-\frac{n}{m}} g\Big(\frac{|x-y|^m}{2s}\Big)\frac{1}{s}\cdot\frac{\mathrm{d}r\mathrm{d}s}{s^{-\frac{\alpha}{2}+1}}\\
&\quad + C\int_t^\infty\int_0^t s^{-\frac{n}{m}} g\Big(\frac{|x-y|^m}{2s}\Big)\frac{1}{s}\cdot\frac{\mathrm{d}r\mathrm{d}s}{s^{-\frac{\alpha}{2}+1}}\\
&= C\int_0^t s^{-\frac{n}{m}} g\Big(\frac{|x-y|^m}{2s}\Big)\frac{\mathrm{d}s}{s^{-\frac{\alpha}{2}+1}}\\
&\quad + Ct\int_t^\infty s^{-\frac{n}{m}} g\Big(\frac{|x-y|^m}{2s}\Big)\frac{\mathrm{d}s}{s^{-\frac{\alpha}{2}+2}}\\
&= \mathrm{II}_1+\mathrm{II}_2.
\end{aligned}$$

Similar to the estimate of term I, one has

$$\mathrm{II}_1 \le C\frac{1}{|x-y|^{n-\frac{m\alpha}{2}}}\Big(\frac{t}{|x-y|^m}\Big)^{\varepsilon_0}.$$

On the other hand,

$$\begin{aligned}
\mathrm{II}_2 &\le Ct\int_{\frac{t}{|x-y|^m}}^\infty (|x-y|^m u)^{\frac{\alpha}{2}-\frac{n}{m}-2} g\Big(\frac{1}{u}\Big)|x-y|^m\mathrm{d}u\\
&\le C\frac{1}{|x-y|^{n-\frac{m\alpha}{2}}}\frac{t}{|x-y|^m}\int_{\frac{t}{(|x-y|)^m}}^\infty u^{\frac{\alpha}{2}-\frac{n}{m}-2} u^{\frac{n}{m}+\varepsilon_0-\frac{\alpha}{2}}\mathrm{d}u\\
&= C\frac{1}{|x-y|^{n-\frac{m\alpha}{2}}}\Big(\frac{t}{|x-y|^m}\Big)^{\varepsilon_0}.
\end{aligned}$$

Therefore, condition (3.1) is satisfied and then the proof of Lemma 3.1 is completed.

We remark that when L has a Gaussian upper bounds, Lemma 3.1 is proved in [7] for $0<\alpha<1$, and in [13] for $0<\alpha<n$, respectively.

Remark 3.1 Let $0<\alpha<\dfrac{2n}{m}$. Using the formula (1.3), together with properties (1.1) and (1.2) and elementary integration, it can be verified that the kernel $K_\alpha(x,\ y)$ of $L^{-\frac{\alpha}{2}}$ satisfies

$$
\begin{aligned}
|K_\alpha(x,\ y)| &\le C\int_0^\infty t^{\frac{\alpha}{2}}t^{-\frac{n}{m}}g\bigg(\frac{|x-y|^m}{t}\bigg)\frac{\mathrm{d}t}{t}\\
&\le C\frac{1}{|x-y|^{n-\frac{m\alpha}{2}}}
\end{aligned}
\tag{3.2}
$$

for some positive constant C.

Lemma 3.2 *Let $b\in BMO$. Then there exists a positive constant C such that*

$$
M_L^{\#}([b,\ L^{-\frac{\alpha}{2}}]f)(x)\le C\|b\|_*[\mathcal{I}_\alpha(|f|)(x)+M_{L\log L,\alpha}f(x)]. \tag{3.3}
$$

Proof. Since $\mathcal{I}_\alpha f\in A_1$, it suffices to prove that there exists a constant C such that for all $x\in\mathbf{R}^n$ and for all $B\ni x$,

$$
\frac{1}{|B|}\int_B|(I-\mathrm{e}^{-t_BL})[b,\ L^{-\frac{\alpha}{2}}]f(y)|\mathrm{d}y\le C\|b\|_*[M(\mathcal{I}_\alpha(|f|))(x)+M_{L\log L,\alpha}f(x)], \tag{3.4}
$$

where $t_B=r_B^m$, and r_B is the radius of B.

For an arbitrary fixed $x\in\mathbf{R}^n$, choose a ball $B=B(x_0;\ r)=\{y\in\mathbf{R}^n:|x_0-y|<r\}$ which contains x. Let $f_1=f\chi_{2B}$ and $f_2=f-f_1$. One writes

$$
[b,\ L^{-\frac{\alpha}{2}}]f=(b-b_{2B})L^{-\frac{\alpha}{2}}f-L^{-\frac{\alpha}{2}}((b-b_{2B})f_1)-L^{-\frac{\alpha}{2}}((b-b_{2B})f_2)
$$

and

$$
\mathrm{e}^{-t_BL}([b,\ L^{-\frac{\alpha}{2}}]f)=\mathrm{e}^{-t_BL}\big[(b-b_{2B})L^{-\frac{\alpha}{2}}f-L^{-\frac{\alpha}{2}}((b-b_{2B})f_1)-L^{-\frac{\alpha}{2}}((b-b_{2B})f_2)\big].
$$

Then,

$$
\begin{aligned}
\text{LHS of (3.4)}&=\frac{1}{|B|}\int_B|[b,\ L^{-\frac{\alpha}{2}}]f(y)-\mathrm{e}^{-t_BL}[b,\ L^{-\frac{\alpha}{2}}]f(y)|\mathrm{d}y\\
&\le\frac{1}{|B|}\int_B|(b(y)-b_{2B})L^{-\frac{\alpha}{2}}f(y)|\mathrm{d}y\\
&\quad+\frac{1}{|B|}\int_B\big\{|L^{-\frac{\alpha}{2}}((b-b_{2B})f_1)(y)|+|\mathrm{e}^{-t_BL}L^{-\frac{\alpha}{2}}((b-b_{2B})f_1)(y)|\big\}\mathrm{d}y\\
&\quad+\frac{1}{|B|}\int_B|\mathrm{e}^{-t_BL}((b-b_{2B})L^{-\frac{\alpha}{2}}f)(y)|\mathrm{d}y\\
&\quad+\frac{1}{|B|}\int_B|(L^{-\frac{\alpha}{2}}-\mathrm{e}^{-t_BL}L^{-\frac{\alpha}{2}})((b-b_{2B})f_2)(y)|\mathrm{d}y\\
&=\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}.
\end{aligned}
$$

We estimate each integral in turn. Obviously, by (3.2), we have the following pointwise inequality

$$
|L^{-\frac{\alpha}{2}}(f)(x)|\le C\mathcal{I}_\alpha(|f|)(x),\qquad x\in\mathbf{R}^n. \tag{3.5}
$$

For I, by Lemma 2.2, $\mathcal{I}_\alpha f$ satisfies the reverse Hölder's inequality with exponent r, by Lemma 2.2 and (3.5), we have

$$
\begin{aligned}
\mathrm{I}&\le\bigg(\frac{1}{|B|}\int_B|b(y)-b_{2B}|^{r'}\mathrm{d}y\bigg)^{\frac{1}{r'}}\bigg(\frac{1}{|B|}\int_B|L^{-\frac{\alpha}{2}}f(y)|^r\mathrm{d}y\bigg)^{\frac{1}{r}}\\
&\le C\|b\|_*\bigg(\frac{1}{|B|}\int_B|\mathcal{I}_\alpha(|f|)(y)|^r\mathrm{d}y\bigg)^{\frac{1}{r}}
\end{aligned}
$$

$$\leq C\|b\|_*\frac{1}{|B|}\int_B|\mathcal{I}_\alpha(|f|)(y)|\mathrm{d}y$$
$$\leq C\|b\|_*M(\mathcal{I}_\alpha(|f|)(x).$$

To estimate the second integral, note that by Lemmas 2.1, 2.2 and (3.5),

$$|L^{-\frac{\alpha}{2}}((b-b_{2B})f_1)(y)|\leq C\mathcal{I}_\alpha(|(b-b_{2B})f_1|)(y),$$

and

$$|\mathrm{e}^{-t_BL}L^{-\alpha/2}((b-b_{2B})f_1)(y)|\leq CM(\mathcal{I}_\alpha(|(b-b_{2B})f_1|))(y)\leq C\mathcal{I}_\alpha(|(b-b_{2B})f_1|)(y).$$

Hence, by Lemmas 2.2, 2.1, and by the generalized Hölder's inequality (2.2),

$$\begin{aligned}
\mathrm{II}&\leq\frac{1}{|B|}\int_B\mathcal{I}_\alpha(|(b-b_{2B})f_1|)(y)\mathrm{d}y\\
&\leq C\frac{|B|^{\frac{m\alpha}{2n}}}{|B|}\int_{2B}|(b(y)-b_{2B})||f(y)|\mathrm{d}y\\
&\leq C|2B|^{\frac{m\alpha}{2n}}\|(b-b_{2B})\|_{\exp L,2B}\|f\|_{L\log L,2B}\\
&\leq C\|b\|_*M_{L\log L,\alpha}f(x).
\end{aligned}$$

For term III, we have

$$\begin{aligned}
\mathrm{III}&=\frac{1}{|B|}\int_B|\mathrm{e}^{-t_BL}((b-b_{2B})L^{-\frac{\alpha}{2}}f)(y)|\mathrm{d}y\\
&\leq\frac{1}{|B|}\int_B\int_{\mathbf{R}^n}|a_{t_B}(y,\ z)||b(z)-b_{2B}||L^{-\frac{\alpha}{2}}f(z)|\mathrm{d}z\mathrm{d}y\\
&\leq\frac{1}{|B|}\int_B\int_{2B}|a_{t_B}(y,\ z)||b(z)-b_{2B}|\mathcal{I}_\alpha(|f|)(z)\mathrm{d}z\mathrm{d}y\\
&\quad+\sum_{k=1}^{\infty}\frac{1}{|B|}\int_B\int_{2^{k+1}B\setminus2^kB}|a_{t_B}(y,\ z)||b(z)-b_{2B}|\mathcal{I}_\alpha(|f|)(z)\mathrm{d}z\mathrm{d}y\\
&=\mathrm{III}_1+\mathrm{III}_2.
\end{aligned}$$

We now estimate III_1. For $y\in B$, $z\in 2B$, we have

$$|a_{t_B}(y,\ z)|\leq t_B^{-\frac{n}{m}}g\left(\frac{|y-z|^m}{t_B}\right)\leq\frac{g(0)}{t_B^{\frac{n}{m}}}=\frac{C}{r_B^n}=\frac{C}{|2B|}.$$

Similar to the estimate of term I, we obtain

$$\begin{aligned}
\mathrm{III}_1&\leq\frac{C}{|B|}\frac{1}{|2B|}\int_B\int_{2B}|b(z)-b_{2B}|\mathcal{I}_\alpha(|f|)(z)\mathrm{d}z\mathrm{d}y\\
&\leq\frac{C}{|2B|}\int_{2B}|b(z)-b_{2B}|\mathcal{I}_\alpha(|f|)(z)\mathrm{d}z\\
&\leq C\|b\|_*M(\mathcal{I}_\alpha(|f|)(x).
\end{aligned}$$

Regarding III_2, for $y\in B$ and $z\in 2^{k+1}B\setminus 2^kB$, we have

$$|y-z|\geq 2^{k-1}r_B$$

and

$$|a_{t_B}(y,\ z)|\leq t_B^{-\frac{n}{m}}g\left(\frac{|y-z|^m}{t_B}\right)\leq\frac{g(2^{(k-1)m})}{r_B^n}=\frac{g(2^{(k-1)m})2^{(k+1)n}}{|2^{k+1}B|}.$$

Similarly, by Lemma 2.1, we have

$$\begin{aligned}
\mathrm{III}_2 &\le C\sum_{k=1}^{\infty} g(2^{(k-1)m})2^{(k+1)n}\frac{1}{|B|}\frac{1}{|2^{k+1}B|}\int_B\int_{2^{k+1}B\backslash 2^kB}|b(z)-b_{2B}|\mathcal{I}_\alpha(|f|)(z)\mathrm{d}z\mathrm{d}y\\
&\le C\sum_{k=1}^{\infty} g(2^{(k-1)m})2^{(k+1)n}\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|b(z)-b_{2B}|\mathcal{I}_\alpha(|f|)(z)\mathrm{d}z\\
&\le C\sum_{k=1}^{\infty} g(2^{(k-1)m})2^{(k+1)n}\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|b(z)-b_{2^{k+1}B}|\mathcal{I}_\alpha(|f|)(z)\mathrm{d}z\\
&\quad+C\sum_{k=1}^{\infty} g(2^{(k-1)m})2^{(k+1)n}\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|b_{2^{k+1}B}-b_{2B}|\mathcal{I}_\alpha(|f|)(z)\mathrm{d}z\\
&\le C\|b\|_*M(\mathcal{I}_\alpha(|f|)(x).
\end{aligned}$$

Let us see what happens with term IV. By using Lemmas 3.1 and 2.1, one has

$$\begin{aligned}
\mathrm{IV} &\le \frac{1}{|B|}\int_B\int_{(2B)^c}|K_{\alpha,t_B}(y,\ z)||(b(z)-b_B)f(z)|\mathrm{d}z\mathrm{d}y\\
&\le C\sum_{k=1}^{\infty}\int_{2^kr_B\le|x_0-z|<2^{k+1}r_B}\frac{1}{|x_0-z|^{n-\frac{m\alpha}{2}}}\left(\frac{t_B}{|x_0-z|^m}\right)^{\varepsilon_0}|(b(z)-b_B)f(z)|\mathrm{d}z\\
&\le C\sum_{k=1}^{\infty}2^{-km\varepsilon_0}\frac{1}{|2^{k+1}B|^{1-\frac{m\alpha}{2n}}}\int_{2^{k+1}B}|(b(z)-b_B)f(z)|\mathrm{d}z\\
&\le C\sum_{k=1}^{\infty}2^{-km\varepsilon_0}\frac{1}{|2^{k+1}B|^{1-\frac{m\alpha}{2n}}}\int_{2^{k+1}B}|(b(z)-b_{2^{k+1}B})f(z)|\mathrm{d}z\\
&\quad+C\sum_{k=1}^{\infty}2^{-km\varepsilon_0}|b_{2^{k+1}B}-b_B|\frac{1}{|2^{k+1}B|^{1-\frac{m\alpha}{2n}}}\int_{2^{k+1}B}|f(z)|\mathrm{d}z\\
&\le C\sum_{k=1}^{\infty}2^{-km\varepsilon_0}|2^{k+1}B|^{\frac{m\alpha}{2n}}\|b(z)-b_{2^{k+1}B}\|_{\exp L,2^{k+1}B}\|f\|_{L\log L,2^{k+1}B}\\
&\quad+C\|b\|_*\sum_{k=1}^{\infty}2^{-km\varepsilon_0}(k+1)M_\alpha f(x)\\
&\le C\|b\|_*\sum_{k=1}^{\infty}2^{-km\varepsilon_0}M_{L\log L,\alpha}f(x)+C\|b\|_*\sum_{k=1}^{\infty}2^{-km\varepsilon_0}(k+1)M_\alpha f(x)\\
&\le C\|b\|_*M_{L\log L,\alpha}f(x).
\end{aligned}$$

Combining the above estimates I, II, III and IV, we obtain (3.4), and the proof of Lemma 3.2 is completed.

4 Proof of Theorem 1.1

(i) By Lemmas 2.4, 3.2 and 2.2, we have

$$\|[b,\ L^{-\frac{\alpha}{2}}]f\|_q \le c\|\mathcal{M}_L^{\#}([b,\ L^{-\frac{\alpha}{2}}])f\|_q$$

$$\leq c\|b\|_*\|\mathcal{I}_\alpha f\|_q + c\|b\|_*\|\mathcal{M}_{\mathrm{LlogL},\alpha} f\|_q$$
$$\leq c\|b\|_*\|\mathcal{I}_\alpha f\|_q + c\|b\|_*\|M_\alpha M f\|_q$$
$$\leq c\|b\|_*\|f\|_p,$$

where we use the fact that $\mathcal{I}_\alpha$, M_α are both bounded from $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$ for $\dfrac{1}{q} = \dfrac{1}{p} - \dfrac{m\alpha}{2n}$ and $1 < p < \dfrac{2n}{m\alpha}$, M is bounded from $L^p(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$ for $1 < p < \infty$.

(ii) Without loss of generality, we assume that f is a smooth function with compact support. By homogeneity, it suffices to verify that (1.4) is true for $\lambda = 1$, that is,

$$|\{x \in \mathbf{R}^n : |[b,\ L^{-\frac{\alpha}{2}}]f(x)| > 1\}|^{\frac{1}{q}}$$
$$\leq C\Big[\int_{\mathbf{R}^n} \Phi\Big(|f(x)|\|b\|_*\Big)\mathrm{d}x\Big]\Big[1 + \frac{m\alpha}{2n}\log^+ \int_{\mathbf{R}^n} \Phi\Big(|f(x)|\|b\|_*\Big)\mathrm{d}x\Big]. \tag{4.1}$$

Note that

$$|\{x \in \mathbf{R}^n : |[b,\ L^{-\frac{\alpha}{2}}](f)|(x) > 1\}|^{\frac{1}{q}}$$
$$\leq\ \Phi(\Phi(1)) \sup_{t>0} \frac{1}{\Phi\Big(\Phi\Big(\dfrac{1}{t}\Big)\Big)} |\{x \in \mathbf{R}^n : |[b,\ L^{-\frac{\alpha}{2}}]f(x)| > t\}|^{\frac{1}{q}}$$
$$\leq\ \Phi(\Phi(1)) \sup_{t>0} \frac{1}{\Phi\Big(\Phi\Big(\dfrac{1}{t}\Big)\Big)} |\{x \in \mathbf{R}^n : |M([b,\ L^{-\frac{\alpha}{2}}]f)(x)| > t\}|^{\frac{1}{q}}.$$

Let

$$\varphi(t) = \frac{1}{\Phi\Big(\Phi\Big(\dfrac{1}{t}\Big)\Big)}.$$

A straightforward calculation shows that $\varphi(t)$ is a doubling function. So

$$\varphi(2t) \leq C\varphi(t).$$

By Lemmas 2.4, 2.2 and 2.3 in sequence, and note that $t\Phi\Big(\Phi\Big(\dfrac{1}{t}\Big)\Big) \geq 1$, we have

$$|\{x \in \mathbf{R}^n : |[b,\ L^{-\frac{\alpha}{2}}](f)|(x) > 1\}|^{\frac{1}{q}}$$
$$\leq\ \Phi(\Phi(1)) \sup_{t>0} \frac{1}{\Phi\Big(\Phi\Big(\dfrac{1}{t}\Big)\Big)} |\{x \in \mathbf{R}^n : |M_L^{\#}([b,\ L^{-\frac{\alpha}{2}}]f)(x)| > t\}|^{\frac{1}{q}}$$
$$\leq C \sup_{t>0} \frac{1}{\Phi\Big(\Phi\Big(\dfrac{1}{t}\Big)\Big)} \Big|\Big\{x \in \mathbf{R}^n : \mathcal{I}_\alpha(|f|)(x) + M_{L\log L,\alpha} f(x) > \frac{t}{C\|b\|_*}\Big\}\Big|^{\frac{1}{q}}$$
$$\leq C \sup_{t>0} \frac{1}{\Phi\Big(\Phi\Big(\dfrac{1}{t}\Big)\Big)} \Big|\Big\{x \in \mathbf{R}^n : \mathcal{I}_\alpha(|f|)(x) > \frac{t}{C\|b\|_*}\Big\}\Big|^{\frac{1}{q}}$$
$$+ C \sup_{t>0} \frac{1}{\Phi\Big(\Phi\Big(\dfrac{1}{t}\Big)\Big)} \Big|\Big\{x \in \mathbf{R}^n : M_{L\log L,\alpha} f(x) > \frac{t}{C\|b\|_*}\Big\}\Big|^{\frac{1}{q}}$$

$$
\begin{aligned}
&\leq C\sup_{t>0}\frac{1}{\Phi\Big(\Phi\Big(\dfrac{1}{t}\Big)\Big)}\frac{\|b\|_*}{t}\int_{\mathbf{R}^n}|f(x)|\mathrm{d}x\\
&\quad+C\sup_{t>0}\frac{1}{\Phi\Big(\Phi\Big(\dfrac{1}{t}\Big)\Big)}\bigg[\int_{\mathbf{R}^n}\Phi\Big(\frac{\|b\|_*|f(x)|}{t}\Big)\mathrm{d}x\bigg]\bigg[1+\frac{m\alpha}{2n}\log^+\int_{\mathbf{R}^n}\Phi\Big(\frac{\|b\|_*|f(x)|}{t}\Big)\mathrm{d}x\bigg]\\
&\leq C\sup_{t>0}\frac{1}{\Phi\Big(\Phi\Big(\dfrac{1}{t}\Big)\Big)}\frac{1}{t}\int_{\mathbf{R}^n}\|b\|_*|f(x)|\mathrm{d}x+C\sup_{t>0}\frac{1}{\Phi\Big(\Phi\Big(\dfrac{1}{t}\Big)\Big)}\Phi\Big(\Phi\Big(\frac{1}{t}\Big)\Big)\\
&\quad\times\bigg[\int_{\mathbf{R}^n}\Phi(\|b\|_*|f(x)|)\mathrm{d}x\bigg]\bigg[1+\frac{m\alpha}{2n}\log^+\int_{\mathbf{R}^n}\Phi(\|b\|_*|f(x)|)\mathrm{d}x\bigg]\\
&=J_1+J_2. \qquad (4.2)
\end{aligned}
$$

Furthermore, since $t\leq\Phi(t)$, we have

$$
\int_{\mathbf{R}^n}\|b\|_*|f(x)|\mathrm{d}x\leq\bigg[\int_{\mathbf{R}^n}\Phi(\|b\|_*|f(x)|)\mathrm{d}x\bigg]\bigg[1+\frac{m\alpha}{2n}\log^+\int_{\mathbf{R}^n}\Phi(\|b\|_*|f(x)|)\mathrm{d}x\bigg]. \qquad (4.3)
$$

From (4.2) and (4.3) we get

$$
J_1+J_2\leq C\bigg[\int_{\mathbf{R}^n}\Phi(\|b\|_*|f(x)|)\mathrm{d}x\bigg]\bigg[1+\frac{m\alpha}{2n}\log^+\int_{\mathbf{R}^n}\Phi(\|b\|_*|f(x)|)\mathrm{d}x\bigg],
$$

which proves (4.1). The proof of Theorem 1.1 is completed.

Remark 4.1 The heat kernel upper bound (1.1) of Theorem 1.1 is satisfied by large classes of differential operators. We list some of them as follows.

(a) Let V be a nonnegative function on $\mathbf{R}^n$. The Schrödinger operator with potential V is defined by

$$
L=-\Delta+V(x). \qquad (4.4)
$$

By domination, the kernel $a_t(x,\ y)$ of the semigroup $\{\mathrm{e}^{-tL}\}_{t>0}$ has a Gaussian upper bound (see [14]). Therefore, the result for the fractional integrals $L^{-\frac{\alpha}{2}}$, that is Theorem 1.1, holds for the operator L of (4.4) in which V is a nonnegative function on $\mathbf{R}^n$.

Note that unless V satisfies additional conditions, the heat kernel can be a discontinuous function of the space variables and the Hölder continuous estimates may fail to hold.

(b) Let $\boldsymbol{A}=((a_{ij}(x))_{1\leq i,j\leq n}$ be an $n\times n$ matrix of complex with entries $a_{ij}\in L^\infty(\mathbf{R}^n)$ satisfying

$$
\mathrm{Re}\sum a_{ij}(x)\xi_i\xi_j\geq\lambda|\xi|^2
$$

for all $x\in\mathbf{R}^n$, $\boldsymbol{\xi}=(\xi_1,\ \xi_2,\ \cdots,\ \xi_n)\in\mathbf{C}^n$ and some $\lambda>0$. We define a divergence form operator

$$
Lf\equiv-\mathrm{div}(A\nabla f),
$$

which we interpret in the usual weak sense via a sesquilinear form.

It is known that the Gaussian upper bound on the heat kernel e^{-tL} is true when A has real entries, when $n=1,\ 2$ in the case of complex entries, see Chapter 1 of [15].

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