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# Growth of Solutions of Some Linear Difference Equations with Meromorphic Coefficients

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**Abstract:** In this paper, we investigate the properties of solutions of some linear difference equations with meromorphic coefficients, and obtain some estimates on growth and value distribution of these meromorphic solutions.

**Key words:** difference equation, meromorphic function, growth of order, exponent of convergence

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## 1 Introduction and Main Results

In this paper, we use the standard notations and the fundamental results of Nevanlinna's theory (see [1]–[2]). Let $f$ be a meromorphic function in the whole complex plane, we denote by $\sigma(f)$, $\lambda(f)$ and $\lambda\left(\dfrac{1}{f}\right)$ the order, the exponent of convergence of zeros and poles of $f(z)$, respectively.

Nevanlinna's theory has been widely applied to the field of complex difference. Many researchers studied the properties of meromorphic solutions of the following linear difference equation by this theory

$$A_n(z)f(z+c_n)+\cdots+A_1(z)f(z+c_1)+A_0(z)f(z)=0, \tag{1.1}$$

where $n\in\mathbf{N}$, $c_j$ $(j=1,\cdots,n)$ are nonzero complex numbers which are different from each other, and obtained lots of results concerning the growth and value distribution of meromorphic solutions of (1.1) (see [3]–[9]). Therein Chiang and Feng[4] considered the

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case when there is only one dominating coefficient among all entire coefficients of (1.1), and obtained the following result:

**Theorem A**[4] *Let $A_0(z), \cdots, A_n(z)$ be entire functions. If there exists an integer $l$ $(0 \le l \le n)$ such that*

$$\sigma(A_l) > \max_{\substack{0\le j\le n\\ j\ne l}}\{\sigma(A_j)\},$$

*then every meromorphic solution $f(\not\equiv 0)$ of* (1.1) *satisfies*

$$\sigma(f) \ge \sigma(A_l) + 1.$$

When most coefficients of (1.1) have the same order, Qi *et al.*[9] studied the properties of meromorphic solutions of the following linear difference equation

$$f(z+n) + \sum_{j=0}^{n-1}\{P_j(\mathrm{e}^{A(z)}) + Q_j(\mathrm{e}^{-A(z)})\}f(z+j) = 0, \tag{1.2}$$

and obtained the following results:

**Theorem B**[9] *Let $P_j(z)$ and $Q_j(z)$ $(j = 0, 1, \cdots, n-1)$ be polynomials, $A(z)$ be a polynomial of degree $k(\ge 1)$. If*

$$\deg(P_0) > \deg(P_j) \quad or \quad \deg(Q_0) > \deg(Q_j), \qquad j = 1, \cdots, n-1,$$

*then each nontrivial meromorphic solution $f(z)$ with finite order of* (1.2) *satisfies*

$$\sigma(f) = \lambda(f-a) \ge k+1,$$

*and so $f$ assumes every nonzero complex value $a \in \mathbf{C}$ infinitely often.*

**Theorem C**[9] *Suppose that the assumptions of Theorem B are satisfied. If $f(z)$ is a nontrivial entire solution with finite order of* (1.2) *that satisfies $\lambda(f) \le k$, then $\sigma(f) = k+1$.*

Comparing Theorem A with Theorem B and Theorem C, we pose the following questions:

**Question 1.1** When all coefficients of (1.1) have the form $P_j(\mathrm{e}^{A(z)}) + Q_j(\mathrm{e}^{-A(z)}) + R_j(z)$ $(j = 0, \cdots, n)$, where $P_j$, $Q_j$ and $A$ are polynomials, $R_j$ are meromorphic functions, and satisfy $\deg(P_l) > \max_{\substack{0\le j\le n\\ j\ne l}}\{\deg(P_j)\}$ or $\deg(Q_l) > \max_{\substack{0\le j\le n\\ j\ne l}}\{\deg(Q_j)\}$, does the conclusion of Theorem B hold?

**Question 1.2** Theorem C provided a criterion which guarantee that each entire solution of (1.2) has the smallest order. Then under the assumptions of Question 1.1, what else condition can guarantee that each meromorphic solution of (1.1) has the smallest order?

In this paper, we investigate the above questions and obtain the following results.

**Theorem 1.1** *Let $A_j(z) = P_j(\mathrm{e}^{A(z)}) + Q_j(\mathrm{e}^{-A(z)}) + R_j(z)$ $(j = 0, 1, \cdots, n)$, where $A(z)$ are polynomials with degree $k(\ge 1)$, $P_j(z)$ and $Q_j(z)$ $(j = 0, 1, \cdots, n-1)$ are polynomials, $R_j(z)$ are meromorphic functions of $\sigma(R_j) < k$ and $A_j(z) - R_j(z) \not\equiv 0$. If there exists an integer $l \in \{0, 1, \cdots, n\}$ such that*

$$\deg(P_l) > \deg(P_j) \quad or \quad \deg(Q_l) > \deg(Q_j), \qquad j = 0, 1, \cdots, n, \ j \ne l,$$

*then every meromorphic solution* $f(\not\equiv 0)$ *with finite order of* (1.1) *satisfies* $\sigma(f) = \lambda(f-\varphi) \geq k+1$, *where* $\varphi(z)(\not\equiv 0)$ *is a meromorphic function with* $\sigma(\varphi) < k+1$.

**Theorem 1.2** *Let* $A_j(z)$ $(j = 0, 1, \cdots, n)$ *and* $l$ *satisfy the conditions of Theorem* 1.1. *If* $f(\not\equiv 0)$ *is a meromorphic solution with finite order of* (1.1) *that satisfies* $\max\left\{\lambda(f),\ \lambda\Big(\dfrac{1}{f}\Big)\right\} \leq k$, *then* $\sigma(f) = k+1$.

Considering the non-homogeneous linear difference equation

$$A_n(z)f(z+c_n) + \cdots + A_1(z)f(z+c_1) + A_0(z)f(z) = F(z), \tag{1.3}$$

we obtain the following result.

**Theorem 1.3** *Let* $A_j(z)$ $(j = 0, 1, \cdots, n)$ *and* $l$ *satisfy the conditions of Theorem* 1.1, *and let* $F(z)(\not\equiv 0)$ *be a meromorphic function with* $\sigma(F) < k+1$. *Then each meromorphic solution* $f(z)$ *with finite order of* (1.3) *satisfies* $\lambda(f) = \sigma(f) \geq k+1$ *with at most one possible exceptional solution* $f_0$ *satisfying* $\sigma(f_0) < k+1$.

## 2 Lemmas

We need the following lemmas for the proof of the above theorems.

**Lemma 2.1**[4] *Let* $\eta_1$, $\eta_2$ *be two arbitrary complex numbers, and* $f(z)$ *be a meromorphic function of finite order. Let* $\varepsilon > 0$ *be given. Then there exists a subset* $E \subset (1, +\infty)$ *with finite logarithmic measure such that for all* $|z| = r \notin E \cup [0, 1]$, *we have*

$$\exp\{-r^{\sigma(f)-1+\varepsilon}\} \leq \left|\frac{f(z+\eta_1)}{f(z+\eta_2)}\right| \leq \exp\{r^{\sigma(f)-1+\varepsilon}\}.$$

**Lemma 2.2**[10] *Let* $f(z)$ *be a non-constant meromorphic function. Then for all irreducible rational functions in* $f$

$$R(z,\ f) = \frac{P(z,\ f)}{Q(z,\ f)} = \frac{\sum\limits_{i=0}^{p} a_i(z)f^i}{\sum\limits_{j=0}^{q} b_j(z)f^j}$$

*with meromorphic coefficients* $a_i(z)$, $b_j(z)$ $(i = 0, \cdots, p,\ j = 0, \cdots, q)$, *we have*

$$T(r,\ R(z,\ f)) = \max\{p,\ q\}T(r,\ f) + O(\varPsi(r)) + S(r,\ f),$$

*where* $\varPsi(r) = \max\limits_{i,j}\{T(r,\ a_i),\ T(r,\ b_j)\}$.

**Lemma 2.3**[10] *Let* $g\colon [0, +\infty) \to \mathbf{R}$, $h\colon [0, +\infty) \to \mathbf{R}$ *be monotone increasing functions such that* $g(r) \leq h(r)$ *outside of an exceptional set* $E$ *of finite logarithmic measure. Then, for any* $\alpha > 1$, *there exists an* $r_0 > 0$ *such that* $g(r) \leq h(\alpha r)$ *holds for all* $r > r_0$.

**Lemma 2.4**[11] *Let $f_j(z)$ $(j=1,\cdots,n+1,\ n\geq 2)$ be meromorphic functions, $g_j(z)$ $(j=1,\cdots,n)$ be entire functions, and satisfy*

(i) $\sum\limits_{j=0}^{n} f_j(z)\mathrm{e}^{g_j(z)} \equiv f_{n+1}$;

(ii) *when* $1\leq j<k\leq n$, $g_j(z)-g_k(z)$ *is not a constant*;

(iii) *when* $1\leq j\leq n+1$, $1\leq h<k\leq n$,

$$T(r,\ f_j)=o\{T(r,\ \exp\{g_h-g_k\})\},\qquad r\to\infty,\ r\notin E,$$

*where $E\subset(1,\ \infty)$ is of finite linear measure or finite logarithmic measure.*
*Then $f_j(z)\equiv 0$ $(j=1,\cdots,n+1)$.*

**Lemma 2.5**[12] *Let $f(z)$ be a meromorphic function of order $\sigma(f)=\sigma<\infty$. Then for any given $\varepsilon>0$, there exists a set $E\subset(1,\ \infty)$ of finite linear measure such that for all $|z|=r\notin[0,1]\cup E$, and $r$ sufficiently large, we have*

$$\exp\{-r^{\sigma+\varepsilon}\}\leq|f(z)|\leq\exp\{r^{\sigma+\varepsilon}\}.$$

**Lemma 2.6**[4] *Let $f$ be a non-constant meromorphic function with finite order, and $\eta$ be a nonzero complex number. Then for each $\varepsilon>0$, we have*

$$T(r,\ f(z+\eta))=T(r,\ f)+O(r^{\sigma(f)-1+\varepsilon})+O(\log r).$$

**Lemma 2.7** *Let $A_j$ $(j=0,\cdots,n)$ be meromorphic functions, and $f(\not\equiv 0)$ be a meromorphic solution with finite order of the difference equation*

$$A_n(z)f(z+c_n)+\cdots+A_0(z)f(z+c_0)=0, \tag{2.1}$$

*where $c_0,\cdots,c_n$ are distinct complex numbers. If $\sigma(f)>\max\limits_{0\leq j\leq n}\{\sigma(A_j)\}+1$, then*

$$\max\Big\{\lambda(f),\ \lambda\Big(\frac{1}{f}\Big)\Big\}\geq\sigma(f)-1.$$

*Proof.* Suppose that $\max\Big\{\lambda(f),\ \lambda\Big(\dfrac{1}{f}\Big)\Big\}<\sigma(f)-1$. From the Hadamard factorization theorem we get

$$f(z)=\frac{d_1(z)}{d_2(z)}\mathrm{e}^{h(z)}=d(z)\mathrm{e}^{h(z)}, \tag{2.2}$$

where $d_1(z)$ and $d_2(z)$ respectively are the canonical products (or polynomials) formed by zeros or poles of $f(z)$, such that

$$\sigma(d_1)=\lambda(d_1)=\lambda(f),\qquad\sigma(d_2)=\lambda(d_2)=\lambda\Big(\frac{1}{f}\Big), \tag{2.3}$$

and $h(z)$ is a polynomial. By (2.2) and (2.3), we get

$$\sigma(f)=\deg h. \tag{2.4}$$

Let

$$h(z)=a_mz^m+a_{m-1}z^{m-1}+\cdots+a_0, \tag{2.5}$$

where $a_m,\cdots,a_0$ are constants, $a_m\neq 0$, and $m$ is a positive integer. Then, by (2.4) and the conditions of Lemma 2.7, we get $m\geq 2$.

Substituting (2.2) into (2.1), we get

$$\sum_{j=0}^{n} A_j(z)d(z+c_j)\mathrm{e}^{h(z+c_j)} = 0. \tag{2.6}$$

By Lemma 2.6 and (2.3)–(2.5), we get

$$\begin{aligned}\sigma(A_j(z)d(z+c_j)) \le\ & \max\{\sigma(A_j),\ \sigma(d_1),\ \sigma(d_2)\}\\ & < \sigma(f)-1\\ & = m-1, \qquad j=0,\cdots,n.\end{aligned} \tag{2.7}$$

On the other hand, by (2.5), we get

$$\deg\{h(z+c_j)-h(z+c_t)\} = m-1 \ge 1, \qquad c_j \ne c_t,\ 0 \le j < t \le n. \tag{2.8}$$

Thus, by (2.7) and (2.8), we get

$$T(r,\ A_j(z)d(z+c_j)) = o\{T(r,\ \mathrm{e}^{h(z+c_k)-h(z+c_t)})\}, \qquad 0 \le j \le n,\ 0 \le k < t \le n. \tag{2.9}$$

Then by Lemma 2.4, (2.6), (2.8) and (2.9), we get

$$A_j(z)d(z+c_j) \equiv 0, \qquad j=0,\cdots,n.$$

This is a contradiction. Lemma 2.7 is thus proved.

**Lemma 2.8**[8] *Let $f(z)$ be a meromorphic solution with finite order of* (1.3). *If*

$$\max\{\sigma(F),\ \sigma(A_j),\ j=0,1,\cdots,n\} < \sigma(f),$$

*then $\lambda(f)=\sigma(f)$.*

# 3 Proofs of Results

## 3.1 Proof of Theorem 1.1

Suppose that

$$\begin{aligned} A(z) &= a_k z^k + a_{k-1}z^{k-1} + \cdots + a_0,\\ P_j(z) &= a_{jp_j} z^{p_j} + a_{jp_{j-1}} z^{p_{j-1}} + \cdots + a_{j0}, \qquad j=0,1,\cdots,n,\\ Q_j(z) &= b_{jq_j} z^{q_j} + b_{jq_{j-1}} z^{q_{j-1}} + \cdots + b_{j0}, \qquad j=0,1,\cdots,n.\end{aligned}$$

Let $f(z)$ be a nontrivial meromorphic solution of (1.1) such that $\sigma(f)=\sigma<\infty$. Firstly, we prove $\sigma(f)\ge k+1$. From Lemma 2.1, for any given $\varepsilon>0$, there exists a set $E_1 \subset (1,\infty)$ of finite logarithmic measure such that for all $z$ satisfying $|z|=r\notin[0,1]\cup E_1$, we have

$$\left|\frac{f(z+c_j)}{f(z+c_l)}\right| \le \exp\{r^{\sigma-1+\varepsilon}\}, \qquad j\ne l. \tag{3.1}$$

Set $\sigma_1=\max\{\sigma(R_j)\colon j=0,\cdots,n\}$. Then by Lemma 2.5, for any given $\varepsilon$ $\left(0<\varepsilon<\dfrac{k-\sigma_1}{2}\right)$, there exists a set $E_2\subset(1,\ \infty)$ of finite linear measure such that for all $|z|=r\notin[0,1]\cup E_2$ and $r$ sufficiently large, we have

$$|R_j(z)| \le \exp\{r^{\sigma_1+\varepsilon}\}, \qquad j=0,1,\cdots,n. \tag{3.2}$$

Now we discuss the following two cases.

Case 1. If $\deg(P_l) > \deg(P_j)$ $(j = 0, 1, \cdots, n, j \neq l)$, then we take a suitable $z$ such that $|z| = r$ and $a_k z^k = |a_k| r^k$, so for $r$ sufficiently large and $r \notin [0, 1] \cup E_1 \cup E_2$, we have

$$\begin{aligned}|A_j(z)| &\leq |P_j(\mathrm{e}^{A(z)})| + |Q_j(\mathrm{e}^{-A(z)})| + |R_j(z)| \\ &\leq |a_{jp_j}| \exp\{p_j r^k |a_k|(1 + o(1))\}(1 + o(1)) + \exp\{r^{\sigma_1+\varepsilon}\} \\ &= |a_{jp_j}| \exp\{p_j r^k |a_k|(1 + o(1))\}(1 + o(1)), \qquad j \neq l \end{aligned} \tag{3.3}$$

and

$$\begin{aligned}|A_l(z)| &\geq |P_l(\mathrm{e}^{A(z)})| - |Q_l(\mathrm{e}^{-A(z)})| - |R_l(z)| \\ &\geq |a_{lp_l}| \exp\{p_l r^k |a_k|(1 + o(1))\}(1 - o(1)) - \exp\{r^{\sigma_1+\varepsilon}\} \\ &= |a_{lp_l}| \exp\{p_l r^k |a_k|(1 + o(1))\}(1 - o(1)). \end{aligned} \tag{3.4}$$

By (1.1), (3.1), (3.3) and (3.4), we get for all $z$ satisfying $a_k z^k = |a_k| r^k$ and $|z| = r \notin [0, 1] \cup E_1 \cup E_2$, when $r$ sufficiently large,

$$\begin{aligned}&|a_{lp_l}| \exp\{p_l r^k |a_k|(1 + o(1))\}(1 - o(1)) \\ \leq\ & |A_l(z)| \\ \leq\ & \sum_{\substack{j=0 \\ j\neq l}}^{n} |A_j(z)| \left|\frac{f(z + j)}{f(z + l)}\right| \\ \leq\ & M \exp\{r^{\sigma-1+\varepsilon}\} \exp\{N r^k |a_k|(1 + o(1))\}, \end{aligned} \tag{3.5}$$

where $M > 0$ is a constant, $N = \max\{p_j : j = 0, \cdots, n,\ j \neq l\}$. By $p_l > N$ and (3.5), we get, for sufficiently large $r \notin [0, 1] \cup E_1 \cup E_2$,

$$\frac{|a_{lp_l}|}{2M} \exp\{(p_l - N) r^k |a_k|(1 + o(1))\} \leq \exp\{r^{\sigma-1+\varepsilon}\}. \tag{3.6}$$

By (3.6) and Lemma 2.3, we get $k \leq \sigma - 1 + \varepsilon$, which implies $\sigma(f) = \sigma \geq k + 1$.

Case 2. If $\deg(Q_l) > \deg(Q_j)$ $(j = 0, 1, \cdots, n,\ j \neq l)$, then we take a suitable $z$ such that $|z| = r$ and $a_k z^k = -|a_k| r^k$. By the similar method as in the proof of Case 1, we can obtain $\sigma(f) \geq k + 1$.

In the following, we prove that $\lambda(f - \varphi) = \sigma(f)$.

Set $g(z) = f(z) - \varphi(z)$. Substituting $f(z) = g(z) + \varphi(z)$ into (1.1), we obtain

$$A_n(z) g(z + c_n) + \cdots + A_1(z) g(z + c_1) + A_0(z) g(z) = -H(z), \tag{3.7}$$

where

$$H(z) = A_n(z)\varphi(z + c_n) + \cdots + A_1(z)\varphi(z + c_1) + A_0(z)\varphi(z).$$

If $H(z) \equiv 0$, then $\varphi(z)$ is a nonzero meromorphic solution of (1.1). Thus, by the above proof, we get

$$\sigma(\varphi) \geq k + 1.$$

This is absurd. Hence,

$$H(z) \not\equiv 0. \tag{3.8}$$

On the other hand, by Lemma 2.2, we get

$$T(r,\ P_j(\mathrm{e}^A) + Q_j(\mathrm{e}^{-A}) = (p_j + q_j) T(r,\ \mathrm{e}^A) + S(r,\ \mathrm{e}^A), \qquad j = 0, \cdots, n. \tag{3.9}$$

Since $\mathrm{e}^A$ is of the regular growth, by (3.9), we get $T(r, A_j) = (p_j + q_j)T(r, \mathrm{e}^A) + S(r, \mathrm{e}^A)$ $(j = 0, \cdots, n)$. Hence we get

$$\sigma(A_j) = \sigma(\mathrm{e}^A) = k, \qquad j = 0, \cdots, n. \tag{3.10}$$

Then by (3.10) and Lemma 2.6, we get

$$\sigma(H) \leq \max\{k,\ \sigma(\varphi)\} < k + 1 \leq \sigma(f) = \sigma(g). \tag{3.11}$$

So by Lemma 2.8, (3.7), (3.8), (3.10) and (3.11), we obtain

$$\lambda(f - \varphi) = \lambda(g) = \sigma(g) = \sigma(f) \geq k + 1.$$

Theorem 1.1 is thus proved.

## 3.2 Proof of Theorem 1.2

Let $f(\not\equiv 0)$ be a meromorphic solution with finite order of (1.1). Then by Theorem 1.1 we get

$$\sigma(f) \geq k + 1. \tag{3.12}$$

Suppose that $\sigma(f) > k + 1$. By (3.10) we get

$$\sigma(f) > \max_{0 \leq j \leq n} \{\sigma(A_j)\} + 1. \tag{3.13}$$

Then combining with Lemma 2.7 and (3.13), we get

$$\max\left\{\lambda(f), \lambda\left(\frac{1}{f}\right)\right\} \geq \sigma(f) - 1 > k.$$

This contradicts the hypothesis of Theorem 1.2. Hence we get $\sigma(f) = k + 1$. Theorem 1.2 is thus proved.

## 3.3 Proof of Theorem 1.3

Suppose that $f_0$ is a meromorphic solution of (1.3) with $\sigma(f_0) < k + 1$. If $f^*(z)(\not\equiv f_0(z))$ is another meromorphic solution of (1.3) satisfying $\sigma(f^*) < k + 1$, then

$$\sigma(f^* - f_0) \leq \max\{\sigma(f^*,\ \sigma(f_0)\} < k + 1,$$

and $f^* - f_0$ is a solution of the corresponding homogeneous equation (1.1) to (1.3). By Theorem 1.1, we have

$$\sigma(f^* - f_0) \geq k + 1,$$

a contradiction. Hence (1.3) possesses at most one exceptional solution $f_0$ with $\sigma(f_0) < k+1$.

Now Suppose that $f$ is a meromorphic solution of (1.3) with $k + 1 \leq \sigma(f) < \infty$. Combining (3.10), we have

$$\sigma(f) > \max\{\sigma(A_j),\ \sigma(F)\}.$$

Hence, by Lemma 2.8, we get

$$\lambda(f) = \sigma(f).$$

Theorem 1.3 is thus proved.

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