Communications in Mathematical Research
**34**(1)(2018), 23–35

# Extensions of Modules with ACC on $d$-annihilators

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Communicated by Du Xian-kun

**Abstract:** A unitary right $R$-module $M_R$ satisfies acc on $d$-annihilators if for every sequence $(a_n)_n$ of elements of $R$ the ascending chain $\mathrm{Ann}_M(a_1) \subseteq \mathrm{Ann}_M(a_1a_2) \subseteq \mathrm{Ann}_M(a_1a_2a_3) \subseteq \cdots$ of submodules of $M_R$ stabilizes. In this paper we first investigate some triangular matrix extensions of modules with acc on $d$-annihilators. Then we show that under some additional conditions, the Ore extension module $M[x]_{R[x;\alpha,\delta]}$ over the Ore extension ring $R[x;\,\alpha,\delta]$ satisfies acc on $d$-annihilators if and only if the module $M_R$ satisfies acc on $d$-annihilators. Consequently, several known results regarding modules with acc on $d$-annihilators are extended to a more general setting.

**Key words:** triangular matrix extension, Ore extension, acc on $d$-annihilator

**2010 MR subject classification:** 16S99

**Document code:** A

**Article ID:** 1674-5647(2018)01-0023-13

**DOI:** 10.13447/j.1674-5647.2018.01.03

## 1 Introduction

Throughout this paper all rings $R$ are associative with identity and all modules $M_R$ are unitary right $R$-modules. The set of all positive integers is denoted by $\mathbf{N}_+$. Let $\alpha$ be an endomorphism and $\delta$ an $\alpha$-derivation of a ring $R$. We denote by $R[x;\,\alpha,\delta]$ the Ore extension whose elements are the polynomials over $R$, the addition is defined as usual and the multiplication is subject to the relation $xa = \alpha(a)x + \delta(a)$ for any $a \in R$. Clearly, polynomial rings $R[x]$, skew polynomial rings $R[x;\,\alpha]$ and differential polynomial rings $R[x;\,\delta]$ are special

**Received date:** June 23, 2016.

**Foundation item:** The NSF (11471108) of China, the NSF (2015JJ2051, 2016JJ2050) of Hunan Province, and the Teaching Reform Foundation (G21316) of Hunan Province.

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Ore extension rings. Given a right $R$-module $M_R$, we can make $M[x]$ into a right $R[x;\,\alpha,\delta]$-module by allowing polynomials from $R[x;\,\alpha,\delta]$ to act on polynomials in $M[x]$ in the obvious way, and apply the above twist whenever necessary. The verification that this defines a valid $R[x;\,\alpha,\delta]$-module structure on $M[x]$ is almost identical to the verification that $R[x;\,\alpha,\delta]$ is a ring and it is straightforward (see [1]).

For an element $a \in R$, $\mathrm{Ann}_M(a) = \{m \in M_R \mid ma = 0\}$ denotes the annihilator of $a$ in $M_R$. Following Frohn[2], a module $M_R$ is said to satisfy acc on $d$-annihilators if for every sequence $(a_n)_n$ of elements of $R$, the ascending chain $\mathrm{Ann}_M(a_1) \subseteq \mathrm{Ann}_M(a_1a_2) \subseteq \cdots$ of submodules of $M_R$ stabilizes. If $R_R$ satisfies acc on $d$-annihilators, then we say that the ring $R$ is a ring satisfying acc on $d$-annihilators. Clearly, strongly Laskerian modules satisfy acc on $d$-annihilators, and if $M_R$ satisfies acc on $d$-annihilators, so is every submodule of $M_R$ (see [2]). Visweswaran[3] showed that the zero-dimension rings with acc on $d$-annihilators are exactly the perfect rings. So in order to characterize the perfect rings $R$, it is important to consider the modules $R_R$ with acc on $d$-annihilators. Hence find more examples of modules with acc on $d$-annihilators is meaningful in module theory. It is well known that, in the module theory literature, many surprising examples and counterexamples have been produced via the triangular matrix extensions. So in this paper we first investigate the relationship between the acc on $d$-annihilators property of $M_R$ and that of the various triangular matrix extension modules over $M_R$, and then obtain more examples of modules with acc on $d$-annihilators.

Polynomial extension of modules with acc on $d$-annihilators was studied by Frohn. He proved in [2] that if $R$ is reduced and satisfies acc on $d$-annihilators, then the polynomial ring $R[X]$ for any set $X$ of indeterminates also has acc on $d$-annihilators. We generalize this result. In Section 3, we consider the acc on $d$-annihilators property of the Ore extension modules $M[x]_{R[x;\,\alpha,\delta]}$ over the Ore extension rings $R[x;\,\alpha,\delta]$. We show that if $M_R$ is an $(\alpha,\delta)$-compatible reduced module, then the Ore extension module $M[x]_{R[x;\,\alpha,\delta]}$ satisfies acc on $d$-annihilators if and only if $M_R$ satisfies acc on $d$-annihilators. So the Frohn's recent work (see [2], Corollary 2.4]) is extended to a more generally setting.

# 2 Triangular Matrix Extension Modules

Let $R$ be a ring and $M_R$ a right $R$-module. Let

$$U_n(R) = \left\{ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \mid a_{ij} \in R \right\}$$

and

$$U_n(M) = \left\{ \begin{pmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ 0 & m_{22} & \cdots & m_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & m_{nn} \end{pmatrix} \mid m_{ij} \in M_R \right\}.$$

Then $U_n(M)$ is a right $U_n(R)$-module under usual matrix operations.

**Proposition 2.1** *Let $R$ be a ring and $M_R$ a right $R$-module. Then the following statements are equivalent*:

(1) *$M_R$ satisfies acc on $d$-annihilators*;

(2) *$U_n(M)_{U_n(R)}$ satisfies acc on $d$-annihilators.*

*Proof.* (1) $\Rightarrow$ (2). Suppose that $M_R$ satisfies acc on $d$-annihilators. We proceed by induction on $n$ to show that the right $U_n(R)$-module $U_n(M)$ also satisfies acc on $d$-annihilators. Let $n = 2$. Put

$$\boldsymbol{A}_i = \begin{pmatrix} a_i & b_i \\ 0 & c_i \end{pmatrix}, \qquad i = 1, 2, \cdots$$

be a sequence of elements of $U_2(R)$. Since $M_R$ satisfies acc on $d$-annihilators, there exists a $k \in \mathbf{N}_+$ such that for any positive integer $l > k$,

$$\mathrm{Ann}_M(a_1a_2\cdots a_k) = \mathrm{Ann}_M(a_1a_2\cdots a_k\cdots a_l)$$

and

$$\mathrm{Ann}_M(c_1c_2\cdots c_k) = \mathrm{Ann}_M(c_1c_2\cdots c_k\cdots c_l).$$

Consider the sequence $(c_{k+m})_m$ of elements of $R$. By the condition that $M_R$ satisfies acc on $d$-annihilators, we can find a positive integer $p \in \mathbf{N}_+$ such that for any $q > p$,

$$\mathrm{Ann}_M(c_{k+1}c_{k+2}\cdots c_{k+p}) = \mathrm{Ann}_M(c_{k+1}c_{k+2}\cdots c_{k+p}\cdots c_{k+q}).$$

Now we show that for any positive integer $v \in \mathbf{N}_+$,

$$\mathrm{Ann}_{U_2(M)}(\boldsymbol{A}_1\boldsymbol{A}_2\cdots \boldsymbol{A}_{k+p}) = \mathrm{Ann}_{U_2(M)}(\boldsymbol{A}_1\boldsymbol{A}_2\cdots \boldsymbol{A}_{k+p}\cdots \boldsymbol{A}_{k+p+v}).$$

For $v = 1$, if $\boldsymbol{w} = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in \mathrm{Ann}_{U_2(M)}(\boldsymbol{A}_1\boldsymbol{A}_2\cdots \boldsymbol{A}_{k+p+1})$, then

$$\begin{aligned} 0 &= \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}\begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix}\cdots\begin{pmatrix} a_{k+p+1} & b_{k+p+1} \\ 0 & c_{k+p+1} \end{pmatrix} \\ &= \begin{pmatrix} xa_1a_2\cdots a_{k+p+1} & xu + yc_1c_2\cdots c_{k+p+1} \\ 0 & zc_1c_2\cdots c_{k+p+1} \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} u = {} & a_1a_2\cdots a_{k+p}b_{k+p+1} + a_1a_2\cdots a_{k+p-1}b_{k+p}c_{k+p+1} + \cdots \\ & + a_1a_2\cdots a_kb_{k+1}c_{k+2}c_{k+3}\cdots c_{k+p+1} + \cdots + a_1b_2c_3c_4\cdots c_{k+p+1} + b_1c_2c_3\cdots c_{k+p+1}. \end{aligned}$$

Hence

$$x \in \mathrm{Ann}_M(a_1a_2\cdots a_{k+p+1}) = \mathrm{Ann}_M(a_1a_2\cdots a_{k+p}) = \cdots = \mathrm{Ann}_M(a_1a_2\cdots a_k),$$

and

$$z \in \mathrm{Ann}_M(c_1c_2\cdots c_{k+p+1}) = \mathrm{Ann}_M(c_1c_2\cdots c_{k+p}) = \cdots = \mathrm{Ann}_M(c_1c_2\cdots c_k).$$

Then

$$\begin{aligned} 0 &= xu + yc_1c_2\cdots c_{k+p+1} \\ &= x(a_1a_2\cdots a_{k+p}b_{k+p+1} + \cdots + a_1a_2\cdots a_kb_{k+1}c_{k+2}\cdots c_{k+p+1} + \cdots \end{aligned}$$

$$
\begin{aligned}
&\quad + b_1c_2c_3\cdots c_{k+p+1}) + yc_1c_2\cdots c_{k+p+1} \\
&= x(a_1a_2\cdots a_{k-1}b_kc_{k+1}\cdots c_{k+p+1} + a_1a_2\cdots a_{k-2}b_{k-1}c_kc_{k+1}\cdots c_{k+p+1} + \cdots \\
&\quad + b_1c_2c_3\cdots c_{k+p+1}) + yc_1c_2\cdots c_{k+p+1} \\
&= [x(a_1a_2\cdots a_{k-1}b_k + \cdots + b_1c_2c_3\cdots c_k) + yc_1c_2\cdots c_k]c_{k+1}\cdots c_{k+p+1}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&[x(a_1a_2\cdots a_{k-1}b_k + \cdots + b_1c_2c_3\cdots c_k) + yc_1c_2\cdots c_k] \\
&\in \mathrm{Ann}_M(c_{k+1}c_{k+2}\cdots c_{k+p+1}) = \mathrm{Ann}_M(c_{k+1}c_{k+2}\cdots c_{k+p}).
\end{aligned}
$$

Hence

$$[x(a_1a_2\cdots a_{k-1}b_k + \cdots + b_1c_2c_3\cdots c_k) + yc_1c_2\cdots c_k]c_{k+1}c_{k+2}\cdots c_{k+p} = 0.$$

By a routine computations, we obtain

$$\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}\begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix}\cdots\begin{pmatrix} a_{k+p} & b_{k+p} \\ 0 & c_{k+p} \end{pmatrix} = 0.$$

Hence

$$\mathrm{Ann}_{U_2(M)}(\boldsymbol{A}_1\boldsymbol{A}_2\cdots\boldsymbol{A}_{k+p+1}) = \mathrm{Ann}_{U_2(M)}(\boldsymbol{A}_1\boldsymbol{A}_2\cdots\boldsymbol{A}_{k+p}).$$

Similarly, we can show that for any positive integer $v \in \mathbf{N}_+$,

$$\mathrm{Ann}_{U_2(M)}(\boldsymbol{A}_1\boldsymbol{A}_2\cdots\boldsymbol{A}_{k+p}) = \mathrm{Ann}_{U_2(M)}(\boldsymbol{A}_1\boldsymbol{A}_2\cdots\boldsymbol{A}_{k+p+v}).$$

Therefore $U_2(M)$ satisfies acc on $d$-annihilators.

Next we assume that the result is true for $n-1$, and let

$$\boldsymbol{B}_n^i = \begin{pmatrix} a_{11}^i & a_{12}^i & \cdots & a_{1n}^i \\ 0 & a_{22}^i & \cdots & a_{2n}^i \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn}^i \end{pmatrix}, \qquad i = 1, 2, \cdots$$

be a sequence of elements of $U_n(R)$. In the following we show that

$$\mathrm{Ann}_{U_n(M)}(\boldsymbol{B}_n^1) \subseteq \mathrm{Ann}_{U_n(M)}(\boldsymbol{B}_n^1\boldsymbol{B}_n^2) \subseteq \cdots$$

stabilizes. Put

$$\boldsymbol{B}_n^i = \begin{pmatrix} a_{11}^i & a_{12}^i & \cdots & a_{1n}^i \\ 0 & a_{22}^i & \cdots & a_{2n}^i \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn}^i \end{pmatrix} = \begin{pmatrix} \boldsymbol{B}_{n-1}^i & \boldsymbol{C}^i \\ 0 & a_{nn}^i \end{pmatrix},$$

where $\boldsymbol{B}_{n-1}^i$ is a $(n-1)\times(n-1)$ upper triangular matrix and $\boldsymbol{C}^i = (a_{1n}^i, a_{2n}^i, \cdots, a_{(n-1)n}^i)^{\mathrm{T}}$. By the induction hypothesis, we can find a positive integer $m \in \mathbf{N}_+$ such that for any $s > m$,

$$
\begin{aligned}
\mathrm{Ann}_{U_{n-1}(M)}(\boldsymbol{B}_{n-1}^1\boldsymbol{B}_{n-1}^2\cdots\boldsymbol{B}_{n-1}^s) &= \mathrm{Ann}_{U_{n-1}(M)}(\boldsymbol{B}_{n-1}^1\boldsymbol{B}_{n-1}^1\cdots\boldsymbol{B}_{n-1}^m), \\
\mathrm{Ann}_M(a_{nn}^1a_{nn}^2\cdots a_{nn}^s) &= \mathrm{Ann}_M(a_{nn}^1a_{nn}^2\cdots a_{nn}^m),
\end{aligned}
$$

and a positive integer $u \in \mathbf{N}_+$ such that for any $v > u$,

$$\mathrm{Ann}_M(a_{nn}^{m+1}a_{nn}^{m+2}\cdots a_{nn}^{m+v}) = \mathrm{Ann}_M(a_{nn}^{m+1}a_{nn}^{m+2}\cdots a_{nn}^{m+u}).$$

Then by using the same way as above, we can show that for any positive integer $w \in \mathbf{N}_+$,

$$\mathrm{Ann}_{U_n(M)}(\boldsymbol{B}_n^1\boldsymbol{B}_n^2\cdots\boldsymbol{B}_n^{m+u+w}) = \mathrm{Ann}_{U_n(M)}(\boldsymbol{B}_n^1\boldsymbol{B}_n^2\cdots\boldsymbol{B}_n^{m+u}).$$

Hence

$$\mathrm{Ann}_{U_n(M)}(\boldsymbol{B}_n^1 \subseteq \mathrm{Ann}_{U_n(M)}(\boldsymbol{B}_n^1\boldsymbol{B}_n^2) \subseteq \cdots$$

stabilizes. Therefore $U_n(M)_{U_n(R)}$ satisfies acc on $d$-annihilators by induction.

(2) $\Rightarrow$ (1). It is trivial.

The proof is completed.

Let $L_n(R)$ denote the lower triangular matrix ring over $R$, and let

$$L_n(M) = \left\{ \begin{pmatrix} m_{11} & 0 & \cdots & 0 \\ m_{21} & m_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} \end{pmatrix} \mid m_{ij} \in M_R \right\}.$$

Then $L_n(M)$ is a right $L_n(R)$-module under usual matrix operations.

**Corollary 2.1** *The following statements are equivalent*:

(1) *$M_R$ satisfies acc on $d$-annihilators*;

(2) *$L_n(M)_{L_n(R)}$ satisfies acc on $d$-annihilators.*

*Proof.* It is similar to the proof as given in the Proposition 2.1.

Let $R$ be a ring and $M_R$ a right $R$-module. Let

$$S_n(R) = \left\{ \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \mid a,\ a_{ij} \in R \right\},$$

$$S_n(M) = \left\{ \begin{pmatrix} m & m_{12} & \cdots & m_{1n} \\ 0 & m & \cdots & m_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & m \end{pmatrix} \mid m,\ m_{ij} \in M_R \right\},$$

$$G_n(R) = \left\{ \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ 0 & a_1 & \cdots & a_{n-1} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_1 \end{pmatrix} \mid a_i \in R \right\},$$

$$G_n(M) = \left\{ \begin{pmatrix} m_1 & m_2 & \cdots & m_n \\ 0 & m_1 & \cdots & m_{n-1} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & m_1 \end{pmatrix} \mid m_i \in M_R \right\}.$$

The following two corollaries give more examples of modules satisfying acc on $d$-annihilators.

**Corollary 2.2** *The following statements are equivalent*:

(1) *The right $R$-module $M_R$ satisfies acc on $d$-annihilators*;

(2) *The right $S_n(R)$-module $S_n(M)$ satisfies acc on d-annihilators*;

(3) *The right $G_n(R)$-module $G_n(M)$ satisfies acc on d-annihilators.*

*Proof.* Employing the same method in the proof of Proposition 2.1, we complete the proof.

**Corollary 2.3** *The following statements are equivalent*:

(1) *$R$ satisfies acc on d-annihilators*;

(2) *$U_n(R)$ satisfies acc on d-annihilators*;

(3) *$L_n(R)$ satisfies acc on d-annihilators*;

(4) *$S_n(R)$ satisfies acc on d-annihilators*;

(5) *$G_n(R)$ satisfies acc on d-annihilators*;

(6) *The trivial extension $R \ltimes R$ of $R$ by $R$ satisfies acc on d-annihilators*;

(7) *$R[x]/(x^n)$ satisfies acc on d-annihilators.*

*Proof.* The equivalence (1) $\Leftrightarrow$ (2) follows by Proposition 2.1. The equivalence (1) $\Leftrightarrow$ (3) follows by Corollary 2.1. The equivalence (1) $\Leftrightarrow$ (4), (1) $\Leftrightarrow$ (5) and (1) $\Leftrightarrow$ (6) follow by Corollary 2.2. The equivalence (1) $\Leftrightarrow$ (7) follows by Corollary 2.2 and the fact that $R[x]/(x^n) \cong G_n(R)$.

Let $R$ be a ring and $M_R$ a right $R$-module. Let

$$W(R) = \left\{ \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \mid a_{ij} \in R \right\},$$

$$W(M) = \left\{ \begin{pmatrix} m_{11} & 0 & 0 \\ m_{21} & m_{22} & m_{23} \\ 0 & 0 & m_{33} \end{pmatrix} \mid m_{ij} \in M_R \right\}.$$

Then $W(M)$ is a right $W(R)$-module under usual matrix operations. In fact, $W(M)$ possesses the similar form of both the lower triangular matrix module and the upper triangular matrix module. A natural problem asks if the acc on $d$-annihilators property of such a module coincides with that of $M_R$. This inspire us to consider the acc on $d$-annihilators property of $W(M)_{W(R)}$.

**Proposition 2.2** *Let $R$ be a ring and $M_R$ a right $R$-module. Then the following statements are equivalent*:

(1) *$M_R$ satisfies acc on d-annihilators*;

(2) *$W(M)_{W(R)}$ satisfies acc on d-annihilators.*

*Proof.* It suffices to show that (1) $\Rightarrow$ (2). Let

$$\boldsymbol{A}_i = \begin{pmatrix} a_i & 0 & 0 \\ x_i & b_i & y_i \\ 0 & 0 & c_i \end{pmatrix}, \qquad i = 1, 2, \cdots$$

be a sequence of elements of $W(R)$. Since $M_R$ satisfies acc on $d$-annihilators, there exists some $k \in \mathbf{N}_+$ such that for all positive integer $l > k$,

$$\mathrm{Ann}_M(a_1a_2\cdots a_k) = \mathrm{Ann}_M(a_1a_2\cdots a_k\cdots a_l),$$
$$\mathrm{Ann}_M(b_1b_2\cdots b_k) = \mathrm{Ann}_M(b_1b_2\cdots b_k\cdots b_l),$$
$$\mathrm{Ann}_M(c_1c_2\cdots c_k) = \mathrm{Ann}_M(c_1c_2\cdots c_k\cdots c_l).$$

Consider the sequences $(a_{k+n})_n$, $(b_{k+n})_n$ and $(c_{k+n})_n$ of elements of $R$, there exists a $p \in \mathbf{N}_+$ such that for all $q > p$,

$$\mathrm{Ann}_M(a_{k+1}a_{k+2}\cdots a_{k+p}) = \mathrm{Ann}_M(a_{k+1}a_{k+2}\cdots a_{k+p}\cdots a_{k+q}),$$
$$\mathrm{Ann}_M(b_{k+1}b_{k+2}\cdots b_{k+p}) = \mathrm{Ann}_M(b_{k+1}b_{k+2}\cdots b_{k+p}\cdots b_{k+q}),$$
$$\mathrm{Ann}_M(c_{k+1}c_{k+2}\cdots c_{k+p}) = \mathrm{Ann}_M(c_{k+1}c_{k+2}\cdots c_{k+p}\cdots c_{k+q}).$$

Now we show that for any positive integer $v \in \mathbf{N}_+$,

$$\mathrm{Ann}_{W(M)}(\boldsymbol{A}_1\boldsymbol{A}_2\cdots\boldsymbol{A}_{k+p}) = \mathrm{Ann}_{W(M)}(\boldsymbol{A}_1\boldsymbol{A}_2\cdots\boldsymbol{A}_{k+p}\cdots\boldsymbol{A}_{k+p+v}),$$

which implies that $\mathrm{Ann}_{W(M)}(\boldsymbol{A}_1) \subseteq \mathrm{Ann}_{W(M)}(\boldsymbol{A}_1\boldsymbol{A}_2) \subseteq \cdots$ stabilizes. First, we show that

$$\mathrm{Ann}_{W(M)}(\boldsymbol{A}_1\boldsymbol{A}_2\cdots A_{k+p}) = \mathrm{Ann}_{W(M)}(\boldsymbol{A}_1\boldsymbol{A}_2\cdots A_{k+p}\boldsymbol{A}_{k+p+1}).$$

Suppose that $\boldsymbol{X} = \begin{pmatrix} d & 0 & 0 \\ s & e & t \\ 0 & 0 & f \end{pmatrix} \in \mathrm{Ann}_{W(M)}(\boldsymbol{A}_1\boldsymbol{A}_2\cdots\boldsymbol{A}_{k+p}\boldsymbol{A}_{k+p+1})$. Then

$$\begin{aligned}
0 &= \boldsymbol{X}\boldsymbol{A}_1\boldsymbol{A}_2\cdots\boldsymbol{A}_{k+p}\boldsymbol{A}_{k+p+1} \\
&= \begin{pmatrix} d & 0 & 0 \\ s & e & t \\ 0 & 0 & f \end{pmatrix}\begin{pmatrix} a_1 & 0 & 0 \\ x_1 & b_1 & y_1 \\ 0 & 0 & c_1 \end{pmatrix}\cdots\begin{pmatrix} a_{k+p+1} & 0 & 0 \\ x_{k+p+1} & b_{k+p+1} & y_{k+p+1} \\ 0 & 0 & c_{k+p+1} \end{pmatrix} \\
&= \begin{pmatrix} da_1a_2\cdots a_{k+p+1} & 0 & 0 \\ sa_1a_2\cdots a_{k+p+1}+eu & eb_1b_2\cdots b_{k+p+1} & er+tc_1c_2\cdots c_{k+p+1} \\ 0 & 0 & fc_1c_2\cdots c_{k+p+1} \end{pmatrix},
\end{aligned}$$

where

$$\begin{aligned}
u = {} & x_1a_2a_3\cdots a_{k+p+1} + b_1x_2a_3\cdots a_{k+p+1} + \cdots + b_1b_2\cdots b_{k-1}x_ka_{k+1}\cdots a_{k+p+1} + \cdots \\
& + b_1b_2\cdots b_{k+p-1}x_{k+p}a_{k+p+1} + b_1b_2\cdots b_{k+p}x_{k+p+1}, \\
r = {} & b_1\cdots b_{k+p}y_{k+p+1} + b_1\cdots b_{k+p-1}y_{k+p}c_{k+p+1} + \cdots + b_1\cdots b_{k-1}y_kc_{k+1}\cdots c_{k+p+1} + \cdots \\
& + b_1y_2c_3c_4\cdots c_{k+p+1} + y_1c_2c_3\cdots c_{k+p+1}.
\end{aligned}$$

Hence

$$d \in \mathrm{Ann}_M(a_1a_2\cdots a_{k+p+1}) = \mathrm{Ann}_M(a_1a_2\cdots a_k),$$
$$e \in \mathrm{Ann}_M(b_1b_2\cdots b_{k+p+1}) = \mathrm{Ann}_M(b_1b_2\cdots b_k),$$
$$f \in \mathrm{Ann}_M(c_1c_2\cdots c_{k+p+1}) = \mathrm{Ann}_M(c_1c_2\cdots c_k).$$

By using the same way as the proof of Proposition 2.1, we also have

$$\begin{aligned}
& sa_1a_2\cdots a_k + e(x_1a_2a_3\cdots a_k + b_1x_2a_3a_4\cdots a_k + \cdots + b_1b_2\cdots b_{k-1}x_k) \\
\in {} & \mathrm{Ann}_M(a_{k+1}a_{k+2}\cdots a_{k+p+1}) = \mathrm{Ann}_M(a_{k+1}a_{k+2}\cdots a_{k+p})
\end{aligned}$$

and

$$e(b_1b_2\cdots b_{k-1}y_k + b_1b_2\cdots b_{k-2}y_{k-1}c_k + \cdots + b_1y_2c_3\cdots c_k + y_1c_2c_3\cdots c_k) + tc_1c_2\cdots c_k$$
$$\in \mathrm{Ann}_M(c_{k+1}c_{k+2}\cdots c_{k+p+1}) = \mathrm{Ann}_M(c_{k+1}c_{k+2}\cdots c_{k+p}).$$

Then by a routine computations, we can show that

$$\boldsymbol{X}\boldsymbol{A}_1\boldsymbol{A}_2\cdots\boldsymbol{A}_{k+p} = \boldsymbol{0},$$

and so

$$\mathrm{Ann}_{W(M)}(\boldsymbol{A}_1\boldsymbol{A}_2\cdots\boldsymbol{A}_{k+p+1}) = \mathrm{Ann}_{W(M)}(\boldsymbol{A}_1\boldsymbol{A}_2\cdots\boldsymbol{A}_{k+p}).$$

Similarly, we can show that

$$\begin{aligned}\mathrm{Ann}_{W(M)}(\boldsymbol{A}_1\boldsymbol{A}_2\cdots\boldsymbol{A}_{k+p}) &= \mathrm{Ann}_{W(M)}(\boldsymbol{A}_1\boldsymbol{A}_2\cdots\boldsymbol{A}_{k+p}\boldsymbol{A}_{k+p+1})\\ &= \mathrm{Ann}_{W(M)}(\boldsymbol{A}_1\boldsymbol{A}_2\cdots\boldsymbol{A}_{k+p}\boldsymbol{A}_{k+p+1}\boldsymbol{A}_{k+p+2})\\ &= \cdots\end{aligned}$$

Therefore $W(M)$ satisfies acc on $d$-annihilators.

Let $R$ be a ring and $M_R$ a right $R$-module. Then under usual matrix operations, we obtain that $W^1(M) = \begin{pmatrix} M & 0 & 0 \\ 0 & M & 0 \\ M & M & M \end{pmatrix}$ is a right $W^1(R) = \begin{pmatrix} R & 0 & 0 \\ 0 & R & 0 \\ R & R & R \end{pmatrix}$ module, $W^2(M) = \begin{pmatrix} M & 0 & M \\ 0 & M & M \\ 0 & 0 & M \end{pmatrix}$ is a right $W^2(R) = \begin{pmatrix} R & 0 & R \\ 0 & R & R \\ 0 & 0 & R \end{pmatrix}$ module, $W^3(M) = \begin{pmatrix} M & 0 & 0 \\ 0 & M & 0 \\ M & 0 & M \end{pmatrix}$ is a right $W^3(R) = \begin{pmatrix} R & 0 & 0 \\ 0 & R & 0 \\ R & 0 & R \end{pmatrix}$ module, and $W^4(M) = \begin{pmatrix} M & 0 & M \\ 0 & M & 0 \\ 0 & 0 & M \end{pmatrix}$ is a right $W^4(R) = \begin{pmatrix} R & 0 & R \\ 0 & R & 0 \\ 0 & 0 & R \end{pmatrix}$ module.

**Proposition 2.3** *Let $R$ be a ring and $M_R$ a right $R$-module. Then the following statements are equivalent*:

(1) *The right $R$-module $M_R$ satisfies acc on $d$-annihilators*;
(2) *The right $W^1(R)$-module $W^1(M)$ satisfies acc on $d$-annihilators*;
(3) *The right $W^2(R)$-module $W^2(M)$ satisfies acc on $d$-annihilators*;
(4) *The right $W^3(R)$-module $W^3(M)$ satisfies acc on $d$-annihilators*;
(5) *The right $W^4(R)$-module $W^4(M)$ satisfies acc on $d$-annihilators.*

*Proof.* By analogy with the proof of Proposition 2.2, we complete the proof.

**Corollary 2.4** *Let $R$ be a ring. Then the following statements are equivalent*:

(1) *$R$ satisfies acc on $d$-annihilators*;
(2) *$W(R)$ satisfies acc on $d$-annihilators*;

(3) $W^1(R)$ *satisfies acc on d-annihilators*;
(4) $W^2(R)$ *satisfies acc on d-annihilators*;
(5) $W^3(R)$ *satisfies acc on d-annihilators*;
(6) $W^4(R)$ *satisfies acc on d-annihilators.*

**Example 2.1** Let $\mathbb{Z}_4 = \{\overline{0},\ \overline{1},\ \overline{2},\ \overline{3}\}$ denote the ring of integers modulo 4. One directly verifies that $\mathbb{Z}_4$ is a commutative ring with acc on $d$-annihilators. According to Corollaries 2.3 and 2.4, the rings

$$\begin{pmatrix} \mathbb{Z}_4 & 0 & 0 \\ \mathbb{Z}_4 & \mathbb{Z}_4 & 0 \\ \mathbb{Z}_4 & \mathbb{Z}_4 & \mathbb{Z}_4 \end{pmatrix}, \quad \begin{pmatrix} \mathbb{Z}_4 & 0 & 0 \\ 0 & \mathbb{Z}_4 & 0 \\ \mathbb{Z}_4 & 0 & \mathbb{Z}_4 \end{pmatrix}, \quad \begin{pmatrix} \mathbb{Z}_4 & 0 & 0 \\ \mathbb{Z}_4 & \mathbb{Z}_4 & \mathbb{Z}_4 \\ 0 & 0 & \mathbb{Z}_4 \end{pmatrix}$$

are all rings satisfying acc on $d$-annihilators.

Following Hamimou *et al.*[4], a ring $R$ is right strongly Hopfian if the chain of right annihilators $\mathrm{Ann}_R(a) \subseteq \mathrm{Ann}_R(a^2) \subseteq \cdots$ stabilizes for each $a \in R$. Based on Corollaries 2.3 and 2.4, we can derive the following:

**Corollary 2.5** *Let $R$ be a ring. If $R$ satisfies acc on d-annihilators, then the following hold*:
(1) $U_n(R)$ *is a right strongly Hopfian ring*;
(2) $L_n(R)$ *is a right strongly Hopfian ring*;
(3) $W(R)$ *is a right strongly Hopfian ring*;
(4) $W^i(R)$ $(i = 1, 2, 3, 4)$ *is a right strongly Hopfian ring*;
(5) $S_n(R)$ *is a right strongly Hopfian ring*;
(6) $G_n(R)$ *is a right strongly Hopfian ring*;
(7) *The trivial extension $R \bowtie R$ of $R$ by $R$ is a right strongly Hopfian ring*;
(8) $R[x]/(x^n)$ *is a right strongly Hopfian ring.*

# 3 Ore Extension Modules

In the Ore extension $R[x; \alpha, \delta]$, we have

$$x^n a = \sum_{i=0}^{n} f_i^n(a) x^i \qquad (n \geq 0),$$

where $f_i^n \in End(R, +)$ denote the map which is the sum of all possible words in $\alpha$, $\delta$ built with $i$ letters $\alpha$ and $n - i$ letters $\delta$ (see [5]).

The following definition appears in [1].

**Definition 3.1** *Given a module $M_R$, an endomorphism $\alpha\colon R \longrightarrow R$ and an $\alpha$-derivation $\delta\colon R \longrightarrow R$, we say that $M_R$ is $\alpha$-compatible if for each $m \in M_R$ and $r \in R$, one has $mr = 0 \Leftrightarrow m\alpha(r) = 0$. Moreover, we say that $M_R$ is $\delta$-compatible if for each $m \in M_R$ and $r \in R$, one has $mr = 0 \Rightarrow m\delta(r) = 0$. If $M_R$ is both $\alpha$-compatible and $\delta$-compatible, we say that $M_R$ is $(\alpha, \delta)$-compatible.*

Note that if $M_R$ is $\alpha$-compatible (resp. $\delta$-compatible), then $M_R$ is $\alpha^i$-compatible (resp. $\delta^i$-compatible) for all $i \geq 1$. It is clear that if $M_R$ is $\alpha$-compatible (resp. $\delta$-compatible), then so is any submodule of $M_R$. The following definition appears in [6].

**Definition 3.2** *Let $M_R$ be a right $R$-module. We say that $M_R$ is reduced, if, for any $m \in M_R$ and any $a \in R$, $ma = 0$ implies $mR \cap Ma = 0$.*

Clearly, if $M_R$ is reduced, then for all $m \in M_R$ and $a \in R$, $ma = 0$ implies $mRa = 0$ and $ma^2 = 0$ implies $ma = 0$.

As a immediate consequence of Definitions 3.1 and 3.2, we obtain the following lemma.

**Lemma 3.1** *Let $M_R$ be an $(\alpha, \delta)$-compatible reduced module. Then the following hold:*

(1) *$ma = 0$ if and only if $m\alpha^n(a) = 0$, where $n$ is a positive integer;*

(2) *$mab = 0$ implies $mf_i^j(a)f_s^t(b) = 0$;*

(3) *$mab = 0$ implies $mba = 0$ and $mRaRb = 0$.*

The next lemma is known and very useful, we leave the proof for the reader.

**Lemma 3.2** *Let $M_R$ be a reduced module and $X = \{a_1, a_2, \cdots, a_n\} \subseteq R$ be a finite subset of $R$. Then for any $m \in M_R$, $mX = 0$ if and only if $m(Ra_1R + Ra_2R + \cdots + Ra_nR) = 0$, where $Ra_1R + Ra_2R + \cdots + Ra_nR$ denotes the ideal of $R$ generated by $a_1$, $a_2$, $\cdots$, $a_n$.*

**Lemma 3.3** *Let $R$ be a ring and $M_R$ a reduced module satisfying acc on d-annihilators. Then for every sequence $(A_n)_n$ of finitely generated ideals of $R$, the ascending chain $\mathrm{Ann}_M(A_1) \subseteq \mathrm{Ann}_M(A_1A_2) \subseteq \cdots$ stabilizes.*

*Proof.* Since $M_R$ is reduced, for any $m \in M_R$ and any $a,\ b \in R$, by Lemma 3.1, $mab = 0$ implies $mba = 0$ and $mRaRb = MRbRa = 0$. Then similar to the proof of Theorem 2.3(b) in [2], we complete the proof.

**Proposition 3.1** *Let $\alpha$ be an endomorphism and $\delta$ an $\alpha$-derivation of a ring $R$. If $M_R$ is an $(\alpha, \delta)$-compatible reduced module, then the following statements are equivalent:*

(1) *$M_R$ satisfies acc on d-annihilators;*

(2) *The right $R[x; \alpha, \delta]$-module $M[x]$ satisfies acc on d-annihilators.*

*Proof.* (1) $\Rightarrow$ (2). For any $f(x) = \sum\limits_{i=0}^{n} a_ix^i \in R[x; \alpha, \delta]$, we denote by $A_f$ the ideal of $R$ generated by the coefficients of $f(x)$. Suppose that $f(x) = \sum\limits_{i=0}^{m} a_ix^i$ and $g(x) = \sum\limits_{j=0}^{n} b_jx^j$ be two polynomials in $R[x; \alpha, \delta]$. We first show that $\mathrm{Ann}_M(A_fA_g) = \mathrm{Ann}_M(A_{fg})$. Note that

$$f(x)g(x) = \left(\sum_{i=0}^{m} a_ix^i\right)\left(\sum_{j=0}^{n} b_jx^j\right) = \sum_{k=0}^{m+n}\left(\sum_{s+t=k}\left(\sum_{i=s}^{m} a_if_s^i(b_t)\right)\right)x^k.$$

If $r \in \mathrm{Ann}_M(A_fA_g)$, then

$$ra_ib_j = 0, \qquad 0 \leq i \leq m,\ 0 \leq j \leq n.$$

Since $M_R$ is $(\alpha,\delta)$-compatible, by Lemma 3.1, we have

$$ra_i f_s^i(b_t)=0, \qquad 0\le i\le m,\ 0\le t\le n,\ s\le i,$$

and so

$$r\left(\sum_{s+t=k}\left(\sum_{i=s}^{m} a_i f_s^i(b_t)\right)\right)=0, \qquad 0\le k\le m+n.$$

Hence by Lemma 3.2, we obtain $r\in \mathrm{Ann}_M(A_{fg})$ and so $\mathrm{Ann}_M(A_fA_g)\subseteq \mathrm{Ann}_M(A_{fg})$. We now turn our attention to proving $\mathrm{Ann}_M(A_fA_g)\supseteq \mathrm{Ann}_M(A_{fg})$. Let $r\in \mathrm{Ann}_M(A_{fg})$. Then we have the following system of equations:

$$r\left(\sum_{s+t=k}\left(\sum_{i=s}^{m} a_i f_s^i(b_t)\right)\right)=0, \qquad k=0,1,2,\cdots,m+n.$$

For $k=m+n$, we have

$$ra_m\alpha^m(b_n)=0.$$

Then by Lemma 3.1, we obtain

$$ra_mb_n=0.$$

For $k=m+n-1$, we have

$$r(a_m\alpha^m(b_{n-1})+a_{m-1}\alpha^{m-1}(b_n)+a_mf_{m-1}^m(b_n))=0. \tag{3.1}$$

Multiplying (3.1) on the right side by $a_m$, then by Lemma 3.1, we obtain

$$ra_m\alpha^m(b_{n-1})a_m=0,$$

and so

$$ra_mb_{n-1}a_m=0.$$

Since $M_R$ is reduced, we have

$$ra_mb_{n-1}=0.$$

Note that $ra_mb_n=0$ implies $ra_mf_{m-1}^m(b_n)=0$ and $ra_mb_{n-1}=0$ implies $ra_m\alpha^m(b_{n-1})=0$. Thus (3.1) becomes

$$ra_{m-1}\alpha^{m-1}(b_n)=0.$$

Then by Lemma 3.1, we have

$$ra_{m-1}b_n=0.$$

For $k=m+n-2$, we have

$$r(a_m\alpha^m(b_{n-2})+\sum_{i=m-1}^{m} a_if_{m-1}^i(b_{n-1})+\sum_{i=m-2}^{m} a_if_{m-2}^i(b_n))=0. \tag{3.2}$$

Multiplying (3.2) on the right side by $a_m$ and using Lemma 3.1, we obtain

$$ra_m\alpha^m(b_{n-2})a_m=0,$$

and so

$$ra_mb_{n-2}a_m=0.$$

Since $M_R$ is reduced, we have

$$ra_mb_{n-2}=0.$$

Note that $ra_mb_{n-2}=0$ implies $ra_m\alpha^m(b_{n-2})=0$, $ra_mb_{n-1}=0$ implies $ra_mf_{m-1}^m(b_{n-1})=0$, $ra_mb_n=0$ implies $ra_m\alpha^m(b_n)=0$ and $ra_{m-1}b_n=0$ implies $ra_{m-1}f_{m-2}^{m-1}(b_n)=0$. Thus (3.2) becomes

$$r(a_{m-1}\alpha^{m-1}(b_{n-1})+a_{m-2}\alpha^{m-2}(b_n))=0. \tag{3.3}$$

Multiplying (3.3) on the right side by $a_{m-1}$, then by Lemma 3.1, we can show that

$$ra_{m-1}b_{n-1}=0.$$

Hence (3.3) becomes

$$ra_{m-2}\alpha^{m-2}(b_n)=0.$$

Thus

$$ra_{m-2}b_n=0.$$

Continuing this procedure yields that

$$ra_ib_j=0, \qquad 0\le i\le m,\ 0\le j\le n.$$

Thus, for each $\sum\limits_{i=0}^{m}r_ia_iu_i\in A_f$, $\sum\limits_{j=0}^{n}s_jb_jv_j\in A_g$, it is easy to see that

$$r\left(\sum_{i=0}^{m}r_ia_iu_i\right)\left(\sum_{j=0}^{n}s_jb_jv_j\right)=0.$$

Hence $r\in \mathrm{Ann}_M(A_fA_g)$ and so

$$\mathrm{Ann}_M(A_{fg})\subseteq \mathrm{Ann}_M(A_fA_g).$$

Therefore $\mathrm{Ann}_M(A_{fg})=\mathrm{Ann}_M(A_fA_g)$ is proved. So by Lemma 3.3, it suffices to prove that $Ann(f(x))=Ann(f(x)g(x))$ in $M[x]$ whenever $Ann(A_f)=Ann(A_{fg})$ in $M_R$. Let $f(x)=\sum\limits_{r=0}^{p}a_rx^r$, $f(x)g(x)=\sum\limits_{j=0}^{n}c_jx^j\in R[x;\alpha,\delta]$ and $m(x)=\sum\limits_{i=0}^{m}m_ix^i\in \mathrm{Ann}_{M[x]}(f(x)g(x))$. Then

$$0=m(x)(f(x)g(x))=\left(\sum_{i=0}^{m}m_ix^i\right)\left(\sum_{j=0}^{n}c_jx^j\right)=\sum_{k=0}^{m+n}\left(\sum_{s+t=k}\left(\sum_{i=s}^{m}m_if_s^i(c_t)\right)\right)x^k.$$

Thus we obtain a system of equations:

$$\sum_{s+t=k}\left(\sum_{i=s}^{m}m_if_s^i(c_t)\right)=0, \qquad k=0,1,\cdots,m+n.$$

By using the same way as above, we can show that

$$m_ic_j=0, \qquad 0\le i\le m,\ 0\le j\le n.$$

Then by Lemma 3.2, we obtain

$$m_i\in \mathrm{Ann}_M(A_{fg})=\mathrm{Ann}_M(A_f), \qquad 0\le i\le m.$$

Hence

$$m_ia_r=0, \qquad 0\le i\le m,\ 0\le r\le p.$$

Then by a routine computations we can show that

$$m(x)f(x)=0.$$

Hence $m(x)\in \mathrm{Ann}_{M[x]}(f(x))$ and so

$$\mathrm{Ann}_{M[x]}(f(x))=\mathrm{Ann}_{M[x]}(f(x)g(x)).$$

Therefore $M[x]$ satisfies acc on $d$-annihilators.

(2) $\Rightarrow$ (1). Note that for any $a \in R$, $\mathrm{Ann}_M(a) = \mathrm{Ann}_{M[x]}(a) \cap M$. Hence the proof of (2) $\Rightarrow$ (1) is trivial.

**Corollary 3.1** *Let $R$ be a ring and $M_R$ a reduced right $R$-module. Then we have the following results*:

(1) *Let $\alpha$ be an endomorphism of $R$. If $M_R$ is $\alpha$-compatible, then the skew polynomial module $M[x]$ over the skew polynomial ring $R[x;\alpha]$ satisfies acc on $d$-annihilators if and only if $M_R$ satisfies acc on $d$-annihilators*;

(2) *Let $\delta$ be a derivation of $R$. If $M_R$ is $\delta$-compatible, then the differential polynomial module $M[x]$ over the differential ring $R[x;\delta]$ satisfies acc on $d$-annihilators if and only if $M_R$ satisfies acc on $d$-annihilators.*

**Corollary 3.2** *Let $R$ be a ring. If $R$ is an $(\alpha,\delta)$-compatible reduced ring, then the Ore extension ring $R[x;\alpha,\delta]$ satisfies acc on $d$-annihilators if and only if $R$ satisfies acc on $d$-annihilators.*

The following corollary is a generalization of Corollary 2.4(iii) in [2].

**Corollary 3.3** *Let $R$ be a reduced ring. Then the polynomial ring $R[x]$ satisfies acc on $d$-annihilators if and only if $R$ satisfies acc on $d$-annihilators.*

We show that if $M_R$ is $(\alpha,\delta)$-compatible and reduced, then the right $R[x;\alpha,\delta]$-module $M[x]$ satisfies acc on $d$-annihilators if and only if $M_R$ satisfies acc on $d$-annihilators (see Proposition 3.1). Let $M_R$ be a module with acc on $d$-annihilators. If $M_R$ does not be $(\alpha,\delta)$-compatible or not be reduced, can one provide a counterexample that the Ore extension module $M[x]_{R[x;\alpha,\delta]}$ does not has acc on $d$-annihilators? We do not know the answer and thus conclude with the following open problem:

**Question 3.1** Let $M_R$ be a module with acc on $d$-annihilators. If $M_R$ is not $(\alpha,\delta)$-compatible or not reduced, does there exist an Ore extension module $M[x]$ over the Ore extension ring $R[x;\alpha,\delta]$ that does not has acc on $d$-annihilators?

# References


[1] Annin S. Associated primes over Ore extension rings. *J. Algebra Appl.*, 2004, **3**(2): 2511–2528.

[2] Frohn D. Modules with $n$-acc and the acc on certain types of annihilators. *J. Algebra*, 2002, **256**(2): 467–483.

[3] Visweswaran S. Some results on modules satisfying (C). *J. Ramanujan Math. Soc.*, 1996, **11**(2): 161–174.

[4] Hmaimou A, Kaidi A, Sanchez Campos E. Generalized fitting modules and rings. *J. Algebra*, 2007, **308**(1): 199–214.

[5] Lam T Y, Leroy A, Matczuk, J. Primeness, semiprimeness and prime radical of Ore extensions. *Comm. Algebra*, 1997, **25**(80): 2459–2506.

[6] Lee T K, Zhou Y. Reduced modules, rings, modules, algebras and abelian groups, 365–377, Lecture Notes in Pure and Appl. Math., 236, Dekker, New York, 2004.