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One Parameter Deformation of Symmetric Toda Lattice Hierarchy

Xiong Zhen
(*Department of Mathematics and Computer, Yichun University, Yichun, Jiangxi, 336000*)

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Abstract: In this paper, we study one parameter deformation of full symmetric Toda hierarchy. This deformation is induced by Hom-Lie algebras, or is the applications of Hom-Lie algebras. We mainly consider three kinds of deformation, and give solutions to deformations respectively under some conditions.
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1 Introduction

Consider the following equation given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{L} = \boldsymbol{BL} - \boldsymbol{LB} = [\boldsymbol{B},\ \boldsymbol{L}], \tag{1.1}$$

where $\boldsymbol{L}$ is an $n \times n$ symmetric real tridiagonal matrix, and $\boldsymbol{B}$ is the skew symmetric matrix obtained from L by

$$\boldsymbol{B} = \boldsymbol{L}_{>0} - \boldsymbol{L}_{<0},$$

where $\boldsymbol{L}_{>0(<0)}$ denotes the strictly upper (lower) triangular part of $\boldsymbol{L}$. In order to study the Toda lattice of statistical mechanics, the equation (1.1) was introduced by Flaschka[1], and this further was studied by Kodama *et al.*[2],[3].

The notion of Hom-Lie algebras was introduced by Hartwig *et al.*[4] as part of a study of deformations of the Witt and the Virasoro algebras. In a Hom-Lie algebra, the Jacobi identity is twisted by a linear map, called the Hom-Jacobi identity. Some q-deformations of the Witt and the Virasoro algebras have the structure of a Hom-Lie algebra (see [4] and [5]). Because of close relation to discrete and deformed vector fields and differential calculus

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E-mail address: 205137@jxycu.edu.cn (Xiong Z).

(see [4], [6] and [7]), more people pay special attention to this algebraic structure and their representations (see [8] and [9]).

We give an application of Hom-Lie algebras. Define a smooth map

$$\beta : R_1 \longrightarrow GL(V), \qquad \boldsymbol{\beta}(s) \in GL(V),$$

where R_1 is a subset of $\mathbf{R}$, $s \in R_1$. In this paper, $R_1 = \mathbf{R} \setminus \{0\}$, or $R_1 = \mathbf{R}$.

We mainly consider the following system:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{L} = \boldsymbol{\beta}(s)\boldsymbol{B}\boldsymbol{\beta}(s)^{-1}\boldsymbol{L}\boldsymbol{\beta}(s)^{-1} - \boldsymbol{\beta}(s)\boldsymbol{L}\boldsymbol{\beta}(s)^{-1}\boldsymbol{B}\boldsymbol{\beta}(s)^{-1} = [\boldsymbol{B},\ \boldsymbol{L}]_{\boldsymbol{\beta}(s)}, \tag{1.2}$$

where $s \in R_1$, and s is not dependent on variable t. $[\,\cdot\,,\ \cdot\,]_{\boldsymbol{\beta}(s)}$ is just a Hom-Lie bracket, and $(\mathfrak{gl}(V),\ [\,\cdot\,,\ \cdot\,]_{\boldsymbol{\beta}(s)},\ \mathrm{Ad}_{\boldsymbol{\beta}(s)})$ is a Hom-Lie algebra (see [9]), where $\mathrm{Ad}_{\boldsymbol{\beta}(s)}(\boldsymbol{L}) = \boldsymbol{\beta}(s)\boldsymbol{L}\boldsymbol{\beta}(s)^{-1}$.

We study system (1.2) which is based on the following points:

(1) (1.2) is one parameter deformation of (1.1). Deformation theory is a very important field in singularity theory and bifurcation theory, and has many applications in science and engineering (see [10] and [11]).

(2) (1.2) is equivariant under the action of Lie group

$$\{\mathrm{Ad}_{\boldsymbol{\beta}(s)} \mid s \in R_1\} : \mathrm{Ad}_{\boldsymbol{\beta}(s)} \circ [\boldsymbol{B},\ \boldsymbol{L}]_{\boldsymbol{\beta}(s)} = [\mathrm{Ad}_{\boldsymbol{\beta}(s)}(\boldsymbol{B}),\ \mathrm{Ad}_{\boldsymbol{\beta}(s)}(\boldsymbol{L})]_{\boldsymbol{\beta}(s)}.$$

This kind of differential equations is very important in equation theory and bifurcation theory (see [10] and [11]).

(3) For a Hom-Lie algebra $(\mathfrak{gl}(V),\ [\,\cdot\,,\ \cdot\,]_{\boldsymbol{\beta}(s)},\ \mathrm{Ad}_{\boldsymbol{\beta}(s)})$, when $\boldsymbol{\beta}(s) = \boldsymbol{I}_n$, it is just a Lie algebra $(\mathfrak{gl}(V),\ [\,\cdot\,,\ \cdot\,])$, where $\boldsymbol{I}_n$ is the $n \times n$ identity matrix.

The general framework is organized as follows: we first introduce the relevant definitions: one parameter deformation, Γ-equivariant and so on; then, we give definitions of $\boldsymbol{\beta}(s)$ and prove that $\{\mathrm{Ad}_{\boldsymbol{\beta}(s)} \mid s \in R_1\}$ is a Lie group. Second, we give three kinds of one parameter deformation of (1.1). Then, we study these deformations respectively and give solutions. At last, some problems are given.

2 Preliminaries

We first give some definitions, one can find these definitions easily in [10] and [11].

Definition 2.1 *An equation*

$$g(x,\ s) = 0,$$

where x is an unknown variable, and the equation depends on an parameter $s(\in \mathbf{R})$. For a fixed s_0, let $g_1(x) = g(x,\ s_0)$. Then we call $g(x,\ s)$ is one parameter deformation of $g_1(x)$.

Definition 2.2 *A smooth map $g : \mathbf{R}^n \times \mathbf{R} \longrightarrow \mathbf{R}^n$ is Γ-equivariant, if for any $\gamma \in \Gamma$, Γ is a Lie group, we have*

$$g(\gamma x,\ s) = \gamma g(x,\ s),$$

where γx is the Lie group Γ action on $\mathbf{R}^n$.

Definition 2.3 *Let $f, g : X \longrightarrow Y$ be continuous maps. We say that f is homotopic to g if there exists a homotopy of f to g, that is, a map $H : X \times [0,\ 1] \longrightarrow Y$ such that $H(x,\ 0) = f(x)$ and $H(x,\ 1) = g(x)$.*

Let

$$G = \{\mathrm{Ad}_{\boldsymbol{\beta}(s)} \mid s \in R_1\}.$$

We know that

$$\mathrm{Ad}_{\boldsymbol{\beta}(s)}(\boldsymbol{B}) = \boldsymbol{\beta}(s)\boldsymbol{B}\boldsymbol{\beta}(s)^{-1}.$$

Then we have

$$\mathrm{Ad}^{-1}_{\boldsymbol{\beta}(s)} = \mathrm{Ad}_{\boldsymbol{\beta}(s)^{-1}}, \qquad \mathrm{Ad}_{\boldsymbol{\beta}(s_1)} \circ \mathrm{Ad}_{\boldsymbol{\beta}(s_2)} = \mathrm{Ad}_{\boldsymbol{\beta}(s_1)\boldsymbol{\beta}(s_2)}.$$

So, G is a Lie group.

Proposition 2.1 *A map $F_1 : \mathbf{R} \setminus \{0\} \longrightarrow G$ is given by $\boldsymbol{F}_1(r) = \mathrm{Ad}_{\boldsymbol{\beta}(r)}$, where $\boldsymbol{\beta}(r) = r\boldsymbol{I}_n$. Then $\boldsymbol{F}_1$ is a homomorphism from a Lie group $(\mathbf{R} \setminus \{0\},\ \times)$ to a Lie group G.*

Proof. It is obvious that $(R_1, \times)$ is a Lie group. We have

$$\boldsymbol{F}_1(r_1 r_2) = \mathrm{Ad}_{\boldsymbol{\beta}(r_1 r_2)} = \mathrm{Ad}_{r_1 r_2 \boldsymbol{I}_n} = \mathrm{Ad}_{\boldsymbol{\beta}(r_1)} \circ \mathrm{Ad}_{\boldsymbol{\beta}(r_2)} = \boldsymbol{F}_1(r_1) \circ \boldsymbol{F}_1(r_2).$$

In particular, we have

$$\boldsymbol{F}_1(1) = \mathrm{Ad}_{\boldsymbol{I}_n} = \boldsymbol{I}_n.$$

Proposition 2.2 *A map $F_2 : \mathbf{R} \longrightarrow G$ is given by $\boldsymbol{F}_2(\theta) = \mathrm{Ad}_{\boldsymbol{\beta}(\theta)}$, where*

$$\boldsymbol{\beta}(\theta) = \begin{pmatrix} \boldsymbol{I}_{n-2} & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

Then $\boldsymbol{F}_2$ is a homomorphism from a Lie group $(\mathbf{R}, +)$ to a Lie group G.

Proof. It is obvious that $(\mathbf{R}, +)$ is a Lie group. We have

$$\boldsymbol{\beta}(\theta_1 + \theta_2) = \begin{pmatrix} \boldsymbol{I}_{n-2} & 0 & 0 \\ 0 & \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ 0 & \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{pmatrix} = \boldsymbol{\beta}(\theta_1)\boldsymbol{\beta}(\theta_2).$$

The following is correct:

$$\boldsymbol{F}_2(\theta_1 + \theta_2) = \mathrm{Ad}_{\boldsymbol{\beta}(\theta_1+\theta_2)} = \mathrm{Ad}_{\boldsymbol{\beta}(\theta_1)\boldsymbol{\beta}(\theta_2)} = \mathrm{Ad}_{\boldsymbol{\beta}(\theta_1)} \circ \mathrm{Ad}_{\boldsymbol{\beta}(\theta_2)} = \boldsymbol{F}_2(\theta_1) \circ \boldsymbol{F}_2(\theta_2),$$

$$\boldsymbol{F}_2(0) = \mathrm{Ad}_{\boldsymbol{\beta}(0)} = \boldsymbol{I}_n.$$

The proof is completed.

Remark 2.1 Define $H : SO(n) \times [0, 1] \longrightarrow SO(n)$ by

$$\boldsymbol{H}(\theta,\ s) = \begin{pmatrix} \boldsymbol{I}_{n-2} & 0 & 0 \\ 0 & \cos(s\theta) & -\sin(s\theta) \\ 0 & \sin(s\theta) & \cos(s\theta) \end{pmatrix},$$

then $\boldsymbol{\beta}(\theta)$ is homotopic to $\boldsymbol{I}_n$. Similarly, when

$$\boldsymbol{\beta}_1(\theta) = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & \boldsymbol{I}_{n-2} & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix},$$

we have $\boldsymbol{\beta}_1(\theta)$ is homotopic to $\boldsymbol{I}_n$. Then

$$\begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & \boldsymbol{I}_{n-2} & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

is homotopic to

$$\begin{pmatrix} \boldsymbol{I}_{n-2} & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

So, we just study

$$\boldsymbol{\beta}(\theta) = \begin{pmatrix} \boldsymbol{I}_{n-2} & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

Now, we let

$$\boldsymbol{\beta}(\lambda) = \begin{pmatrix} \boldsymbol{I}_{n-2} & 0 & 0 \\ 0 & \cosh(\lambda) & \sinh(\lambda) \\ 0 & \sinh(\lambda) & \cosh(\lambda) \end{pmatrix},$$

where

$$\cosh(\lambda) = \frac{\mathrm{e}^{\lambda} + \mathrm{e}^{-\lambda}}{2}, \qquad \sinh(\lambda) = \frac{\mathrm{e}^{\lambda} - \mathrm{e}^{-\lambda}}{2}.$$

We have

$$\begin{aligned} &\cosh^2(\lambda) - \sinh^2(\lambda) = 1, \\ &2\sinh(\lambda)\cosh(\lambda) = \sinh(2\lambda), \\ &\cosh^2(\lambda) + \sinh^2(\lambda) = \cosh(2\lambda), \\ &\boldsymbol{\beta}(\lambda)^{-1} = \begin{pmatrix} \boldsymbol{I}_{n-2} & 0 & 0 \\ 0 & \cosh(\lambda) & -\sinh(\lambda) \\ 0 & -\sinh(\lambda) & \cosh(\lambda) \end{pmatrix}. \end{aligned}$$

Obviously, by a direct calculation, we also have

$$\begin{pmatrix} \cosh(\lambda_1) & \sinh(\lambda_1) \\ \sinh(\lambda_1) & \cosh(\lambda_1) \end{pmatrix} \begin{pmatrix} \cosh(\lambda_2) & \sinh(\lambda_2) \\ \sinh(\lambda_2) & \cosh(\lambda_2) \end{pmatrix} = \begin{pmatrix} \cosh(\lambda_1+\lambda_2) & \sinh(\lambda_1+\lambda_2) \\ \sinh(\lambda_1+\lambda_2) & \cosh(\lambda_1+\lambda_2) \end{pmatrix}.$$

Proposition 2.3 *With above notations, a map $F_3 : \mathbf{R} \longrightarrow G$ is given by $\boldsymbol{F}_3(\lambda) = \mathrm{Ad}_{\boldsymbol{\beta}(\lambda)}$. Then $\boldsymbol{F}_3$ is a homomorphism from a Lie group $(\mathbf{R}, +)$ to a Lie group G.*

Proof. By a direct calculation, we have

$$\boldsymbol{F}_3(\lambda_1+\lambda_2) = \mathrm{Ad}_{\boldsymbol{\beta}(\lambda_1+\lambda_2)} = \mathrm{Ad}_{\boldsymbol{\beta}(\lambda_1)\boldsymbol{\beta}(\lambda_2)} = \mathrm{Ad}_{\boldsymbol{\beta}(\lambda_1)} \circ \mathrm{Ad}_{\boldsymbol{\beta}(\lambda_2)} = \boldsymbol{F}_3(\lambda_1) \circ \boldsymbol{F}_3(\lambda_2).$$

At the same time, we have

$$\boldsymbol{F}_3(0) = \mathrm{Ad}_{\boldsymbol{\beta}(0)} = \boldsymbol{I}_n.$$

Remark 2.2 We can also define $H : SO(n) \times [0,1] \longrightarrow SO(n)$ by

$$\boldsymbol{H}(\lambda,\ s) = \begin{pmatrix} \boldsymbol{I}_{n-2} & 0 & 0 \\ 0 & \cosh(s\lambda) & \sinh(s\lambda) \\ 0 & \sinh(s\lambda) & \cosh(s\lambda) \end{pmatrix}.$$

Then $\boldsymbol{\beta}(\lambda)$ is homotopic to $\boldsymbol{I}_n$.

3 Main Results

3.1 Case of $\boldsymbol{\beta}(r)$

In this case, (1.2) has the following form:

$$\frac{\mathrm{d}}{\mathrm{d}t} = [\boldsymbol{B},\ \boldsymbol{L}]_{\boldsymbol{\beta}(r)} = \frac{1}{r}(\boldsymbol{BL} - \boldsymbol{LB}). \tag{3.1}$$

We have the following theorem.

Theorem 3.1 *System* (3.1) *is integrable. If* $\boldsymbol{L}$ *is a solution of system* (1.1), *then* $\frac{1}{r}\boldsymbol{L}$ *is a solution of system* (3.1).

So, in this case, deformation system (3.1) does not change properties of the solution of system (1.1).

3.2 Case of $\boldsymbol{\beta}(\theta)$

We just consider $n = 2$. Let

$$\boldsymbol{L} = \begin{pmatrix} a & c \\ c & b \end{pmatrix},$$

where a, b, c are functions which are independent of variable t. Then system (1.1) can be written as the following form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{L} = [\boldsymbol{B},\ \boldsymbol{L}] = \begin{pmatrix} 2c^2 & bc - ac \\ bc - ac & -2c^2 \end{pmatrix}.$$

System (1.2) has the following form:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{L} &= [\boldsymbol{B},\ \boldsymbol{L}]_{\boldsymbol{\beta}(\theta)} \\ &= \cos\theta\cos(2\theta)\begin{pmatrix} 2c^2 & bc - ac \\ bc - ac & -2c^2 \end{pmatrix} + 2\sin\theta\cos^2\theta\begin{pmatrix} ac - bc & 2c^2 \\ 2c^2 & bc - ac \end{pmatrix}. \end{aligned} \tag{3.2}$$

By a direct calculation, we have the following results.

Proposition 3.1 *In this case, we have*

$$\begin{aligned} &\mathrm{tr}([\boldsymbol{B},\ \boldsymbol{L}]_{\boldsymbol{\beta}(\theta)}) = 0, \\ &[\boldsymbol{B},\ \boldsymbol{L}]^{\mathrm{T}}_{\boldsymbol{\beta}(\theta)} = [\boldsymbol{B},\ \boldsymbol{L}]_{\boldsymbol{\beta}(\theta)}, \\ &[\boldsymbol{B},\ \boldsymbol{L}]_{\boldsymbol{\beta}(\theta+\pi)} = -[\boldsymbol{B},\ \boldsymbol{L}]_{\boldsymbol{\beta}(\theta)}. \end{aligned}$$

Theorem 3.2 *System* (3.2) *is integrable. If* $\boldsymbol{L}=\begin{pmatrix} a & c \\ c & b \end{pmatrix}$ *is a solution of system* (1.1), *then*

$$\cos\theta\cos(2\theta)\begin{pmatrix} a & c \\ c & b \end{pmatrix}+2\sin\theta\cos^2\theta\begin{pmatrix} -c & a \\ a & c \end{pmatrix}+\textit{constant matrix}$$

is a solution of system (3.2).

Remark 3.1 When $\theta=\dfrac{\pi}{2}+k\pi$, $k\in\mathbf{Z}$, the solution of system (3.2) is a constant matrix. So, in this case, parameter θ has changed properties of the solution of system (1.1).

3.3 Case of $\boldsymbol{\beta}(\lambda)$

In this case, we just consider $n=2$. By a direct calculation, the system (1.2) has the following form:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{L}&=[\boldsymbol{B},\ \boldsymbol{L}]_{\boldsymbol{\beta}(\lambda)} \\ &=\cosh(\lambda)\cosh(2\lambda)\begin{pmatrix} 2c^2 & bc-ac \\ bc-ac & -2c^2 \end{pmatrix}+\cosh(\lambda)\sinh(2\lambda)\begin{pmatrix} bc-ac & -2c^2 \\ 2c^2 & bc-ac \end{pmatrix} \\ &\quad+2\sinh(\lambda)\begin{pmatrix} -\cosh(2\lambda)bc & \sinh(2\lambda)ac \\ -\sinh(2\lambda)bc & \cosh(2\lambda)ac \end{pmatrix}. \end{aligned} \tag{3.3}$$

Then, we have the following facts.

Proposition 3.2

$$\begin{aligned} \mathrm{tr}([\boldsymbol{B},\ \boldsymbol{L}]_{\boldsymbol{\beta}(\lambda)})&=2(bc-ac)\sinh(\lambda), \\ \mathrm{tr}([\boldsymbol{B},\ \boldsymbol{L}]^{\mathrm{T}}_{\boldsymbol{\beta}(\lambda)})&=2(bc-ac)\cosh(\lambda). \end{aligned}$$

From the above discussion, we have the following theorem.

Theorem 3.3 *The system* (3.3) *is integrable if and only if* $\lambda=0$.

So, the parameter λ change the integrability of system (1.1).

4 Problems

When $n=3$, we let

$$\boldsymbol{L}=\begin{pmatrix} a & x & 0 \\ x & b & y \\ 0 & y & c \end{pmatrix},\qquad \boldsymbol{B}=\begin{pmatrix} 0 & x & 0 \\ -x & 0 & y \\ 0 & -y & 0 \end{pmatrix},\qquad \boldsymbol{\beta}(\theta)=\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

Therefore, the system (1.2) has the following form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{L} = [\boldsymbol{B},\ \boldsymbol{L}]_{\boldsymbol{\beta}(\theta)} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \tag{4.1}$$

where

$$\begin{aligned}
a_{11} &= 2x^2 \cos\theta,\\
a_{12} &= bx\cos^2\theta - xc\sin^2\theta - ax\cos\theta + xy\sin(2\theta),\\
a_{13} &= \frac{1}{2}(bx - cx)\sin(2\theta) + 2xy\sin^2\theta - ax\sin\theta,\\
a_{21} &= bx\cos^2\theta + xc\sin^2\theta - ax\cos\theta - xy\sin(2\theta),\\
a_{22} &= -2x^2\cos^2\theta + y^2(\cos\theta + \cos(3\theta)) - 2cy\cos^2\theta\sin\theta + by\cos\theta\sin(2\theta),\\
a_{23} &= -x^2\sin(2\theta) + y^2(\sin\theta + \sin(3\theta)) + cy(\cos\theta - \sin\theta\sin(2\theta)) - by\cos\theta\cos(2\theta),\\
a_{31} &= -ax\sin\theta - 2xy\sin^2\theta + \frac{1}{2}(bx - cx)\sin(2\theta),\\
a_{32} &= -x^2\sin(2\theta) - by(\cos\theta - \sin\theta\sin(2\theta)) + y^2(\sin\theta + \sin(3\theta)) + cy\cos\theta\cos(2\theta),\\
a_{33} &= -2x^2\sin^2\theta - 2by\sin\theta\cos^2\theta - y^2(\cos\theta + \cos(3\theta)) + cy\cos\theta\sin(2\theta).
\end{aligned}$$

Thus,

$$\mathrm{tr}([\boldsymbol{B},\ \boldsymbol{L}]_{\boldsymbol{\beta}(\theta)}) \neq 0, \qquad [\boldsymbol{B},\ \boldsymbol{L}]^{\mathrm{T}}_{\boldsymbol{\beta}(\theta)} \neq [\boldsymbol{B},\ \boldsymbol{L}]_{\boldsymbol{\beta}(\theta)}.$$

So, is this system (4.1) integrable? When $n > 3$, system (1.2) is integrable?

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