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On Lie 2-bialgebras

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Abstract: A Lie 2-bialgebra is a Lie 2-algebra equipped with a compatible Lie 2-coalgebra structure. In this paper, we give another equivalent description for Lie 2-bialgebras by using the structure maps and compatibility conditions. We can use this method to check whether a 2-term direct sum of vector spaces is a Lie 2-bialgebra easily.

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1 Introduction

1.1 Background

This paper is a sequel to [1], in which the notion of Lie 2-bialgeras was introduced. The main purpose of this paper is to give an equivalent condition for Lie 2-bialgebras. Generally speaking, a Lie 2-bialgebra is a Lie 2-algebra endowed with a Lie 2-coalgebra structure, satisfying certain compatibility conditions. As we all know, a Lie bialgebra structure on a Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$ consists of a cobracket $\delta\colon \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$, which squares to zero, and satisfies the compatibility condition: for all $x, y, z \in \mathfrak{g}$,

$$\delta([x, y]) = [x, \delta(y)] - [y, \delta(x)].$$

Consequently, one may ask what is a Lie 2-bialgebra. A Lie 2-bialgebra is a pair of 2-terms of L_∞-algebra structure underlying a 2-vector space and its dual. The compatibility conditions are described by big bracket (see [1]). And an L_∞-algebra structure on a $\mathbb{Z}$-graded vector space can be found in [2]–[4]. This description of Lie 2-bialgebras seems to be elegant, but one cannot get directly the maps twisted between them and compatibility conditions. This is what we will explore in this paper.

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This paper is organized as follows: In Section 1, we recall the notion of big bracket, which has a fundamental role in this paper. Then, we introduce the basic concepts in Section 2 which is closely related to our result, that is, Lie 2-algebras and Lie 2-coalgebras, most of which can be found in [3]. Finally, in Section 3, we give an equivalent description of Lie 2-bialgebras, whose compatibility conditions are given by big bracket.

1.2 The Big Bracket

We introduce the following Notations.

(1) Let V be a graded vector space. The degree of a homogeneous vector $\boldsymbol{e}$ is denoted by $|\boldsymbol{e}|$.

(2) On the symmetric algebra $\mathscr{S}^\bullet(V)$, the symmetric product is denoted by $\odot$.

It is now necessary to recall the notion of big bracket underlying the graded vector spaces [1]. Let $V = \bigoplus_{k\in\mathbf{Z}} V_k$ be a $\mathbf{Z}$-graded vector space, and $V[i]$ be its degree-shifted one. Now, we focus on the symmetric algebra $\mathscr{S}^\bullet(V[2]\oplus V^*[1])$, denoted by $\mathscr{S}^\bullet$. In order to equip $\mathscr{S}^\bullet$ with a Lie bracket, i.e., the Schouten bracket, denoted by $\{\,\cdot\,,\,\cdot\,\}$, we define a bilinear map $\{\,\cdot\,,\,\cdot\,\}\colon \mathscr{S}^\bullet\otimes\mathscr{S}^\bullet\to\mathscr{S}^\bullet$ by:

(1) $\{v,\, v'\} = \{\varepsilon,\, \varepsilon'\} = 0$, $\{v,\, \varepsilon\} = (-1)^{|v|}\langle v \mid \varepsilon\rangle$, $v, v' \in V[2]$, $\varepsilon,\, \varepsilon' \in V^*[1]$;

(2) $\{e_1,\, e_2\} = -(-1)^{(|e_1|+3)(|e_2|+3)}\{e_2,\, e_1\}$, $e_i \in \mathscr{S}^\bullet$;

(3) $\{e_1,\, e_2\odot e_3\} = \{e_1,\, e_2\}\odot e_3 + (-1)^{(|e_1|+3)|e_2|}e_2\odot\{e_1,\, e_3\}$, $e_i\in\mathscr{S}^\bullet$.

Clearly, $\{\,\cdot\,,\,\cdot\,\}$ has degree 3, and all homogeneous elements $e_i \in \mathscr{S}^\bullet$ satisfy the following modified Jacobi identity:

$$\{e_1,\, \{e_2,\, e_3\}\} = \{\{e_1,\, e_2\},\, e_3\} + (-1)^{(|e_1|+3)(|e_2|+3)}\{e_2,\, \{e_1,\, e_3\}\}. \tag{1.1}$$

Hence, $(\mathscr{S}^\bullet,\, \odot,\, \{\,\cdot\,,\,\cdot\,\})$ becomes a Schouten algebra, or a Gerstenhaber algebra, see [1] and [4] for more details. Note that the big bracket here is different from that in [5], which is defined on $\mathscr{S}^\bullet(V\oplus V^*)$ without degree shifting.

For element $F \in S^p(V[2])\odot S^q(V^*[1])$, we define the following multilinear map: for all $x_i \in \mathscr{S}^\bullet(V[2])$,

$$D_F\colon \underbrace{\mathscr{S}^\bullet(V[2])\otimes\cdots\otimes\mathscr{S}^\bullet(V[2])}_{q\text{-tuples}} \to \mathscr{S}^\bullet(V[2])$$

by

$$D_F(x_1,\cdots,x_q) = \{\{\cdots\{\{F,\, x_1\},\, x_2\},\cdots,x_{q-1}\},\, x_q\}.$$

Lemma 1.1 *The following equations hold*:

(1) $|D_F(x_1,\, x_2,\, \cdots,\, x_q)| = |x_1| + |x_2| + \cdots + |x_q| + |F| + 3q$;

(2) $D_F(x_1,\, \cdots,\, x_i,\, x_{i+1},\, \cdots,\, x_q) = (-1)^{(|x_i|+3)(|x_{i+1}|+3)}D_F(x_1,\, \cdots,\, x_{i+1},\, x_i,\, \cdots,\, x_q)$.

Proof. Since the degree of big bracket is 3, we apply this fact q-times to obtain (1).

If $q = 2$, by (1.1), we have

$$\begin{aligned}\{x_1,\, \{x_2,\, F\}\} &= \{\{x_1,\, x_2\},\, F\} + (-1)^{(|x_1|+3)(|x_2|+3)}\{x_2,\, \{x_1,\, F\}\}\\ &= (-1)^{(|x_1|+3)(|x_2|+3)}\{x_2,\, \{x_1,\, F\}\}.\end{aligned}$$

It is easy to check that if $\{x_1, \{x_2, F\}\} = \{\{F, x_2\}, x_1\}$, then
$$\{x_2, \{x_1, F\}\} = \{\{F, x_1\}, x_2\};$$
and if $\{x_1, \{x_2, F\}\} = -\{\{F, x_2\}, x_1\}$, then
$$\{x_2, \{x_1, F\}\} = -\{\{F, x_1\}, x_2\}.$$
By induction, we conclude the proof.

Lemma 1.2 *For any $E \in S^k(V[2]) \odot S^l(V^*[1])$, $F \in S^p(V[2]) \odot S^q(V^*[1])$, we have for all $x_i \in \mathscr{S}^\bullet(V[2])$,*
$$\begin{aligned}&D_{\{E,F\}}(x_1, \cdots, x_n)\\ =&\sum_{\sigma\in \mathrm{sh}\text{-}(q,l-1)} \varepsilon(\sigma) D_E(D_F(x_{\sigma(1)}, \cdots, x_{\sigma(q)}), x_{\sigma(q+1)}, \cdots, x_{\sigma(n)})\\ &-(-1)^{(|E|+3)(|F|+3)} \sum_{\sigma\in \mathrm{sh}\text{-}(l,q-1)} \varepsilon(\sigma) D_F(D_E(x_{\sigma(1)}, \cdots, x_{\sigma(l)}), x_{\sigma(l+1)}, \cdots, x_{\sigma(n)}),\end{aligned}$$
where $n = q+l-1$, and here $\mathrm{sh}\text{-}(j, n-j)$ denotes the collection of all $(j, n-j)$-shuffles, and $\varepsilon(\sigma)$ means that a sign change $(-1)^{(|x_i|+3)(|x_{i+1}|+3)}$ happens if the place of two successive elements x_i, x_{i+1} are changed.

Proof. If $n = 1$, by (1.1), we get that
$$\{\{E, F\}, x\} = \{E, \{F, x\}\} - (-1)^{(|E|+3)(|F|+3)}\{F, \{E, x\}\}.$$
If $n \geq 2$, by (1.1) and Lemma 1.1, the result can be derived easily.

2 Lie 2-algebras and Lie 2-coagebras

2.1 Lie 2-algebras

We now pay special attention to L_∞-algebra structure restricted to 2-terms $V = \theta \oplus \mathfrak{g}$, where θ is of degree 1, and $\mathfrak{g}$ is of degree 0, while the shifted vector space $V[2]$ and $V^*[1]$ should be considered. One can read [1] and [6] for more details of L_∞-algebras, where the notion of L_∞-algebra is called an SH (strongly homotopy) Lie algebras. And the degrees of elements in $\mathfrak{g}$, θ, $\mathfrak{g}^*$, and θ^* can be easily obtained by a straight computation (see [1] and [4]). The following concept is taken from [1] and [6]:

Definition 2.1 *A Lie 2-algebra structure on a 2-graded vector spaces $\mathfrak{g}$ and θ consists of the following maps*:

(1) *a linear map $\phi\colon \theta \to \mathfrak{g}$;*
(2) *a bilinear skew-symmetric map $[\cdot, \cdot]\colon \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{g}$;*
(3) *a bilinear skew-symmetric map $\cdot \succ \cdot\colon \mathfrak{g} \wedge \theta \to \theta$;*
(4) *a trilinear skew-symmetric map $h\colon \mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g} \to \theta$,*

such that the following equations are satisfied: *for all $x, y, z, w \in \mathfrak{g}$, $u, v \in \theta$,*

(a) $[[x, y], z] + \text{c.p.} + \phi h(x, y, z) = 0$;
(b) $y \succ (x \succ u) - x \succ (y \succ u) + [x, y] \succ u + h(\phi(u), x, y) = 0$;

(c) $\phi(u) \succ v + \phi(v) \succ u = 0$;

(d) $\phi(x \succ u) = [x, \phi(u)]$;

(e) $h([x, y], z, w) + \text{c.p.} = -w \succ h(x, y, z) + \text{c.p.}$,

where c.p. *stands for cyclic permutation.*

In the sequel, we denote a Lie 2-algebra by $(\theta, \mathfrak{g}; \phi, [\cdot, \cdot], \cdot \succ \cdot, h)$ or simply $(\theta, \mathfrak{g})$. One may be confused with these notions since in [1] the same notions are used to denote the strict Lie 2-algebras, which are different from our weak sense. But in this paper, the notions denote the weak cases without other statements.

We should point out that the notion of Lie 2-algebras stands for different meaning in different literatures. The notion of semidirect Lie 2-algebras is not a special case of our Lie 2-algebras in [2], where Baez and Crans treat semidirect Lie 2-algebras as a 2-vector space endowed with a skew-symmetric bilinear map satisfying the Jacobi identity up to a completely antisymmetric trilinear map called Jacobiator, which also makes sense in terms of its Jacobiator identity. By contract our definition of Lie 2-algebras is that of 2-term L_∞-algebra in [1] and [6]. The reader should distinguish these concepts. However, Baez and Crans[2] have given a one-to-one correspondence between the notion of Lie 2-algebras and 2-term L_∞-algebra.

Before we prove a proposition, we give the following lemma.

Lemma 2.1 *There is a bijection between the linear maps ϕ, $[\cdot, \cdot]$, $\cdot \succ \cdot$ and h of Lie 2-algebra and the data ε_{01}^{10}, ε_{00}^{12}, ε_{11}^{01} and ε_{10}^{03}, where $\varepsilon_{01}^{10} \in \theta^* \odot \mathfrak{g}$, $\varepsilon_{00}^{12} \in (\odot^2 \mathfrak{g}^*) \odot \mathfrak{g}$, $\varepsilon_{11}^{01} \in \mathfrak{g}^* \odot \theta^* \odot \theta$ and $\varepsilon_{10}^{03} \in (\odot^3 \mathfrak{g}^*) \odot \theta$.*

Proof. If the data ε_{01}^{10}, ε_{00}^{12}, ε_{11}^{01} and ε_{10}^{03} are given, then we let $\phi(u) = D_{\varepsilon_{01}^{10}}(u)$, $[x, y] = D_{\varepsilon_{00}^{12}}(x, y)$, $x \succ u = D_{\varepsilon_{11}^{01}}(x, u)$, $h(x, y, z) = D_{\varepsilon_{10}^{03}}(x, y, z)$ for all $x, y, z \in \mathfrak{g}$, $u, v \in \theta$.

Conversely, for all $x, y, z \in \mathfrak{g}$, $u \in \theta$, if the structure maps ϕ, $[\cdot, \cdot]$, $\cdot \succ \cdot$ and h are given, we take $\varepsilon_{01}^{10} = f_\theta \odot \phi(u)$, where $\langle u \mid f_\theta \rangle = 1$. So, we have

$$D_{\varepsilon_{01}^{10}}(u) = \{\varepsilon_{01}^{10}, u\} = \phi(u).$$

Similarly, we take $\varepsilon_{00}^{12} = f_{\mathfrak{g}}^{11} \odot f_{\mathfrak{g}}^{21} \odot [x, y]$, $\varepsilon_{11}^{01} = f_{\mathfrak{g}} \odot f_\theta \odot (x \succ u)$ and $\varepsilon_{10}^{03} = f_{\mathfrak{g}}^{12} \odot f_{\mathfrak{g}}^{22} \odot f_{\mathfrak{g}}^{32} \odot h(x, y, z)$ such that

$$\langle x \mid f_{\mathfrak{g}}^{21} \rangle = \langle y \mid f_{\mathfrak{g}}^{11} \rangle = 1,$$
$$\langle u \mid f_\theta \rangle = \langle x \mid f_{\mathfrak{g}} \rangle = 1,$$
$$\langle x \mid f_{\mathfrak{g}}^{32} \rangle = \langle y \mid f_{\mathfrak{g}}^{22} \rangle = \langle z \mid f_{\mathfrak{g}}^{12} \rangle = 1.$$

By the language of big bracket, a Lie 2-algebra can be described in a beautiful manner:

Proposition 2.1 *A Lie 2-algebra structure on a pair of graded vector spaces θ and $\mathfrak{g}$ is a solution $l = \varepsilon_{01}^{10} + \varepsilon_{00}^{12} + \varepsilon_{11}^{01} + \varepsilon_{10}^{03}$ to the equation*

$$\{l, l\} = 0,$$

where $\varepsilon_{01}^{10} \in \theta^* \odot \mathfrak{g}$, $\varepsilon_{00}^{12} \in (\odot^2 \mathfrak{g}^*) \odot \mathfrak{g}$, $\varepsilon_{11}^{01} \in \mathfrak{g}^* \odot \theta^* \odot \theta$ and $\varepsilon_{10}^{03} \in (\odot^3 \mathfrak{g}^*) \odot \theta$. Here the bracket stands for the big bracket described in Section 1.2. Moreover, if $\varepsilon_{10}^{03} = 0$, we call it a strict Lie 2-algebra.

Proof. By Lemma 2.1, it is easy to see that $\phi(u) = D_{\varepsilon_{01}^{10}}(u)$, $[x, y] = D_{\varepsilon_{00}^{12}}(x, y)$, $x \succ u = D_{\varepsilon_{11}^{01}}(x, u)$ and $h(x, y, z) = D_{\varepsilon_{10}^{03}}(x, y, z)$ for all $x, y, z \in \mathfrak{g}$, $u, v \in \theta$.

Since $l = \varepsilon_{01}^{10} + \varepsilon_{00}^{12} + \varepsilon_{11}^{01} + \varepsilon_{10}^{03}$, we get

$$\begin{aligned}\{l, l\} = {} & \{\varepsilon_{00}^{12}, \varepsilon_{00}^{12}\} + \{\varepsilon_{11}^{01}, \varepsilon_{11}^{01}\} + 2\{\varepsilon_{01}^{10}, \varepsilon_{00}^{12}\} + 2\{\varepsilon_{01}^{10}, \varepsilon_{11}^{01}\} \\ & + 2\{\varepsilon_{01}^{10}, \varepsilon_{10}^{03}\} + 2\{\varepsilon_{00}^{12}, \varepsilon_{11}^{01}\} + 2\{\varepsilon_{00}^{12}, \varepsilon_{10}^{03}\} + 2\{\varepsilon_{11}^{01}, \varepsilon_{10}^{03}\}.\end{aligned}$$

By Lemma 1.2, we have

$$\begin{aligned}D_{\{l,l\}}(x, y, z) &= D_{\{\varepsilon_{00}^{12}, \varepsilon_{00}^{12}\}}(x, y, z) + 2D_{\{\varepsilon_{01}^{10}, \varepsilon_{10}^{03}\}}(x, y, z) \\ &= 2(D_{\varepsilon_{00}^{12}}(D_{\varepsilon_{00}^{12}}(x, y), z) + \text{c.p.}) + 2D_{\varepsilon_{01}^{10}}(D_{\varepsilon_{10}^{03}}(x, y, z)) \\ &= 2([[x, y], z] + \text{c.p.} + \phi(h(x, y, z))); \\ D_{\{l,l\}}(x, y, u) &= 2D_{\{\varepsilon_{01}^{10}, \varepsilon_{10}^{03}\}}(x, y, u) + D_{\{\varepsilon_{11}^{01}, \varepsilon_{11}^{01}\}}(x, y, u) + 2D_{\{\varepsilon_{00}^{12}, \varepsilon_{11}^{01}\}}(x, y, u) \\ &= 2D_{\varepsilon_{10}^{03}}(D_{\varepsilon_{01}^{10}}(u), x, y) + 2(-D_{\varepsilon_{11}^{01}}(D_{\varepsilon_{11}^{01}}(x, u), y) + D_{\varepsilon_{11}^{01}}(D_{\varepsilon_{11}^{01}}(y, u), x)) \\ &\quad + 2D_{\varepsilon_{11}^{01}}(D_{\varepsilon_{00}^{12}}(x, y), u) \\ &= 2(h(\phi(u), x, y) + y \succ (x \succ u) - x \succ (y \succ u) + [x, y] \succ u); \\ D_{\{l,l\}}(x, u) &= 2D_{\{\varepsilon_{01}^{10}, \varepsilon_{00}^{12}\}}(x, u) + 2D_{\{\varepsilon_{01}^{10}, \varepsilon_{11}^{01}\}}(x, u) \\ &= 2D_{\varepsilon_{00}^{12}}(D_{\varepsilon_{01}^{10}}(u), x) + 2D_{\varepsilon_{01}^{10}}(D_{\varepsilon_{11}^{01}}(x, u) \\ &= 2([\phi(u), x] + \phi(x \succ u)); \\ D_{\{l,l\}}(u, v) &= 2D_{\{\varepsilon_{01}^{10}, \varepsilon_{11}^{01}\}}(u, v) \\ &= 2(D_{\varepsilon_{11}^{01}}(D_{\varepsilon_{01}^{10}}(u), v) + D_{\varepsilon_{11}^{01}}(D_{\varepsilon_{01}^{10}}(v), u)) \\ &= 2(\phi(u) \succ v + \phi(v) \succ u); \\ D_{\{l,l\}}(x, y, z, w) &= 2D_{\{\varepsilon_{00}^{12}, \varepsilon_{10}^{03}\}}(x, y, z, w) + 2D_{\{\varepsilon_{11}^{01}, \varepsilon_{10}^{03}\}}(x, y, z, w) \\ &= 2(D_{\varepsilon_{10}^{03}}(D_{\varepsilon_{00}^{12}}(x, y), z, w) + \text{c.p.} + D_{\varepsilon_{11}^{01}}(D_{\varepsilon_{10}^{03}}(x, y, z), w) + \text{c.p.}) \\ &= 2(h([x, y], z, w) + \text{c.p.} + w \succ h(x, y, z) + \text{c.p.}).\end{aligned}$$

Hence, $\{l, l\} = 0$ if and only if the right hand side of these equations vanish, which implies that $(\theta, \mathfrak{g})$ is a Lie 2-algebra.

Remark 2.1 Note that in [1], a strict Lie 2-algebra is equivalent to a Lie algebra crossed module. Similarly, one may ask what is a Lie 2-algebra crossed module, this work has been solved in [8]. The reader can read this for more details.

Example 2.1 Let V_3 be a 3-dimensional vector space. Then we can construct a Lie 2-algebra as follows:

$\mathscr{O}\colon \mathbf{R} \to V_3$ is the trivial map;

$[\cdot, \cdot]\colon V_3 \times V_3 \to V_3$ is the crossed product;

$\cdot \succ \cdot\colon V_3 \wedge \mathbf{R} \to \mathbf{R}$ is given by $\boldsymbol{\alpha} \succ k = k(\boldsymbol{\alpha}\boldsymbol{e})$, where $\boldsymbol{e} = (1, 1, 1)$;

$h\colon V_3 \wedge V_3 \wedge V_3 \to \mathbf{R}$ is its mixed product.

One can easily check that $\mathbf{R} \oplus V_3$ becomes a Lie 2-algebra.

2.2 Lie 2-coalgebras

As we all know, if $(\mathfrak{g}^*, [\cdot, \cdot]_*)$ is a Lie algebra, then $(\mathfrak{g}, \delta)$ is a Lie coalgebra, where $\langle x \mid [\xi, \varsigma]_*\rangle = -\langle \delta(x) \mid \xi \wedge \varsigma\rangle$ for all $x \in \mathfrak{g}$, $\xi, \varsigma \in \mathfrak{g}^*$. Similar to the relation between Lie algebras and Lie coalgebras, if $(\mathfrak{g}^*, \theta^*)$ is a Lie 2-algebra, then we call $(\theta, \mathfrak{g})$ a Lie 2-coalgebra. Besides, we have the following:

Proposition 2.2 *A Lie 2-coalgebra structure on a pair of graded vector spaces θ and $\mathfrak{g}$ is a solution $c = \varepsilon^{10}_{01} + \varepsilon^{00}_{21} + \varepsilon^{11}_{10} + \varepsilon^{01}_{30} \in \mathscr{S}^{(-4)}$ to the equation*

$$\{c, c\} = 0,$$

where $\varepsilon^{10}_{01} \in \theta^* \odot \mathfrak{g}$, $\varepsilon^{00}_{21} \in \theta^* \odot (\odot^2\theta)$, $\varepsilon^{11}_{10} \in \mathfrak{g}^* \odot \mathfrak{g} \odot \theta$ and $\varepsilon^{01}_{30} \in \mathfrak{g}^* \odot (\odot^3\theta)$.

We would give an equivalent condition of a Lie 2-coalgebra by the language of maps and compatibility conditions. The following notations are taken from [1]:

(1) $W_k = \{w \in \mathfrak{g} \wedge (\wedge^{k-1}\theta)\colon \iota_\xi \iota_{\phi^*\varsigma} w = -\iota_\varsigma \iota_{\phi^*}\xi w\}$, $k \geq 1$, $\xi, \varsigma \in \mathfrak{g}^*$;

(2) The bilinear map: for all $x \in \mathfrak{g}$, $u \in \theta$,

$$D_\phi\colon \wedge^\bullet (\mathfrak{g} \oplus \theta) \to \wedge^\bullet(\mathfrak{g} \oplus \theta)$$

defined by

$$D_\phi(x + u) = \phi(u)$$

is a degree-0 derivation with respect to the wedge product.

The maps and compatibility conditions of a Lie 2-coalgebra can be summarized as follows.

Theorem 2.1 *A Lie 2-coalgebra structure on $(\theta, \mathfrak{g})$ is equivalent to the following linear maps $\delta\colon \mathfrak{g} \to W_2 \subset \mathfrak{g} \wedge \theta$, $\omega\colon \theta \to \theta \wedge \theta$, and $\eta\colon \mathfrak{g} \to \theta \wedge \theta \wedge \theta$ such that*

(1) $D_\phi\omega = \delta\phi$;

(2) $\omega^2 = \eta\phi$;

(3) $(\omega + \delta)\delta = D_\phi\eta$;

(4) $\omega\eta = \eta\delta$.

Here we regard both ω and δ as degree-1 derivations on $\wedge^\bullet(\mathfrak{g}\oplus\theta)$, and η as degree-2 by letting $\omega|_\mathfrak{g} = 0$, $\delta|_\theta = 0$ and $\eta|_\theta = 0$.

Proof. According to Proposition 2.1, a Lie 2-coalgebra structure on $(\theta, \mathfrak{g})$ is equivalent to the fact that $(\mathfrak{g}^*, \theta^*)$ is a Lie 2-algebra, which consists of the following linear maps:

$\phi^T\colon \mathfrak{g}^* \to \theta^*$;

$[\cdot, \cdot]_*\colon \theta^* \wedge \theta^* \to \theta^*$;

$\cdot \triangleright \cdot\colon \theta^* \wedge \mathfrak{g}^* \to \mathfrak{g}^*$;

$m\colon \theta^* \wedge \theta^* \wedge \theta^* \to \mathfrak{g}^*$,

such that for all $\xi, \varsigma \in \mathfrak{g}^*$, $\kappa, \kappa_1, \kappa_2, \kappa_3, \kappa_4 \in \theta^*$,

(a) $[[\kappa_1, \kappa_2]_*, \kappa_3]_* + \text{c.p.} - \phi^T m(\kappa_1, \kappa_2, \kappa_3) = 0$;

(b) $\kappa_2 \triangleright (\kappa_1 \triangleright \xi) - \kappa_1 \triangleright (\kappa_2 \triangleright \xi) + [\kappa_1, \kappa_2]_* \triangleright \xi - m(\phi^T(\xi), \kappa_1, \kappa_2) = 0$;

(c) $\phi^T(\xi) \triangleright \varsigma = -\phi^T(\varsigma) \triangleright \xi$;

(d) $\phi^T(\kappa \triangleright \xi) = [\kappa, \phi^T(\xi)]_*$;

(e) $m([\kappa_1,\,\kappa_2]_*,\,\kappa_3,\,\kappa_4)+\text{c.p.}=-\kappa_4\triangleright m(\kappa_1,\,\kappa_2,\,\kappa_3)+\text{c.p.}$

Then a triple of linear maps $(\delta,\,\omega,\,\eta)$ is defined by: for all $x\in\mathfrak{g}$, $u\in\theta$, $\xi,\varsigma\in\mathfrak{g}^*$, $\kappa,\kappa_1,\kappa_2,\kappa_3\in\theta^*$,

$$\langle\delta(x)\mid\xi\wedge\kappa\rangle=\langle x\mid\kappa\triangleright\xi\rangle,$$
$$\langle\omega(u)\mid\kappa_1\wedge\kappa_2\rangle=-\langle u\mid[\kappa_1,\,\kappa_2]_*\rangle,$$
$$\langle\eta(x)\mid\kappa_1\wedge\kappa_2\wedge\kappa_3\rangle=-\langle x\mid m(\kappa_1,\,\kappa_2,\,\kappa_3)\rangle.$$

Note that

$$\begin{aligned}\langle\delta\phi(u)\mid\xi\wedge\kappa\rangle&=\langle\phi(u)\mid\kappa\triangleright\xi\rangle\\&=-\langle u\mid\phi^T(\kappa\triangleright\xi)\rangle,\\\langle D_\phi\omega(u)\mid\xi\wedge\kappa\rangle&=-\langle\omega(u)\mid\phi^T(\xi)\wedge\kappa\rangle\\&=-\langle u\mid[\kappa,\,\phi^T(\xi)]_*\rangle.\end{aligned}$$

So, $D_\phi\omega=\delta\phi$ is equivalent to (d).

$$\begin{aligned}\langle\omega^2(u)\mid\kappa_1\wedge\kappa_2\wedge\kappa_3\rangle&=-\langle\omega(u)\mid[\kappa_1,\,\kappa_2]_*\wedge\kappa_3+\text{c.p.}\rangle\\&=\langle u\mid[[\kappa_1,\,\kappa_2]_*,\,\kappa_3]_*+\text{c.p.}\rangle,\\\langle\eta\phi(u)\mid\kappa_1\wedge\kappa_2\wedge\kappa_3\rangle&=-\langle\phi(u)\mid m(\kappa_1,\,\kappa_2,\,\kappa_3)\rangle\\&=\langle u\mid\phi^T m(\kappa_1,\,\kappa_2,\,\kappa_3)\rangle.\end{aligned}$$

Hence, $\omega^2=\eta\phi$ is equivalent to (a).

$$\begin{aligned}\langle\omega\delta(x)\mid\xi\wedge\kappa_1\wedge\kappa_2\rangle&=\langle\omega\delta(x)\mid\kappa_1\wedge\kappa_2\wedge\xi\rangle\\&=-\langle\delta(x)\mid[\kappa_1,\,\kappa_2]_*\wedge\xi\rangle\\&=\langle x\mid[\kappa_1,\,\kappa_2]_*\triangleright\xi\rangle,\\\langle\delta\delta(x)\mid\xi\wedge\kappa_1\wedge\kappa_2\rangle&=\langle\delta(x)\mid(\kappa_1\triangleright\xi)\wedge\kappa_2-(\kappa_2\triangleright\xi)\wedge\kappa_1\rangle\\&=\langle x\mid\kappa_2\triangleright(\kappa_1\triangleright\xi)-\kappa_1\triangleright(\kappa_2\triangleright\xi)\rangle,\\\langle D_\phi\eta(x)\mid\xi\wedge\kappa_1\wedge\kappa_2\rangle&=-\langle\eta(x)\mid\phi^T(\xi)\wedge\kappa_1\wedge\kappa_2\rangle\\&=\langle x\mid m(\phi^T(\xi),\,\kappa_1,\,\kappa_2)\rangle.\end{aligned}$$

Therefore, $(\omega+\delta)\delta=D_\phi\eta$ if and only if (b) holds.

$$\begin{aligned}\langle\omega\eta(x)\mid\kappa_1\wedge\kappa_2\wedge\kappa_3\wedge\kappa_4\rangle&=-\langle\eta(x)\mid[\kappa_1,\,\kappa_2]_*\wedge\kappa_3\wedge\kappa_4+c.p.\rangle\\&=\langle x\mid m([\kappa_1,\,\kappa_2]_*,\,\kappa_3,\,\kappa_4)+c.p.\rangle,\\\langle\eta\delta(x)\mid\kappa_1\wedge\kappa_2\wedge\kappa_3\wedge\kappa_4\rangle&=-\langle\delta(x)\mid m(\kappa_1,\,\kappa_2,\,\kappa_3)\wedge\kappa_4+c.p.\rangle\\&=-\langle x\mid\kappa_4\triangleright m(\kappa_1,\,\kappa_2,\,\kappa_3)+c.p.\rangle.\end{aligned}$$

Thus, $\omega\eta=\eta\delta$ if and only if (e) holds.

Meanwhile, since $\iota_\xi\iota_{\phi^*\varsigma}w=(\xi\wedge\phi^*(\varsigma))w$, we have that $\delta(x)\in W_2$ if and only if (c) holds.

3 Lie 2-bialgebras

3.1 Basic Concepts

The following concept is taken from [1].

Definition 3.1 *A Lie 2-bialgebra structure on a pair of graded vector spaces θ and $\mathfrak{g}$ is a solution $\epsilon = \varepsilon_{01}^{10} + \varepsilon_{00}^{12} + \varepsilon_{11}^{01} + \varepsilon_{10}^{03} + \varepsilon_{21}^{00} + \varepsilon_{10}^{11} + \varepsilon_{30}^{01} \in \mathscr{S}^{(-4)}$ to the equation*

$$\{\epsilon,\ \epsilon\} = 0,$$

where $\varepsilon_{01}^{10} \in \theta^* \odot \mathfrak{g}$, $\varepsilon_{21}^{00} \in \theta^* \odot (\odot^2\theta)$, $\varepsilon_{10}^{11} \in \mathfrak{g}^* \odot \mathfrak{g} \odot \theta$, $\varepsilon_{30}^{01} \in \mathfrak{g}^* \odot (\odot^3\theta)$, $\varepsilon_{00}^{12} \in (\odot^2\mathfrak{g}^*) \odot \mathfrak{g}$, $\varepsilon_{11}^{01} \in \mathfrak{g}^* \odot \theta^* \odot \theta$ and $\varepsilon_{10}^{03} \in (\odot^3\mathfrak{g}^*) \odot \theta$.

In particular, if both ε_{10}^{03} and ε_{30}^{01} vanish, we call it a strict Lie 2-bialgebra.

It is known that if $(\mathfrak{g}, [\cdot, \cdot], \delta)$ is a Lie bialgebra, then $(\mathfrak{g}, [\cdot, \cdot])$ is a Lie algebra and $(\mathfrak{g}, \delta)$ is a Lie coalgebra. Similarly, we have the following lemma which can also be found in [1].

Lemma 3.1 *If $(\theta, \mathfrak{g}; \epsilon)$ is a Lie 2-bialgebra, then $(\theta, \mathfrak{g}; l)$, where $l = \varepsilon_{01}^{10} + \varepsilon_{00}^{12} + \varepsilon_{11}^{01} + \varepsilon_{10}^{03}$ is a Lie 2-algebra, and $(\theta, \mathfrak{g}; c)$, where $c = \varepsilon_{01}^{10} + \varepsilon_{21}^{00} + \varepsilon_{10}^{11} + \varepsilon_{30}^{11} \in \mathscr{S}^{(-4)}$ is a Lie 2-coalgebra.*

In the view of the proof of Lemma 3.2 below, this fact can be obtained easily.

3.2 Main Theorem

Before we state and prove our main theorem, we give the following lemma.

Lemma 3.2 *If $(\theta, \mathfrak{g}; \epsilon)$ is a Lie 2-bialgebra, then it is equivalent to the following equations:*

$$\{\varepsilon_{00}^{12} + \varepsilon_{11}^{01} + \varepsilon_{10}^{03} + \varepsilon_{01}^{10},\ \varepsilon_{00}^{12} + \varepsilon_{11}^{01} + \varepsilon_{10}^{03} + \varepsilon_{01}^{10}\} = 0, \tag{3.1}$$

$$\{\varepsilon_{01}^{10} + \varepsilon_{21}^{00} + \varepsilon_{10}^{11} + \varepsilon_{30}^{01},\ \varepsilon_{01}^{10} + \varepsilon_{21}^{00} + \varepsilon_{10}^{11} + \varepsilon_{30}^{01}\} = 0, \tag{3.2}$$

$$\begin{aligned}&\{\varepsilon_{00}^{12},\ \varepsilon_{10}^{11}\} + \{\varepsilon_{00}^{12},\ \varepsilon_{30}^{01}\} + \{\varepsilon_{11}^{01},\ \varepsilon_{21}^{00}\}\\ &\qquad + \{\varepsilon_{11}^{01},\ \varepsilon_{10}^{11}\} + \{\varepsilon_{11}^{01},\ \varepsilon_{30}^{01}\} + \{\varepsilon_{10}^{03},\ \varepsilon_{21}^{00}\} + \{\varepsilon_{10}^{03},\ \varepsilon_{10}^{11}\} = 0.\end{aligned} \tag{3.3}$$

Proof. Let $a = \varepsilon_{00}^{12} + \varepsilon_{11}^{01} + \varepsilon_{10}^{03}$ and $b = \varepsilon_{21}^{00} + \varepsilon_{10}^{11} + \varepsilon_{30}^{01}$. Then we have

$$\begin{aligned}\{\varepsilon, \varepsilon\} &= \{l + c - \varepsilon_{01}^{10}, l + c - \varepsilon_{01}^{10}\}\\ &= \{l, l\} + \{c, c\} + 2\{l, c\} - 2\{l, \varepsilon_{01}^{10}\} - 2\{c, \varepsilon_{01}^{10}\} + \{\varepsilon_{01}^{10}, \varepsilon_{01}^{10}\}\\ &= \{l, l\} + \{c, c\} + 2\{a, b\}.\end{aligned}$$

By examining each component, we have that $\{l, l\} \in S^p(V^*[1]) \odot V[2]$, $\{c, c\} \in S^q(V[2]) \odot V^*[1]$ and $\{a, b\} \in S^k(V^*[1]) \odot S^l(V[2])$, where $p, q, k, l \geq 2$.

Hence, we have $\{\epsilon, \epsilon\} = 0$ if and only if $\{l, l\} = 0$, $\{c, c\} = 0$ and $\{a, b\} = 0$. Expanding these three terms gives the desired result. The proof is completed.

Our main theorem is now ready to be stated.

Theorem 3.1 *Given a Lie 2-coalgebra structure (δ, ω, η) on a Lie 2-algebra $(\theta, \mathfrak{g}; \phi, [\cdot, \cdot], \cdot \succ \cdot, h)$, it forms a Lie 2-bialgebra if and only if the following equations are satisfied: for all $x, y, z \in \mathfrak{g}$, $u \in \theta$,*

(1) $\delta([x, y]) = [\![x, \delta(y)]\!] - [\![\delta(x), y]\!]$;

(2) $\eta([x, y]) = x \succ \eta(y) - y \succ \eta(x)$;

(3) $\omega(x \succ u) = x \succ \omega(u) + \delta(x) \succ u$;

(4) $\omega h(x, y, z) = h(\delta(x)y, z) + \text{c.p.}$

Proof. Let $\varepsilon = \varepsilon_{00}^{12} + \varepsilon_{11}^{01} + \varepsilon_{10}^{03} + \varepsilon_{01}^{10} + \varepsilon_{21}^{00} + \varepsilon_{10}^{11} + \varepsilon_{30}^{01} \in \mathscr{S}^{(-4)}$.

According to Proposition 2.1, $\{l, l\} = 0$ is equivalent to the fact that $(\theta, \mathfrak{g})$ is a Lie 2-algebra, where $\phi(u) = D_{\varepsilon_{01}^{10}}(u)$, $[x, y] = D_{\varepsilon_{00}^{12}}(x, y)$, $x \succ u = D_{\varepsilon_{11}^{01}}(x, u)$ and $h(x, y, z) = D_{\varepsilon_{10}^{03}}(x, y, z)$ for all $x, y, z \in \mathfrak{g}$, $u \in \theta$. And Proposition 2.2 implies that $\{c, c\} = 0$ is equivalent to $(\theta, \mathfrak{g})$ being a Lie 2-coalgebra, i.e., $(\mathfrak{g}^*, \theta^*)$ is a Lie 2-algebra.

Hence, by Lemma 3.2, it suffices to prove that (3.3) is equivalent to the four conditions in this theorem.

For any $E \in S^k(V^*[1]) \odot S^l(V[2])$, we introduce a multilinear map $\overset{\vee}{D}_E$ dual to D_E by: for all $\xi_i \in \mathscr{S}^{\bullet}(V^*[1])$,

$$\overset{\vee}{D}_E(\xi_1, \xi_2, \cdots, \xi_l) = \{\{\cdots\{\{E, \xi_1\}, \xi_2\}, \cdots, \xi_{l-1}\}, \xi_l\}.$$

Then the triple of linear maps (δ, ω, η) is introduced by: for all $x \in \mathfrak{g}$, $u \in \theta$, $\kappa, \kappa_1, \kappa_2, \kappa_3 \in \theta^*$, $\xi \in \mathfrak{g}^*$,

$$\begin{aligned}
\langle \delta(x) \mid \xi \wedge \kappa \rangle &= \langle x \mid \kappa \triangleright \xi \rangle \\
&= \left\{ x, \overset{\vee}{D}_{\varepsilon_{10}^{11}}(\kappa, \xi) \right\} \\
&= \{x, \{\{\varepsilon_{10}^{11}, \kappa\}, \xi\}\} \\
&= \{\{x, \{\varepsilon_{10}^{11}, \kappa\}\}, \xi\} \\
&= \{\{\{x, \varepsilon_{10}^{11}\}, \kappa\}, \xi\} \\
&= \{\xi, \{\{x, \varepsilon_{10}^{11}\}, \kappa\}\} \\
&= -\{\{\{\varepsilon_{10}^{11}, x\}, \xi\}, \kappa\} \\
&= -\{\{D_{\varepsilon_{10}^{11}}(x), \xi\}, \kappa\}, \\
\langle \omega(u) \mid \kappa_1 \wedge \kappa_2 \rangle &= -\langle u \mid [\kappa_1, \kappa_2]_* \rangle \\
&= -\left\{ u, \overset{\vee}{D}_{\varepsilon_{21}^{00}}(\kappa_1, \kappa_2) \right\} \\
&= -\{u, \{\{\varepsilon_{21}^{00}, \kappa_1\}, \kappa_2\}\} \\
&= -\{\{u, \{\varepsilon_{21}^{00}, \kappa_1\}\}, \kappa_2\} \\
&= -\{\{\{u, \varepsilon_{21}^{00}\}, \kappa_1\}, \kappa_2\} \\
&= \{\{\{\varepsilon_{21}^{00}, u\}, \kappa_1\}, \kappa_2\} \\
&= \{\{D_{\varepsilon_{21}^{00}}(u), \kappa_1\}, \kappa_2\}, \\
\langle \eta(x) \mid \kappa_1 \wedge \kappa_2 \wedge \kappa_3 \rangle &= -\langle x \mid m(\kappa_1, \kappa_2, \kappa_3) \rangle \\
&= -\left\{ x, \overset{\vee}{D}_{\varepsilon_{30}^{01}}(\kappa_1, \kappa_2, \kappa_3) \right\} \\
&= -\{x, \{\{\{\varepsilon_{30}^{01}, \kappa_1\}, \kappa_2\}, \kappa_3\}\} \\
&= -\{\{x, \{\{\varepsilon_{30}^{01}, \kappa_1\}, \kappa_2\}\}, \kappa_3\} \\
&= -\{\{\{x, \{\varepsilon_{30}^{01}, \kappa_1\}\}, \kappa_2\}, \kappa_3\} \\
&= -\{\{\{\{\varepsilon_{30}^{01}, x\}, \kappa_1\}, \kappa_2\}, \kappa_3\} \\
&= -\{\{\{D_{\varepsilon_{30}^{01}}(x), \kappa_1\}, \kappa_2\}, \kappa_3\}.
\end{aligned}$$

Since the left hand side of (3.3) belongs to $(\odot^2\mathfrak{g}^*)\odot\mathfrak{g}\odot\theta+(\odot^2\theta)\odot\mathfrak{g}^*\odot\theta^*+(\odot^2\mathfrak{g}^*)\odot(\odot^3\theta)+(\odot^3\mathfrak{g}^*)\odot(\odot^2\theta)$, we have

$$
\begin{aligned}
&\{\{D_{\{\varepsilon^{12}_{00},\,\varepsilon^{11}_{10}\}+\{\varepsilon^{12}_{00},\,\varepsilon^{01}_{30}\}+\{\varepsilon^{01}_{11},\,\varepsilon^{00}_{21}\}+\{\varepsilon^{01}_{11},\,\varepsilon^{11}_{10}\}+\{\varepsilon^{01}_{11},\,\varepsilon^{01}_{30}\}+\{\varepsilon^{03}_{10},\,\varepsilon^{00}_{21}\}+\{\varepsilon^{03}_{10},\,\varepsilon^{11}_{10}\}}(x,\,y),\,\xi\},\,\kappa\}\\
&=\{\{D_{\{\varepsilon^{12}_{00},\,\varepsilon^{11}_{10}\}}(x,\,y)+D_{\{\varepsilon^{01}_{11},\,\varepsilon^{11}_{10}\}}(x,\,y),\,\xi\},\,\kappa\}\\
&=\{\{D_{\varepsilon^{11}_{10}}(D_{\varepsilon^{12}_{00}}(x,\,y))+D_{\varepsilon^{12}_{00}+\varepsilon^{01}_{11}}(D_{\varepsilon^{11}_{10}}(x),\,y)-D_{\varepsilon^{12}_{00}+\varepsilon^{01}_{11}}(D_{\varepsilon^{11}_{10}}(y),\,x),\,\xi\},\,\kappa\}\\
&=\langle-\delta([x,\,y])+[\![x,\,\delta(y)]\!]-[\![\delta(x),\,y]\!]\mid\xi\wedge\kappa\rangle,\\
&\{\{D_{\{\varepsilon^{12}_{00},\,\varepsilon^{11}_{10}\}+\{\varepsilon^{12}_{00},\,\varepsilon^{01}_{30}\}+\{\varepsilon^{01}_{11},\,\varepsilon^{00}_{21}\}+\{\varepsilon^{01}_{11},\,\varepsilon^{11}_{10}\}+\{\varepsilon^{01}_{11},\,\varepsilon^{01}_{30}\}+\{\varepsilon^{03}_{10},\,\varepsilon^{00}_{21}\}+\{\varepsilon^{03}_{10},\,\varepsilon^{11}_{10}\}}(x,\,u),\,\kappa_1\},\,\kappa_2\}\\
&=\{\{D_{\{\varepsilon^{01}_{11},\,\varepsilon^{00}_{21}\}}(x,\,y)+D_{\{\varepsilon^{01}_{11},\,\varepsilon^{11}_{10}\}}(x,\,y),\,\kappa_1\},\,\kappa_2\}\\
&=\{\{D_{\varepsilon^{00}_{21}}(D_{\varepsilon^{01}_{11}}(x,\,u))+D_{\varepsilon^{01}_{11}}(D_{\varepsilon^{00}_{21}}(u),\,x)+D_{\varepsilon^{01}_{11}}(D_{\varepsilon^{11}_{10}}(x),\,u),\,\kappa_1\},\,\kappa_2\}\\
&=\langle\omega(x\succ u)-x\succ\omega(u)-\delta(x)\succ u\mid\kappa_1\wedge\kappa_2\rangle,\\
&\{\{\{D_{\{\varepsilon^{12}_{00},\,\varepsilon^{11}_{10}\}+\{\varepsilon^{12}_{00},\,\varepsilon^{01}_{30}\}+\{\varepsilon^{01}_{11},\,\varepsilon^{00}_{21}\}+\{\varepsilon^{01}_{11},\,\varepsilon^{11}_{10}\}+\{\varepsilon^{01}_{11},\,\varepsilon^{01}_{30}\}+\{\varepsilon^{03}_{10},\,\varepsilon^{00}_{21}\}+\{\varepsilon^{03}_{10},\,\varepsilon^{11}_{10}\}}(x,\,y),\,\kappa_1\},\,\kappa_2\},\,\kappa_3\}\\
&=\{\{\{D_{\{\varepsilon^{12}_{00},\,\varepsilon^{01}_{30}\}}(x,\,y)+D_{\{\varepsilon^{01}_{11},\,\varepsilon^{01}_{30}\}}(x,\,y),\,\kappa_1\},\,\kappa_2\},\,\kappa_3\}\\
&=\{\{\{D_{\varepsilon^{01}_{30}}(D_{\varepsilon^{12}_{00}}(x,\,y))+D_{\varepsilon^{01}_{11}}(D_{\varepsilon^{01}_{30}}(x),\,y)-D_{\varepsilon^{01}_{11}}(D_{\varepsilon^{01}_{30}}(y),\,x),\,\kappa_1\},\,\kappa_2\},\,\kappa_3\}\\
&=\langle-\eta([x,\,y])-y\succ\eta(x)+x\succ\eta(y)\mid\kappa_1\wedge\kappa_2\wedge\kappa_3\rangle,\\
&\{\{D_{\{\varepsilon^{12}_{00},\,\varepsilon^{11}_{10}\}+\{\varepsilon^{12}_{00},\,\varepsilon^{01}_{30}\}+\{\varepsilon^{01}_{11},\,\varepsilon^{00}_{21}\}+\{\varepsilon^{01}_{11},\,\varepsilon^{11}_{10}\}+\{\varepsilon^{01}_{11},\,\varepsilon^{01}_{30}\}+\{\varepsilon^{03}_{10},\,\varepsilon^{00}_{21}\}+\{\varepsilon^{03}_{10},\,\varepsilon^{11}_{10}\}}(x,\,y,\,z),\,\kappa_1\},\,\kappa_2\}\\
&=\{\{D_{\{\varepsilon^{03}_{10},\,\varepsilon^{00}_{21}\}}(x,\,y,\,z)+D_{\{\varepsilon^{03}_{10},\,\varepsilon^{11}_{10}\}}(x,\,y,\,z),\,\kappa_1\},\,\kappa_2\}\\
&=\{\{D_{\varepsilon^{00}_{21}}(D_{\varepsilon^{03}_{10}}(x,\,y,\,z))+D_{\varepsilon^{03}_{10}}(D_{\varepsilon^{11}_{10}}(x),\,y,\,z)+c.p.,\,\kappa_1\},\,\kappa_2\}\\
&=\langle\omega h(x,\,y,\,z)-h(\delta(x),\,y,\,z)+c.p.\mid\kappa_1\wedge\kappa_2\rangle.
\end{aligned}
$$

Hence, it follows that (3.3) is equivalent to that the triple $(\delta,\,\omega,\,\eta)$ satisfies four compatibility conditions. This concludes the proof.

In the following, we give two examples of Lie 2-bialgebras to end up this paper. The first is a strict one.

Example 3.1 Consider a 2-term complex $(\mathfrak{g}\to\mathfrak{g}/\mathfrak{h},\,\pi)$, where $\mathfrak{g}$ is a Lie algebra and $\mathfrak{h}$ is one of its ideal and π is the canonical map. Equip the trivial action of $\mathfrak{g}$ on $\mathfrak{g}/\mathfrak{h}$, then $(\mathfrak{g}\to\mathfrak{g}/\mathfrak{h},\,\pi)$ is a strict Lie 2-algebra.

As in [3], any Lie 2-bialgebra structure underlying $(\mathfrak{g}\to\mathfrak{g}/\mathfrak{h},\,\pi)$ is equivalently assigning a Lie 2-algebra structure on $((\mathfrak{g}/\mathfrak{h})^*\to\mathfrak{g}^*,\,\pi^T)$.

Example 3.2 Following Example 2.1, we can construct a Lie 2-coalgebra structure on $(\mathbf{R}\to V_3)$. Let $\mathbf{R}^*$ be a dual space of $\mathbf{R}$, then we can equip a Lie bracket $[f,\,g]=fg-gf$ on $\mathbf{R}^*$. Hence $\mathbf{R}^*$ becomes an abelian Lie algebra, then we endow the trivial action of $\mathbf{R}^*$ on V_3^* and the trivial homotopy map. One can check that the maps in Example 2.1 and above make $(\mathbf{R}\to V_3)$ become a Lie 2-bialgebra.

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