Communications in Mathematical Research
34(1)(2018), 65–76

On the Coefficients of Several Classes of Bi-univalent Functions Defined by Convolution

Guo Dong[1] and Li Zong-tao[2]
(*1. Foundation Department, Chuzhou Vocational and Technical College, Chuzhou, Anhui, 239000*)
(*2. Foundation Department, Guangzhou Civil Aviation College, Guangzhou, 510403*)

Communicated by Ji You-qing

Abstract: In this paper, we introduce several new subclasses of the function class Σ of bi-univalent functions analytic in the open unit disc defined by convolution. Furthermore, we investigate the bounds of the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses. The results presented in this paper improve or generalize the recent works of other authors.

Key words: analytic and univalent functions, coefficient, bi-univalent function, Hadamard product, convolution
2010 MR subject classification: 30C45, 30C50, 30C80
Document code: A
Article ID: 1674-5647(2018)01-0065-12
DOI: 10.13447/j.1674-5647.2018.01.07

1 Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{+\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $U = \{z\colon |z| < 1\}$. Further, we denote by S the class of all functions in A which are univalent in U. A function f in S is said to be starlike of order α, $0 \le \alpha < 1$, and is denoted by $S^*(\alpha)$ if $\mathrm{Re}\left\{\dfrac{zf'(z)}{f(z)}\right\} > \alpha$, $z \in U$, and is said to be convex of order α, $0 \le \alpha < 1$, and is denoted by $K(\alpha)$ if $\mathrm{Re}\left\{1 + \dfrac{zf''(z)}{f'(z)}\right\} > \alpha$,

Received date: Nov. 9, 2016.
Foundation item: The NSF (KJ2015A372) of Anhui Provincial Department of Education.
E-mail address: gd791217@163.com (Guo D).

$z \in U$. Mocanu[1] studied linear combinations of the representations of convex and starlike functions and defined the class of α-convex functions. In [2], it was shown that if

$$\mathrm{Re}\left\{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} > 0, \qquad z \in U,$$

then f is in the class of starlike functions $S^*(0)$ for α be a real number and is in the class of convex functions $K(0)$ for $\alpha \geq 1$.

Further, We say that $f(z) \in A$ is α-starlike in U if $f(z)$ satisfies

$$f(z)f'(z)\frac{1+zf''(z)}{f'(z)} \neq 0, \qquad |z| < 1$$

and

$$\mathrm{Re}\left\{\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\alpha} > 0\right\}.$$

For such α-starlike functions, Lewandowski *et al.*[3] proved that all α-starlike functions are univalent and starlike for all α ($\alpha \in \mathbf{R}$).

In [4], it was shown that if

$$\mathrm{Re}\left(\frac{\alpha z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)}\right) > -\frac{\alpha}{2}, \qquad \alpha \geq 0,\ z \in U,$$

then $f \in S^*(0)$.

For the function $f(z) = z + \sum\limits_{n=2}^{+\infty} a_n z^n$ and $g(z) = z + \sum\limits_{n=2}^{+\infty} b_n z^n$, let $(f * g)(z)$ denote the Hadamard product or convolution of $f(z)$ and $g(z)$, defined by

$$(f * g)(z) = z + \sum_{n=2}^{+\infty} a_n b_n z^n. \tag{1.2}$$

For $0 \leq \alpha < 1$ and $\lambda \geq 0$, we let $Q_\lambda(h, \alpha)$ be the subclass of A consisting of functions $f(z)$ of the form (1.1) and functions $h(z)$ given by

$$h(z) = z + \sum_{n=2}^{+\infty} h_n z^n, \qquad h_n > 0 \tag{1.3}$$

and satisfying the analytic criterion:

$$\mathrm{Re}\left[(1-\lambda)\frac{(f*h)(z)}{z} + \lambda(f*h)'(z)\right] > \alpha, \qquad 0 \leq \alpha < 1,\ \lambda \geq 0.$$

It is easy to see that $Q_{\lambda_1}(h, \alpha) \subset Q_{\lambda_2}(h, \alpha)$ for $\lambda_1 > \lambda_2 \geq 0$. Thus, for $\lambda \geq 1$, $0 \leq \alpha < 1$, $Q_\lambda(h,\ \alpha) \subset Q_1(h,\ \alpha) = \{f, h \in A\colon \mathrm{Re}(f*h)'(z) > \alpha,\ 0 \leq \alpha < 1\}$ and hence $Q_\lambda(h,\ \alpha)$ is univalent class (see [5]–[7]).

We note that $Q_\lambda\left(\dfrac{z}{1-z},\ \alpha\right) = Q_\lambda(\alpha)$ (see [8]).

It is well known that every function $f \in S$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z, \qquad z \in U$$

and

$$f(f^{-1}(\omega)) = \omega, \qquad |\omega| < r_0(f),\ r_0(f) \geq \frac{1}{4},$$

where

$$f^{-1}(\omega)=\omega-a_2\omega^2+(2a_2^2-a_3)\omega^3-(5a_2^3-5a_2a_3+a_4)\omega^4+\cdots$$

A function $f\in A$ is said to be bi-univalent in U if both $f(z)$ and $f^{-1}(z)$ are univalent in U.

Let Σ denote the class of all bi-univalent functions in U given by (1.1). The class of bi-univalent functions was introduced by Lewin[9] in 1967 and was showed that $|a_2|<1.51$. Brannan and Clunie[10] conjectured that $|a_2|<\sqrt{2}$ for $f\in\Sigma$. Netanyahu[11] showed that $\max|a_2|=\dfrac{4}{3}$ if $f\in\Sigma$. Recently, many authors investigated bounds for various subclasses of bi-univalent functions (see [12]–[17]).

The object of the present paper is to introduce several subclasses of the function class Σ and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses of the function class Σ employing the techniques used earlier by Peng *et al.*[16]

2 Coefficient Estimates

In the sequel, it is assumed that φ is an analytic function with positive real part in the unit disk U, satisfying $\varphi(0)=1$, $\varphi'(0)>0$, and $\varphi(U)$ is symmetric with respect to the real axis. Such a function has a Taylor series of the form

$$\varphi(z)=1+B_1z+B_2z^2+B_3z^3+\cdots,\qquad B_1>0. \tag{2.1}$$

Suppose that $u(z)$ and $v(z)$ are analytic in the unit disk U with $u(0)=v(0)=0$, $|u(z)|<1$, $|v(z)|<1$, and

$$u(z)=b_1z+\sum_{n=2}^{+\infty}b_nz^n,\quad v(z)=c_1z+\sum_{n=2}^{+\infty}c_nz^n,\qquad |z|<1. \tag{2.2}$$

It is well known that (see [18], P.172)

$$|b_1|\le 1,\quad |b_2|\le 1-|b_1|^2,\quad |c_1|\le 1,\quad |c_2|\le 1-|c_1|^2. \tag{2.3}$$

By a simple calculation, we have

$$\varphi(u(z))=1+B_1b_1z+(B_1b_2+B_2b_1^2)z^2+\cdots,\qquad |z|<1, \tag{2.4}$$

$$\varphi(v(\omega))=1+B_1c_1\omega+(B_1c_2+B_2c_1^2)\omega^2+\cdots,\qquad |\omega|<1. \tag{2.5}$$

Definition 2.1 *A function $f\in\Sigma$ given by* (1.1) *is said to be in the class $M_\Sigma(h,\alpha,\varphi)$, $\alpha\ge 0$, if the following conditions are satisfied*:

$$(1-\alpha)\frac{z(f*h)'(z)}{(f*h)(z)}+\alpha\left(1+\frac{z(f*h)''(z)}{(f*h)'(z)}\right)\prec\varphi(z),\qquad z\in U$$

and

$$(1-\alpha)\frac{\omega((f*h)^{-1})'(\omega)}{(f*h)^{-1}(\omega)}+\alpha\left(1+\frac{\omega((f*h)^{-1})''(\omega)}{((f*h)^{-1})'(\omega)}\right)\prec\varphi(\omega),\qquad \omega\in U,$$

where the function $h(z)$ is given by (1.3) *and $(f*h)^{-1}(\omega)$ is defined by*:

$$(f*h)^{-1}(\omega)=\omega-a_2h_2\omega^2+(2a_2^2h_2^2-a_3h_3)\omega^3+\cdots \tag{2.6}$$

We note that for $h(z) = \dfrac{z}{1-z}$, the class $M_\Sigma(h, \alpha, \varphi)$ reduces to the class $M_\Sigma(\alpha, \varphi)$ studied by Peng *et al.* (see [16], Definition 2.3).

Theorem 2.1 *Let f given by (1.1) be in the class $M_\Sigma(h, \alpha, \varphi)$, $\alpha \ge 0$. Then*

$$|a_2| \le \frac{B_1\sqrt{B_1}}{h_2\sqrt{(1+\alpha)|B_1^2 - (1+\alpha)B_2| + B_1(1+\alpha)^2}} \tag{2.7}$$

and

$$|a_3| \le \begin{cases} \dfrac{B_1}{h_3(1+\alpha)}, & \text{if } |B_2| \le B_1; \\ \dfrac{B_1|B_1^2 - (1+\alpha)B_2| + (1+\alpha)B_1|B_2|}{h_3(1+\alpha)(|B_1^2 - (1+\alpha)B_2| + B_1(1+\alpha))}, & \text{if } |B_2| > B_1. \end{cases} \tag{2.8}$$

Proof. Let $f \in M_\Sigma(h, \alpha, \varphi)$, $\alpha \ge 0$. Then there are analytic functions $u, v\colon U \longrightarrow U$ given by (2.2) such that

$$(1-\alpha)\frac{z(f*h)'(z)}{(f*h)(z)} + \alpha\left(1 + \frac{z(f*h)''(z)}{(f*h)'(z)}\right) = \varphi(u(z)) \tag{2.9}$$

and

$$(1-\alpha)\frac{\omega((f*h)^{-1})'(\omega)}{(f*h)^{-1}(\omega)} + \alpha\left(1 + \frac{\omega((f*h)^{-1})''(\omega)}{((f*h)^{-1})'(\omega)}\right) = \varphi(v(\omega)). \tag{2.10}$$

Now, equating the coefficients in (2.9) and (2.10), we get

$$(1+\alpha)a_2h_2 = B_1b_1, \tag{2.11}$$

$$2(1+2\alpha)a_3h_3 - (1+3\alpha)a_2^2h_2^2 = B_1b_2 + B_2b_1^2, \tag{2.12}$$

$$-(1+\alpha)a_2h_2 = B_1c_1, \tag{2.13}$$

$$(3+5\alpha)a_2^2h_2^2 - 2(1+2\alpha)a_3h_3 = B_1c_2 + B_2c_1^2. \tag{2.14}$$

From (2.11) and (2.13) we get

$$b_1 = -c_1, \tag{2.15}$$

$$a_2^2 = \frac{B_1^2(b_1^2 + c_1^2)}{2h_2^2(1+\alpha)^2}. \tag{2.16}$$

Adding (2.12) and (2.13), we have

$$2(1+\alpha)a_2^2h_2^2 = B_1(b_2 + c_2) + B_2(b_1^2 + c_1^2). \tag{2.17}$$

Substituting (2.15) and (2.16) into (2.17), we get

$$b_1^2 = \frac{B_1(1+\alpha)^2(b_2 + c_2)}{2(1+\alpha)B_1^2 - 2B_2(1+\alpha)^2}. \tag{2.18}$$

Substituting (2.15) and (2.18) into (2.16), we get

$$a_2^2 = \frac{B_1^3(b_2 + c_2)}{2(1+\alpha)h_2^2(B_1^2 - B_2(1+\alpha))}. \tag{2.19}$$

Then, in view of (2.3), we have

$$(1+\alpha)h_2^2|B_1^2 - B_2(1+\alpha)||a_2|^2 \le B_1^3(1 - |b_1|^2). \tag{2.20}$$

From (2.11) and (2.20) we get

$$|a_2| \le \frac{B_1\sqrt{B_1}}{h_2\sqrt{(1+\alpha)|B_1^2 - (1+\alpha)B_2| + B_1(1+\alpha)^2}}.$$

Next, from (2.12) and (2.14) we have

$$4(1+2\alpha)(1+\alpha)a_3h_3 = (3+5\alpha)B_1b_2 + (1+3\alpha)B_1c_2 + 4(1+2\alpha)B_2b_1^2.$$

Then, in view of (2.3), we have

$$4(1+2\alpha)(1+\alpha)h_3|a_3| \le 4(1+2\alpha)B_1 + 4(1+2\alpha)[|B_2| - B_1]|b_1|^2.$$

Notice that

$$|b_1|^2 = \frac{(1+\alpha)^2h_2^2}{B_1^2}|a_2|^2 \le \frac{B_1(1+\alpha)}{|B_1^2-(1+\alpha)B_2|+B_1(1+\alpha)},$$

we get

$$|a_3| \le \begin{cases} \dfrac{B_1}{h_3(1+\alpha)}, & \text{if } |B_2| \le B_1; \\ \dfrac{B_1|B_1^2-(1+\alpha)B_2| + (1+\alpha)B_1|B_2|}{h_3(1+\alpha)(|B_1^2-(1+\alpha)B_2|+B_1(1+\alpha))}, & \text{if } |B_2| > B_1. \end{cases}$$

This completes the proof of Theorem 2.1.

Remark 2.1 Putting $h(z) = \dfrac{z}{1-z}$ in Theorem 2.1, we obtain the results obtained by Peng *et al.* (see [16], Theorem 2.3).

Example 2.1 (1) For

$$h(z) = z + \sum_{n=2}^{+\infty}\left[\frac{1+\iota+\gamma(n-1)}{1+\iota}\right]^m z^n, \qquad \iota,\gamma \ge 0,\ m \in \mathbf{N}, \tag{2.21}$$

this operator contains in turn many interesting operator (see [19]). Theorem 2.1 becomes

$$|a_2| \le \frac{B_1\sqrt{B_1}}{\left[\dfrac{1+\iota+\gamma}{1+\iota}\right]^m \sqrt{(1+\alpha)|B_1^2-(1+\alpha)B_2|+B_1(1+\alpha)^2}}$$

and

$$|a_3| \le \begin{cases} \dfrac{1}{\left[\dfrac{1+\iota+2\gamma}{1+\iota}\right]^m}\dfrac{B_1}{(1+\alpha)}, & \text{if } |B_2| \le B_1; \\ \dfrac{1}{\left[\dfrac{1+\iota+2\gamma}{1+\iota}\right]^m}\dfrac{B_1|B_1^2-(1+\alpha)B_2| + (1+\alpha)B_1|B_2|}{(1+\alpha)(|B_1^2-(1+\alpha)B_2|+B_1(1+\alpha))}, & \text{if } |B_2| > B_1. \end{cases}$$

(2) For

$$h(z) = z + \sum_{n=2}^{+\infty}\mathit{\Gamma}_{n-1}(\alpha_1)z^n, \tag{2.22}$$

where

$$\mathit{\Gamma}_{n-1}(\alpha_1) = \frac{(\alpha_1)_{n-1}\cdots(\alpha_q)_{n-1}}{(\beta_1)_{n-1}\cdots(\beta_s)_{n-1}(1)_{n-1}}, \qquad n \ge 2, \tag{2.23}$$

$q \le s+1$, $\alpha_i \in \mathbf{C}$ $(i = 1,2,\cdots,q)$, and $\beta_j \in \mathbf{C}\backslash\mathbf{Z}_0^-$ $(j = 1,2,\cdots,s)$, where $\mathbf{Z}_0^- = \{0,-1,-2,\cdots\}$, this operator contains in turn many interesting operators (see [20]). Theorem 2.1 becomes

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{|\varGamma_1(\alpha_1)|\sqrt{(1+\alpha)|B_1^2-(1+\alpha)B_2|+B_1(1+\alpha)^2}}$$

and

$$|a_3| \leq \begin{cases} \dfrac{B_1}{|\varGamma_2(\alpha_1)|(1+\alpha)}, & \text{if } |B_2| \leq B_1; \\ \dfrac{B_1|B_1^2-(1+\alpha)B_2|+(1+\alpha)B_1|B_2|}{|\varGamma_2(\alpha_1)|(1+\alpha)(|B_1^2-(1+\alpha)B_2|+B_1(1+\alpha))}, & \text{if } |B_2| > B_1. \end{cases}$$

Definition 2.2 *A function $f \in \Sigma$ given by* (1.1) *is said to be in the class $B_\Sigma(h, \lambda, \varphi)$, $\lambda \geq 0$, if the following conditions are satisfied*:

$$(1-\lambda)\frac{(f*h)(z)}{z} + \lambda(f*h)'(z) \prec \varphi(z), \qquad z \in U$$

and

$$(1-\lambda)\frac{(f*h)^{-1}(\omega)}{\omega} + \lambda((f*h)^{-1})'(\omega) \prec \varphi(\omega), \qquad \omega \in U,$$

where the function $h(z)$ is given by (1.3) *and $(f*h)^{-1}(\omega)$ is given by* (2.6).

We note that for $h(z) = \dfrac{z}{1-z}$, the class $B_\Sigma(h,\, \alpha,\, \varphi)$ reduces to the class $B_\Sigma(\lambda,\, \varphi)$ studied by Peng *et al.* (see [16], Definition 2.5). Also for $\varphi(z) = \left(\dfrac{1+z}{1-z}\right)^\alpha$, the class $B_\Sigma(h,\, \alpha,\, \varphi)$ reduces to the class $B_\Sigma(h,\, \alpha,\, \lambda)$, which is introduced and studied by EI-Ashwah (see [17], Definition 1). And for $\varphi(z) = \dfrac{1+(1-2\beta)}{1-z}$, the class $B_\Sigma(h,\, \alpha,\, \varphi)$ reduces to the class $B_\Sigma(h,\, \beta,\, \lambda)$, which is introduced and studied by EI-Ashwah (see [17], Definition 2].

Theorem 2.2 *Let f given by* (1.1) *be in the class $B_\Sigma(h,\, \lambda,\, \varphi)$, $\lambda \geq 0$. Then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{h_2\sqrt{|(1+2\lambda)B_1^2-(1+\lambda)^2B_2|+(1+\lambda)^2B_1}} \tag{2.24}$$

and

$$|a_3| \leq \begin{cases} \dfrac{B_1}{h_3(1+2\lambda)}, & \textit{if } B_1 \leq \dfrac{(1+\lambda)^2}{1+2\lambda}; \\ \dfrac{B_1|(1+2\lambda)B_1^2-(1+\lambda)^2B_2|+(1+2\lambda)B_1^3}{h_3(1+2\lambda)(|(1+2\lambda)B_1^2-(1+\lambda)^2B_2|+(1+\lambda)^2B_1)}, & \textit{if } B_1 > \dfrac{(1+\lambda)^2}{1+2\lambda}. \end{cases} \tag{2.25}$$

Proof. Let $f(z) \in B_\Sigma(h,\, \lambda,\, \varphi)$, $\lambda \geq 0$. Then there are analytic functions $u, v\colon U \longrightarrow U$ given by (2.2) such that

$$(1-\lambda)\frac{(f*h)(z)}{z} + \lambda(f*h)'(z) = \varphi(u(z)) \tag{2.26}$$

and

$$(1-\lambda)\frac{(f*h)^{-1}(\omega)}{\omega} + \lambda((f*h)^{-1})'(\omega) = \varphi(v(\omega)). \tag{2.27}$$

Since

$$(1-\lambda)\frac{(f*h)(z)}{z} + \lambda(f*h)'(z) = 1 + (1+\lambda)a_2h_2z + (1+2\lambda)a_3h_3z^2 + \cdots$$

and

$$(1-\lambda)\frac{(f*h)^{-1}(\omega)}{\omega}+\lambda((f*h)^{-1})'(\omega)=1-(1+\lambda)a_2h_2\omega+(1+2\lambda)(2a_2^2h_2^2-a_3h_3)\omega^2+\cdots,$$

it follows from (2.4), (2.5), (2.26) and (2.27) that

$$(1+\lambda)a_2h_2=B_1b_1, \tag{2.28}$$

$$(1+2\lambda)a_3h_3=B_1b_2+B_2b_1^2, \tag{2.29}$$

$$-(1+\lambda)a_2h_2=B_1c_1, \tag{2.30}$$

$$2(1+2\lambda)a_2^2h_2^2-(1+2\lambda)a_3h_3=B_1c_2+B_2c_1^2. \tag{2.31}$$

From (2.28) and (2.30) we get

$$b_1=-c_1, \tag{2.32}$$

$$a_2^2=\frac{B_1^2(b_1^2+c_1^2)}{2h_2^2(1+\lambda)^2}. \tag{2.33}$$

By adding (2.29) to (2.31), we have

$$2(1+2\lambda)a_2^2h_2^2=B_1(b_2+c_2)+B_2(b_1^2+c_1^2). \tag{2.34}$$

Substituting (2.32) and (2.33) into (2.34), we get

$$b_1^2=\frac{B_1(1+\lambda)^2(b_2+c_2)}{2(1+2\lambda)B_1^2-2B_2(1+\lambda)^2}. \tag{2.35}$$

Substituting (2.32) and (2.35) into (2.33), we get

$$a_2^2=\frac{B_1^3(b_2+c_2)}{h_2^2(2(1+2\lambda)B_1^2-2B_2(1+\lambda)^2)}. \tag{2.36}$$

Then, in view of (2.3) and (2.32), we have

$$|a_2|^2\le\frac{B_1^3(1-|b_1|^2)}{h_2^2|(1+2\lambda)B_1^2-B_2(1+\lambda)^2|}. \tag{2.37}$$

From (2.28) and (2.37) we get

$$|a_2|\le\frac{B_1\sqrt{B_1}}{h_2\sqrt{|(1+2\lambda)B_1^2-(1+\lambda)^2B_2|+(1+\lambda)^2B_1}}. \tag{2.38}$$

By subtracting (2.29) from (2.31) and a computation using (2.32) finally leads to

$$2(1+2\lambda)a_3h_3=2(1+2\lambda)a_2^2h_2^2+B_1(b_2-c_2). \tag{2.39}$$

Then, in view of (2.3) and (2.32), we have

$$\begin{aligned}2(1+2\lambda)h_3|a_3|&\le 2(1+2\lambda)h_2^2|a_2|^2+B_1(|b_2|+|c_2|)\\&\le 2(1+2\lambda)h_2^2|a_2|^2+2B_1(1-|b_1|^2).\end{aligned}$$

It follows from (2.28) that

$$(1+2\lambda)h_3B_1|a_3|\le h_2^2[(1+2\lambda)B_1-(1+\lambda)^2]|a_2|^2+B_1^2.$$

Notice that (2.38), we have

$$|a_3|\le\begin{cases}\dfrac{B_1}{h_3(1+2\lambda)}, & \text{if } B_1\le\dfrac{(1+\lambda)^2}{1+2\lambda};\\[2ex]\dfrac{B_1|(1+2\lambda)B_1^2-(1+\lambda)^2B_2|+(1+2\lambda)B_1^3}{h_3(1+2\lambda)(|(1+2\lambda)B_1^2-(1+\lambda)^2B_2|+(1+\lambda)^2B_1)}, & \text{if } B_1>\dfrac{(1+\lambda)^2}{1+2\lambda}.\end{cases}$$

This completes the proof of Theorem 2.2.

Remark 2.2 (1) Putting $h(z) = \dfrac{z}{1-z}$ in Theorem 2.2, we obtain the results obtained by Peng *et al.* (see [16], Theorem 2.5).

(2) If let $\varphi(z) = \left(\dfrac{1+z}{1-z}\right)^{\alpha} = 1 + 2\alpha z + 2\alpha^2 z^2 + \cdots$ $(0 < \alpha \le 1)$, then inequalities (2.24) and (2.25) become

$$|a_2| \le \frac{2\alpha}{h_2\sqrt{\alpha|1+2\lambda-\lambda^2| + (1+\lambda)^2}} \tag{2.40}$$

and

$$|a_3| \le \begin{cases} \dfrac{2\alpha}{h_3(1+2\lambda)}, & \text{if } 0 < \alpha \le \dfrac{(1+\lambda)^2}{2(1+2\lambda)}; \\ \dfrac{2\alpha^2|1+2\lambda-\lambda^2| + 4(1+2\lambda)\alpha^2}{h_3(1+2\lambda)(\alpha|1+2\lambda-\lambda^2| + (1+\lambda)^2)}, & \text{if } \dfrac{(1+\lambda)^2}{2(1+2\lambda)} < \alpha \le 1. \end{cases} \tag{2.41}$$

The bounds on $|a_2|$ and $|a_3|$ given in (2.40) and (2.41) are more accurate than that given by Theorem 1 in [17].

(3) If let $\varphi(z) = \dfrac{1+(1-2\beta)z}{1-z} = 1 + 2(1-\beta)z + 2(1-\beta)z^2 + \cdots$ $(0 \le \alpha < 1)$, then inequalities (2.24) and (2.25) become

$$|a_2| \le \frac{2(1-\beta)}{h_2\sqrt{|2(1+2\lambda)(1-\beta) - (1+\lambda)^2| + (1+\lambda)^2}} \tag{2.42}$$

and

$$|a_3| \le \begin{cases} \dfrac{2(1-\beta)}{h_3(1+2\lambda)}, & \text{if } \dfrac{1+2\lambda-\lambda^2}{2(1+2\lambda)} \le \beta < 1; \\ \dfrac{2(1-\beta)|2(1+2\lambda)(1-\beta) - (1+\lambda)^2| + 4(1+2\lambda)(1-\beta)^2}{h_3(1+2\lambda)(|2(1+2\lambda)(1-\beta) - (1+\lambda)^2| + (1+\lambda)^2)}, & \\ & \text{if } 0 \le \beta < \dfrac{1+2\lambda-\lambda^2}{2(1+2\lambda)}. \end{cases} \tag{2.43}$$

The bounds on $|a_2|$ and $|a_3|$ given in (2.42) and (2.43) are more accurate than that given by Theorem 2 in [17].

Definition 2.3 *A function $f \in \Sigma$ given by* (1.1) *is said to be in the class $C_\Sigma(h, \lambda, \varphi)$, $\lambda \ge 0$, if the following conditions are satisfied*:

$$\left(\frac{(f*h)(z)}{z}\right)^{\lambda}((f*h)'(z))^{1-\lambda} \prec \varphi(z), \qquad z \in U$$

and

$$\left(\frac{(f*h)^{-1}(\omega)}{\omega}\right)^{\lambda}[((f*h)^{-1})'(\omega)]^{1-\lambda} \prec \varphi(\omega), \qquad \omega \in U,$$

where the function $h(z)$ is given by (1.3) *and $(f*h)^{-1}(\omega)$ is given by* (2.6).

By applying the method of the proof of Theorem 2.2, we can prove the following result.

Theorem 2.3 *Let f given by* (1.1) *be in the class $C_\Sigma(h, \lambda, \varphi)$, $\lambda \ge 0$. Then*

$$|a_2| \le \frac{B_1\sqrt{2B_1}}{h_2\sqrt{|(\lambda^2 - 5\lambda + 6)B_1^2 - 2(2-\lambda)^2 B_2| + 2(2-\lambda)^2 B_1}}$$

and

$$|a_3| \leq \begin{cases} \dfrac{B_1}{h_3|3-2\lambda|}, & \text{if } B_1 \leq \dfrac{(2-\lambda)^2}{|3-2\lambda|}; \\ \dfrac{2B_1^3|3-2\lambda| + B_1|(\lambda^2-5\lambda+6)B_1^2 - 2(2-\lambda)^2 B_2|}{h_3|3-2\lambda|[|(\lambda^2-5\lambda+6)B_1^2 - 2(2-\lambda)^2 B_2| + 2(2-\lambda)^2 B_1]}, & \text{if } B_1 > \dfrac{(2-\lambda)^2}{|3-2\lambda|}. \end{cases}$$

Definition 2.4 *A function $f \in \Sigma$ given by* (1.1) *is said to be in the class $L_\Sigma(h, \alpha, \varphi)$, $\alpha \geq 0$, if the following conditions are satisfied*:

$$\left(\frac{z(f*h)'(z)}{(f*h)(z)}\right)^{\alpha}\left(1+\frac{z(f*h)''(z)}{(f*h)'(z)}\right)^{1-\alpha} \prec \varphi(z), \qquad z \in U$$

and

$$\left(\frac{\omega((f*h)^{-1})'(\omega)}{(f*h)^{-1}(\omega)}\right)^{\alpha}\left(1+\frac{\omega((f*h)^{-1})''(\omega)}{((f*h)^{-1})'(\omega)}\right)^{1-\alpha} \prec \varphi(\omega), \qquad \omega \in U,$$

where the function $h(z)$ is given by (1.3) *and $(f*h)^{-1}(\omega)$ is given by* (2.6). *We note that for $h(z) = \dfrac{z}{1-z}$, the class $L_\Sigma(h, \alpha, \varphi)$ reduces to the class $L_\Sigma(\alpha, \varphi)$ studied by Peng et al.* (*see* [16], *Definition* 2.4).

By applying the method of the proof of Theorem 2.1, we can prove the following result.

Theorem 2.4 *Let f given by* (1.1) *be in the class $L_\Sigma(h, \alpha, \varphi)$, $\alpha \geq 0$. Then*

$$|a_2| \leq \frac{B_1\sqrt{2B_1}}{h_2\sqrt{|(\alpha^2-3\alpha+4)B_1^2 - 2(2-\alpha)^2 B_2| + 2B_1(2-\alpha)^2}}$$

and

$$|a_3| \leq \begin{cases} \dfrac{2B_1}{h_3(\alpha^2-3\alpha+4)}, & \text{if } |B_2| \leq B_1; \\ \dfrac{2B_1|(\alpha^2-3\alpha+4)B_1^2 - 2(2-\alpha)^2 B_2| + 4(2-\alpha)^2 B_1|B_2|}{h_3(\alpha^2-3\alpha+4)(|(\alpha^2-3\alpha+4)B_1^2 - 2(2-\alpha)^2 B_2| + 2B_1(2-\alpha)^2)}, & \text{if } |B_2| > B_1. \end{cases}$$

Remark 2.3 Putting $h(z) = \dfrac{z}{1-z}$ in Theorem 2.4, we obtain the results obtained by Peng *et al.* (see [16], Theorem 2.4).

Definition 2.5 *A function $f \in \Sigma$ given by* (1.1) *is said to be in the class $ST_\Sigma(h, \alpha, \varphi)$, $\alpha \geq 0$, if the following conditions are satisfied*:

$$\frac{z(f*h)'(z)}{(f*h)(z)} + \alpha\frac{z^2(f*h)''(z)}{(f*h)(z)} \prec \varphi(z), \qquad z \in U$$

and

$$\frac{\omega((f*h)^{-1})'(\omega)}{(f*h)^{-1}(\omega)} + \alpha\frac{\omega^2((f*h)^{-1})''(\omega)}{(f*h)^{-1}(\omega)} \prec \varphi(\omega), \qquad \omega \in U,$$

where the function $h(z)$ is given by (1.3) *and $(f*h)^{-1}(\omega)$ is given by* (2.6). *We note that for $h(z) = \dfrac{z}{1-z}$, the class $ST_\Sigma(h, \alpha, \varphi)$ reduces to the class $ST_\Sigma(\alpha, \varphi)$ studied by Peng et al.* (*see* [16], *Definition* 2.2).

By applying the method of the proof of Theorem 2.1, we can prove the following result.

Theorem 2.5 *Let f given by* (1.1) *be in the class $ST_\Sigma(h, \alpha, \varphi)$, $\alpha \geq 0$. Then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{h_2\sqrt{|(1+4\alpha)B_1^2-(1+2\alpha)^2B_2|+B_1(1+2\alpha)^2}}$$

and

$$|a_3| \leq \begin{cases} \dfrac{B_1}{h_3(1+4\alpha)}, & \text{if } |B_2| \leq B_1; \\ \dfrac{B_1|(1+4\alpha)B_1^2-(1+2\alpha)^2B_2|+(1+2\alpha)^2B_1|B_2|}{h_3(1+4\alpha)(|(1+4\alpha)B_1^2-(1+2\alpha)^2B_2|+(1+2\alpha)^2B_1)}, & \text{if } |B_2| > B_1. \end{cases}$$

Remark 2.4 Putting $h(z) = \dfrac{z}{1-z}$ in Theorem 2.5, we obtain the results obtained by Peng *et al.* (see [16], Theorem 2.2).

Definition 2.6 *A function $f \in \Sigma$ given by* (1.1) *is said to be in the class $B_\Sigma(h, \lambda, k)$, $\lambda \geq 0$, $0 < k \leq 1$, if the following conditions are satisfied*:

$$\left|(1-\lambda)\frac{(f*h)(z)}{z}+\lambda(f*h)'(z)-1\right| < k, \qquad z \in U$$

and

$$\left|(1-\lambda)\frac{(f*h)^{-1}(\omega)}{\omega}+\lambda((f*h)^{-1})'(\omega)-1\right| < k, \qquad \omega \in U,$$

where the function $h(z)$ is given by (1.3) *and $(f*h)^{-1}(\omega)$ is given by* (2.6).

Theorem 2.6 *Let f given by* (1.1) *be in the class $B_\Sigma(h, \lambda, k)$, $\lambda \geq 0$, $0 < k \leq 1$. Then*

$$|a_2| \leq \frac{k}{h_2\sqrt{(1+2\lambda)k+(1+\lambda)^2}}$$

and

$$|a_3| \leq \begin{cases} \dfrac{k}{h_3(1+2\lambda)}, & \text{if } k \leq \dfrac{(1+\lambda)^2}{1+2\lambda}; \\ \dfrac{2k^2}{h_3[(1+2\lambda)k+(1+\lambda)^2]}, & \text{if } k > \dfrac{(1+\lambda)^2}{1+2\lambda}. \end{cases}$$

Proof. Let $f(z) \in B_\Sigma(h, \lambda, k)$, $\lambda \geq 0$, $0 < k \leq 1$. Then there are analytic functions $u, v \colon U \longrightarrow U$ given by (2.2) such that

$$(1-\lambda)\frac{(f*h)(z)}{z}+\lambda(f*h)'(z) = 1-ku(z) \tag{2.44}$$

and

$$(1-\lambda)\frac{(f*h)^{-1}(\omega)}{\omega}+\lambda((f*h)^{-1})'(\omega) = 1-kv(\omega). \tag{2.45}$$

Now, equating the coefficients in (2.44) and (2.45), we get

$$(1+\lambda)a_2h_2 = -kb_1, \tag{2.46}$$

$$(1+2\lambda)a_3h_3 = -kb_2, \tag{2.47}$$

$$-(1+\lambda)a_2h_2 = -kc_1, \tag{2.48}$$

$$2(1+2\lambda)a_2^2h_2^2 - (1+2\lambda)a_3h_3 = -kc_2. \tag{2.49}$$

From (2.46) and (2.48) we get

$$b_1 = -c_1. \tag{2.50}$$

By adding (2.47) to (2.49), we have

$$2(1+2\lambda)a_2^2h_2^2 = -k(b_2+c_2). \tag{2.51}$$

From (2.3) and (2.51) we have

$$|a_2| \le \frac{k}{h_2\sqrt{(1+2\lambda)k+(1+\lambda)^2}}. \tag{2.52}$$

Subtracting (2.47) from (2.49) we have

$$2(1+2\lambda)a_3h_3 = 2(1+2\lambda)a_2^2h_2^2 + k(c_2-b_2). \tag{2.53}$$

Then, in view of (2.3) and (2.53), we have

$$\begin{aligned} 2(1+2\lambda)h_3|a_3| &\le 2(1+2\lambda)h_2^2|a_2|^2 + k(|c_2|+|b_2|) \\ &\le 2(1+2\lambda)h_2^2|a_2|^2 + 2k(1-|b_1|^2). \end{aligned}$$

It follows from (2.46) that

$$(1+2\lambda)h_3k|a_3| \le h_2^2[(1+2\lambda)k-(1+\lambda)^2]|a_2|^2 + k^2.$$

Notice that (2.52), we have

$$|a_3| \le \begin{cases} \dfrac{k}{h_3(1+2\lambda)}, & \text{if } k \le \dfrac{(1+\lambda)^2}{1+2\lambda}; \\ \dfrac{2k^2}{h_3[(1+2\lambda)k+(1+\lambda)^2]}, & \text{if } k > \dfrac{(1+\lambda)^2}{1+2\lambda}. \end{cases}$$

This completes the proof of Theorem 2.6.

References

[1] Mocanu P T. Une propriete de convexite generalisee dans la theorie de la representation conforme. *Mathematica (Cluj)*, 1969, **11**: 127–133.

[2] Miller S S, Mocanu P T, Reade M O. All α-convex functions are univalent and starlike. *Proc. Amer. Math. Soc.*, 1973, **37**: 553–554.

[3] Lewandowski Z, Miller S, Zlotkiewicz E. Gamma-starlike functions. *Annales Universitatis Mariae Curie Sklodowska Sect. A*, 1974, **28**: 53–58.

[4] Li J L, Owa S. Sufficient conditions for starlikness. *Indian. J. Pure. Appl. Math.*, 2002, **33**: 303–318.

[5] Chen M. On the regular functions satisfying $\mathrm{Re}\Big(\dfrac{f(z)}{z} > \alpha\Big)$. *Bull. Inst. Math. Acad. Sinica*, 1975, **3**: 65–70.

[6] Chichra P N. New subclasses of the class of close-to-convex functions. *Proc. Amer. Math. Soc.*, 1977, **62**: 37–43.

[7] MacGregor T H. Functions whose derivative has a positive real part. *Trans. Amer. Math. Soc.*, 1962, **104**: 532–537.

[8] Ding S S, Ling Y, Bao G J. Some properties of a class of analytic functions. *J. Math. Anal. Appl.*, 1995, **195**(1): 71–81.

[9] Lewin M. On a coefficient problem for bi-univalent functions. *Proc. Amer. Math. Soc.*, 1967, **18**(1): 63–68.

[10] Brannan D A, Clunie J G. Aspects of Contemporary Complex Analysis. Proceedings of the NATO Advanced Study Instute Held at the University of Durham, Durham, July 1-20, 1979. New York: Academic Press, 1980.

[11] Netanyahu E. The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$. *Arch. Ration. Mech. Anal.*, 1969, **32**: 100–112.

[12] Lashin A Y. On certain subclasses of analytic and bi-univalent functions. *J. Egyptian Math. Soc.*, 2016, **24**: 220–225.

[13] Ularu N. Coefficient estimates for bi-univalent Bazilevic functions. *Anal. Thoery Appl.*, 2014, **30**(3): 275–280.

[14] Orhan H, Magesh N, Balaji V K. Fekete-Szegö problem for certain class of Ma-Minda bi-univalent functions. *Afr. mat.*, Publishing online: 25 September, 2015.

[15] Srivastava H M, Murugusundaramoorthy G, Vijaya K. Coefficient estimates for some families of bi-bazilevic functions of the Ma-Minda type involving the Hohlov operator. *J. Clssical Analysis*, 2013, **2**(2): 167–181.

[16] Peng Z G, Han Q Q. On the coefficients of several classes of bi-univalent functions. *Acta Math. Sci.*, 2014, **34B**(1): 228–240.

[17] EI-Ashwah R M. Subclasses of bi-univalent functions defined by convolution. *J. Egyptian Math. Soc.*, 2014, **22**: 348–351.

[18] Nehari Z. Conformal Mapping. New York: McGraw-Hill Book Co., 1952.

[19] Catas A, Oros G I, Oros G. Differential subordinations associated with multiplier transformations. *Abstr. Appl. Anal.*, 2008, ID 845724: 1–11.

[20] Dziok J, Srivastava H M. Classes of analytic functions associated with the generalized hypergeometric function. *Appl. Math. Comput.*, 1999, **103**: 1–13.