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The Mod 2 Kauffman Bracket Skein Module of Thickened Torus

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Abstract: Framed links in thickened torus are studied. We define the mod 2 Kauffman bracket skein module of thickened torus and give an expression of a framed link in this module. From this expression we propose a new ambient isotopic invariant of framed links.

Key words: framed link, thickened torus, mod 2 Kauffman bracket skein module

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1 Introduction

We are concerned with framed links in thickened torus $T^2 \times I$ by using skein theory. We will extend the Kauffman bracket skein module to the mod 2 Kauffman bracket skein module and obtain an expression of a framed link as a new ambient isotopic invariant.

Skein relations have their origin in an observation by Alexander[1], Conway found a way to calculate the Alexander polynomial of a link using a so-called skein relation[2]. This is an equation that relates the polynomial of a link to the polynomial of links obtained by changing the crossings in a projection of the original link. Skein modules were introduced by Przytycki in [3]. Skein modules are quotients of free modules over ambient isotopy classes of framed links in a 3-manifold by properly chosen local skein relations. The skein module based on Kauffman bracket skein relation is one of the most extensively studied object of the algebraic topology based on framed links, which is also an important invariant of 3-manifolds. There have been extensive study and application of Kauffman bracket skein

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module (see [4]–[8]).

A convenient way of representing a framed link in an orientable 3-manifold M is in the form of smoothly embedded closed bands ($\sqcup_{j=1}^{k} S_j^1 \times I \hookrightarrow M$), such that bands for different components do not intersect. For a framed link, let ϕ_j be the linking number between the knots $S_j^1 \times \{0\} \hookrightarrow M$ and $S_j^1 \times \{1\} \hookrightarrow M$ associated with each component of the framed link, $j = 1, \cdots, k$, we call $(\phi_1, \cdots, \phi_k)$ the framing of the framed link.

If we work with regular projections of links, then the topology of links is reflected by Reidemeister moves. Regular isotopy is the equivalent relation on link projections generated by the Reidemeister moves of types II and III. The Reidemeister moves of types II and III on the cores of bands extend to the bands themselves, while the type I move dose not extend (it corresponds to a full twist on the band). Consequently, regular isotopy corresponds to ambient isotopy of framed links.

Noted that torus knot is a kind of knot that had been investigated and used widely (see [9]). We are concerned in this paper with the torus knot, which is defined below. Given two generators x_1, x_2 in $\pi_1(T^2)$, where

$$x_1 : S^1 \hookrightarrow T^2, \quad x_1(\mathrm{e}^{\mathrm{i}\theta}) = (\mathrm{e}^{\mathrm{i}\theta}, 1),$$
$$x_2 : S^1 \hookrightarrow T^2, \quad x_2(\mathrm{e}^{\mathrm{i}\theta}) = (1, \mathrm{e}^{\mathrm{i}\theta}),$$

and consider the closed curve

$$\gamma : S^1 \hookrightarrow T^2, \quad \gamma(\mathrm{e}^{\mathrm{i}\theta}) = x_1^p x_2^q.$$

If $(p, q) = (0, 0)$ or p, q are relatively prime, then γ is called a (p, q) knot in T^2, denoted by $K_{(p,q)}$. Obviously,

$$K_{(1,0)} = x_1, \qquad K_{(0,1)} = x_2.$$

This paper is organized by two sections: In Section 2, we cover the necessary definitions and lemmas. The main result and its proof are provided in Section 3.

2 Preliminary

The data that determine a knot in $\mathbf{R}^3$ are usually given by a projection onto a plane. Now we derive it in thickened surface $F \times I$ as in $\mathbf{R}^3$.

Definition 2.1[10] *Let F be a compact orientable surface, L be a framed link in the thickened surface $F \times I$. Suppose $r : \sqcup_{j=1}^{k} S_j^1 \times \{0\} \to \sqcup_{j=1}^{k} S_j^1 \times I$, $p : F \times I \to F \times \{0\}$, we call the composition mapping $p \cdot L \cdot r : \sqcup_{j=1}^{k} S_j^1 \to F$ a projection of L onto F, denoted by ℓ.*

Definition 2.2[10] *A projection ℓ of a framed link L is called regular if*

(1) *ℓ is an immerse;*

(2) *there are only finitely many intersections in ℓ and all intersections are double points;*

(3) *ℓ is transverse to the every intersection point.*

Moreover, if the upper crossing line and the lower crossing line are marked at every double point in a regular projection, then this regular projection of a link is called a link projection.

We work always in the smooth-category. We do not make any distinction between two ambient isotopic framed links, while two framed links, L and L', in M are said to be ambient isotopic if there is a smooth orientation preserving automorphism $h: M \to M$ such that $h(L) = L'$.

Then we give the definition of the Kauffman bracket skein module of $F \times I$ for an oriented surface F and an interval I as follows:

Definition 2.3[3], [11] *The Kauffman bracket skein module of* 3*-manifold* $F \times I$, $S_{2,\infty}(F \times I; R, A)$ *is defined as follows: Let* $\mathcal{L}$ *be the set of unoriented framed links in* $F \times I$ (*including the empty knot* $\emptyset$), R *any commutative ring with identity and* A *an invertible element in* R. *Let* $R\mathcal{L}$ *be the free* R*-module generated by* $\mathcal{L}$, $S_{2,\infty}$ *be the submodule of* $R\mathcal{L}$ *generated by two skein expressions*: $L_+ - AL_0 - A^{-1}L_\infty$, $L \sqcup T_1 + (A^2 + A^{-2})L$, *where the triple* L_+, L_0 *and* L_∞ *as presented by their regular projections* ℓ_+, ℓ_0 *and* ℓ_∞ *on* F *are shown in Fig.* 2.1, *which can be ambient isotopy except within the neighborhood shown, and* T_1 *denotes the trivial framed knot. Set* $S_{2,\infty}(F \times I; R, A) = R\mathcal{L}/S_{2,\infty}$. *The notation is shortened for special case*:

$$S_{2,\infty}(F \times I) = S_{2,\infty}(F \times I; \mathbf{Z}[A^{\pm 1}], A).$$

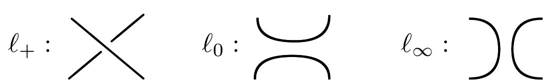

Fig. 2.1 Link projections

From the above definition, we have:

Proposition 2.1 *For a framed link* L *in* $F \times I$, *its expression in* $S_{2,\infty}(F \times I)$ *is an ambient isotopic invariant of* L.

Proof. Suppose that two framed links L and L' are ambient isotopic in $F \times I$, then the link projection of L is obtained from the link projection of L' by a sequence of Reidemeister moves $R2(R2^{-1})$ or $R3(R3^{-1})$. Suppose that L and L' are presented in the free module $S_{2,\infty}(F \times I)$ as $L = \sum\limits_i f_i(A)c_i$ and $L' = \sum\limits_{i'} f'_{i'}(A)c_{i'}$, using skein expressions of Definition 2.3, it is easy to obtain that $\sum\limits_i f_i(A)c_i$ is unchanged in $S_{2,\infty}(F \times I)$ by Reidemeister moves $R2(R2^{-1})$ or $R3(R3^{-1})$, so

$$\sum_i f_i(A)c_i = \sum_{i'} f'_{i'}(A)c_{i'} \in S_{2,\infty}(F \times I).$$

Moreover, the expression of L in the free module is unique. Hence, it is an ambient isotopic invariant of the framed link L.

The following lemmas are used later.

Lemma 2.1[8] $S_{2,\infty}(F \times I; R, A)$ *is a free* R*-module with a basis* $B(F)$ *consisting of links in* F *without contractible components* (*but including the empty knot*).

Lemma 2.2[9] *Suppose that* $K_{(p,q)}$ *and* $K_{(p',q')}$ *are two knots in* T^2. *If* $K_{(p,q)} \cap K_{(p',q')} = \emptyset$, *then* $K_{(p,q)} = K_{(p',q')}$ *or one of them is* $K_{(0,0)}$.

3 Main Results

Beginning with our main ingredient, we consider the special case $T^2 \times I$. Given $S_{2,\infty}(T^2 \times I; R, A)$ the Kauffman bracket skein module of $T^2 \times I$ and

$$S_{2,\infty}(T^2 \times I) = S_{2,\infty}(T \times I;\, \mathbf{Z}[A^{\pm 1}],\, A).$$

For a framed link L in $T^2 \times I$, by Lemma 2.1, its expression in the free module $S_{2,\infty}(T^2 \times I)$ is presented as

$$f = \sum_i f_i(A) \sqcup_{k=1}^{n_i} K_{(p_i,\, q_i)},$$

where $K_{(p_i,\, q_i)}$ is a $(p_i,\, q_i)$ *knot* in T^2.

Definition 3.1 *For any two $K_{(p,\,q)}$ and $K_{(p',\,q')}$ in T^2, we define an equivalent relation $\sim$: $K_{(p,\,q)} \sim K_{(p',\,q')}$ if and only if $p - p' \equiv q - q' \equiv 0 \pmod 2$, then we call quotient module $S_{2,\infty}(T^2 \times I)/\sim$ the mod 2 Kauffman bracket skein module of $T^2 \times I$, denoted by $S^{\mathbf{Z}_2}_{2,\infty}(T^2 \times I)$. The equivalent class of $K_{(p,\,q)}$ is denoted by $\overline{K_{(p,\,q)}}$.*

Theorem 3.1 *Let L be a framed link in $T^2 \times I$. Then there exists a unique $\overline{K_{(p,\,q)}}$ such that L is presented in $S^{\mathbf{Z}_2}_{2,\infty}(T^2 \times I)$ as $L = \sum\limits_i f_i(A) \sqcup_{k=1}^{n_i} \overline{K_{(p,\,q)}}$. Furthermore, $\overline{K_{(p,\,q)}}$ in this expression is an ambient isotopic invariant of L.*

Proof. Suppose that L is a framed link in $T^2 \times I$ with framing $(\phi_1, \cdots, \phi_k)$. By skein expression, we have

$$L^{(1)} = -A^3 L \in S_{2,\infty}(T^2 \times I),$$

where $L^{(1)}$ denotes a link obtained from L by twisting the framing of L by a full twist in a positive direction. It follows that for L, there exists a framed link L' with blackboard framing (this framing is obtained by converting each component to a band lying flat on F), its framing is denoted by $(\phi'_1, \cdots, \phi'_k)$, such that

$$L = (-A^3)^{\left(\sum\limits_{i=1}^{k} \phi_i - \sum\limits_{i=1}^{k} \phi'_i\right)} L' \in S_{2,\infty}(T^2 \times I).$$

Suppose that $\ell : \sqcup_{j=1}^{k} S^1_j \to T^2$ is a link projection of L' on T^2, and its crossing set is $\{x_1, \cdots, x_{V(\ell)}\}$. For an arbitrary crossing $x_i \in \{x_1, \cdots, x_{V(\ell)}\}$, $\ell(x_i, 0) : \sqcup_{j=1}^{k} S^1_j \to T^2$ and $\ell(x_i, \infty) : \sqcup_{j=1}^{k} S^1_j \to T^2$ respectively denote ℓ_0 and ℓ_∞ at crossing x_i, as depicted in Fig. 2.1 within the neighborhood of x_i. So we obtain state $S = (s_1, \cdots, s_{V(\ell)})$, and $s_i = 0$ or ∞ respectively denotes the choice of $\ell(x_i, 0)$ or $\ell(x_i, \infty)$. Obviously, there are $2^{V(\ell)}$ different states.

By the first skein expression of Definition 2.3, we also have

$$L_+ = AL_0 + A^{-1} L_\infty \in S_{2,\infty}(T^2 \times I),$$

it is followed that

$$L' = \sum_S A^{a(S) - b(S)} L_S \in S_{2,\infty}(T^2 \times I),$$

where S denotes one of the states of ℓ;

$$a(S) = \sharp\{s_i \mid s_i \text{ is the component of } S,\ s_i = 0\},$$

$$b(S) = \sharp\{s_i \mid s_i \text{ is the component of } S,\ s_i = \infty\},$$

obviously, $a(S)+b(S)=V(\ell)$; L_S denotes the framed link without crossing in the state S, including contractible components. In fact, L_S is a union of some (p,q) knots in T^2.

Nextly, by the second skein expression of Definition 2.3, we see

$$L\sqcup T_1=(-A^2-A^{-2})L\in S_{2,\infty}(T^2\times I),$$

so we get

$$L_S=(-A^2-A^{-2})^{m(S)}\sqcup_{k=1}^{n(S)}K_{(p_S,q_S)}\in S_{2,\infty}(T^2\times I),$$

where $m(S)$ denotes the number of contractible components, and $n(S)$ denotes the number of uncontractible components.

From the above analysis we obtain

$$L=(-A^3)^{\left(\sum\limits_{i=1}^{k}\phi_i-\sum\limits_{i=1}^{k}\phi_i'\right)}\sum_S A^{a(S)-b(S)}(-A^2-A^{-2})^{m(S)}\sqcup_{k=1}^{n(S)}K_{(p_S,q_S)}\in S_{2,\infty}(T^2\times I).$$

By Lemma 2.1, the expression of L is unique after collecting the link terms.

It remains to show that if there is one different component between two states $S=(s_1,\cdots,s_{V(\ell)})$ and $S'=(s_1',\cdots,s_{V(\ell)}')$, then even though $n_S\neq n_{S'}$, we can obtain $\overline{K_{(p_S,\,q_S)}}=\overline{K_{(p_{S'},\,q_{S'})}}$.

Assume $s_i=0$, $s_i'=\infty$, while for any $j\neq i$, $s_j=s_j'$. Firstly we consider $K_{(p_S,\,q_S)}$ and $K_{(p_{S'},\,q_{S'})}\in\pi_1(T^2)$. We can consider the loops in $\pi_1(T^2)$. We define four paths near the crossing x_i to be α_{WS}, α_{ES}, α_{EN}, α_{WN}: I, 0, $1\to F$, $\alpha_*(0)$, $\alpha_*(1)$.

Without loss of generality we assume that

$$\alpha_{WS}(0)=\alpha_{ES}(1)=\alpha_{WN}(0)=\alpha_{EN}(1)=x_i,$$

which decides the direction of four paths as shown in Fig. 3.1.

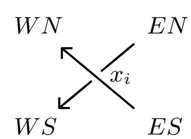

Fig. 3.1 The directions of four paths

We are now in a position to discuss the following three cases:

Case 1. Provided that $\alpha_{WS}(1)=\alpha_{ES}(0)$, $\alpha_{WN}(1)=\alpha_{EN}(0)$, we have

$$\ell=\alpha_{WS}\circ\alpha_{ES}\circ\alpha_{WN}\circ\alpha_{EN},$$

while

$$\ell(x_i,\,0)=\alpha_{WS}\circ\alpha_{ES}\sqcup\alpha_{WN}\circ\alpha_{EN},$$

$$\ell(x_i,\,\infty)=\alpha_{WS}\circ\alpha_{ES}\circ\alpha_{EN}^{-1}\circ\alpha_{WN}^{-1},$$

see Fig. 3.2.

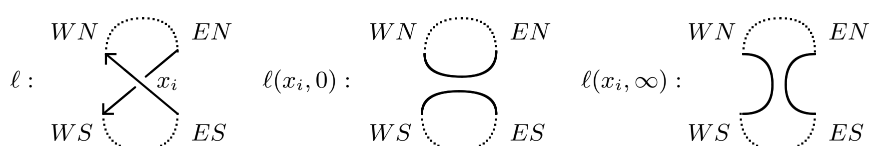

Fig. 3.2 The link projections for Case 1

Notice that in $\ell(x_i,\,0)$, $\alpha_{WN}\circ\alpha_{EN}\cap\alpha_{WS}\circ\alpha_{ES}=\emptyset$. Using Lemma 2.2, we can now derive $\alpha_{WN}\circ\alpha_{EN}=\alpha_{WS}\circ\alpha_{ES}$ or one of $\alpha_{WN}\circ\alpha_{EN}$, $\alpha_{WS}\circ\alpha_{ES}$ is $K_{(0,\,0)}$. The results of these calculations are given in Table 3.1.

It should be noted that the subtraction here is a formal subtraction, and the result of each subtraction is obtained by the analysis of actual $\ell(x_i,\,0)$ and $\ell(x_i,\,\infty)$. Besides, for unoriented (p,q) knot, $K_{(p,\,q)}=K_{(-p,\,-q)}$.

Table 3.1 The results for Case 1

$\ell(x_i,\,0)$	$\ell(x_i,\,\infty)$
$\alpha_{WN}\circ\alpha_{EN},\,\alpha_{WS}\circ\alpha_{ES}$	$\alpha_{WS}\circ\alpha_{ES}\circ\alpha_{EN}^{-1}\circ\alpha_{WN}^{-1}$
$K_{(0,\,0)},\,K_{(0,\,0)}$	$K_{(0,\,0)}-K_{(0,\,0)}=K_{(0,\,0)}$
$K_{(0,\,0)},\,K_{(p,\,q)}$	$K_{(p,q)}-K_{(0,\,0)}=K_{(p,\,q)}$
$K_{(p,\,q)},\,K_{(0,\,0)}$	$K_{(0,\,0)}-K_{(p,\,q)}=K_{(-p,\,-q)}$
$K_{(p,\,q)},\,K_{(p,\,q)}$	$K_{(p,q)}-K_{(p,\,q)}=K_{(0,\,0)}$

It follows from Table 3.1 that there exists a unique $K_{(p,\,q)}\in\pi_1(T^2)$, such that

$$\ell(x_i,0),\,\ell(x_i,\,\infty)\in\{K_{(0,\,0)},\,K_{(p,\,q)}\}.$$

Case 2. Provided that $\alpha_{WS}(1)=\alpha_{EN}(0),\ \alpha_{WN}(1)=\alpha_{ES}(0)$, we have

$$\ell=\alpha_{WS}\circ\alpha_{EN}\cup\alpha_{WN}\circ\alpha_{ES},$$

while

$$\ell(x_i,\,0)=\alpha_{WS}\circ\alpha_{EN}\circ\alpha_{WN}\circ\alpha_{ES},$$
$$\ell(x_i,\,\infty)=\alpha_{WS}\circ\alpha_{EN}\circ\alpha_{ES}^{-1}\circ\alpha_{WN}^{-1},$$

see Fig. 3.3.

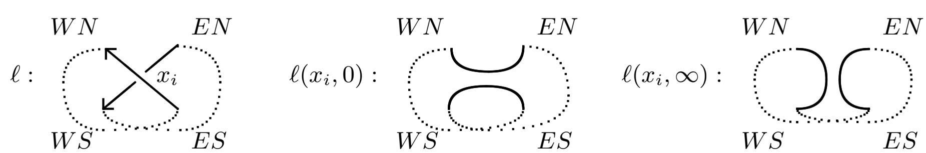

Fig. 3.3 The link projections for Case 2

Assume $\alpha_{WS}\circ\alpha_{EN}=K_{(a,\,b)},\alpha_{WN}\circ\alpha_{ES}=K_{(c,\,d)}$. Then

$$\ell(x_i,\,0)=\alpha_{WS}\circ\alpha_{EN}\circ\alpha_{WN}\circ\alpha_{ES}=K_{(a+c,\,b+d)},$$
$$\ell(x_i,\,\infty)=\alpha_{WS}\circ\alpha_{EN}\circ\alpha_{ES}^{-1}\circ\alpha_{WN}^{-1}=K_{(a-c,\,b-d)}.$$

We see

$$(a+c)-(a-c)\equiv(b+d)-(b-d)\equiv 0\,(\mathrm{mod}\,2).$$

So there exists a unique $\overline{K_{(p,\,q)}}=\overline{K_{(a+c,b+d)}}$ such that $\ell(x_i,\,0),\ell(x_i,\,\infty)\in\{\overline{K_{(p,\,q)}}\}$.

Case 3. Provided that $\alpha_{WS}(1)=\alpha_{WN}(1),\ \alpha_{EN}(0)=\alpha_{ES}(0)$, we have

$$\ell=\alpha_{WS}\circ\alpha_{WN}^{-1}\circ\alpha_{ES}^{-1}\circ\alpha_{EN},$$

while

$$\ell(x_i, 0) = \alpha_{WS} \circ \alpha_{WN}^{-1} \circ \alpha_{EN}^{-1} \circ \alpha_{ES},$$
$$\ell(x_i, \infty) = \alpha_{WS} \circ \alpha_{WN}^{-1} \sqcup \alpha_{EN} \circ \alpha_{ES}^{-1},$$

see Fig. 3.4.

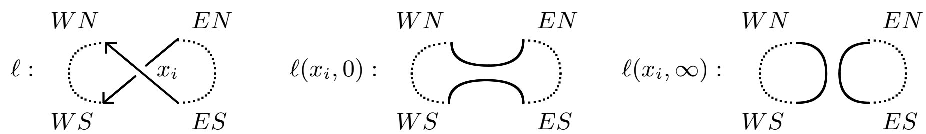

Fig. 3.4 The link projections for Case 3

By Lemma 2.2, we also have the results in the Table 3.2.

Table 3.2 The results for Case 3

$\ell(x_i,\, 0)$	$\ell(x_i,\, \infty)$
$\alpha_{WS} \circ \alpha_{WN}^{-1} \circ \alpha_{EN}^{-1} \circ \alpha_{ES}$	$\alpha_{WS} \circ \alpha_{WN}^{-1},\, \alpha_{EN} \circ \alpha_{ES}^{-1}$
$K_{(0,\,0)} - K_{(0,\,0)} = K_{(0,\,0)}$	$K_{(0,\,0)},\, K_{(0,\,0)}$
$K_{(0,\,0)} - K_{(p,\,q)} = K_{(-p,\,-q)}$	$K_{(0,\,0)},\, K_{(p,\,q)}$
$K_{(p,\,q)} - K_{(0,\,0)} = K_{(p,\,q)}$	$K_{(p,\,q)},\, K_{(0,\,0)}$
$K_{(p,\,q)} - K_{(p,\,q)} = K_{(0,0)}$	$K_{(p,\,q)},\, K_{(p,\,q)}$

It follows from Table 3.2 that there exists a unique $K_{(p,q)} \in \pi_1(T^2)$, such that

$$\ell(x_i, 0),\, \ell(x_i, \infty) \in \{K_{(0,0)}, K_{(p,q)}\}.$$

No matter what the case is, even if there is one different component between two states $S = (s_1, \cdots, s_{V(\ell)})$ and $S' = (s'_1, \cdots, s'_{V(\ell)})$, we have $\overline{K_{(p_S,q_S)}} = \overline{K_{(p_{S'},q_{S'})}}$. Using the same argument, we can easily show that even if there are k $(1 \le k \le V(\ell))$ different components between two states $S = (s_1, \cdots, s_{V(\ell)})$ and $S' = (s'_1, \cdots, s'_{V(\ell)})$, we also have $\overline{K_{(p_S,q_S)}} = \overline{K_{(p_{S'},q_{S'})}}$ in the expression

$$\begin{aligned} L = {} & (-A^3)^{\left(\sum\limits_{i=1}^{k} \phi_i - \sum\limits_{i=1}^{k} \phi'_i\right)} \sum_S A^{a(S)-b(S)} (-A^2 - A^{-2})^{m(S)} \sqcup_{k=1}^{n_S} \overline{K_{(p_S,q_S)}} \\ & \in S_{2,\infty}^{\mathbf{Z}_2}(T^2 \times I). \end{aligned}$$

Hence, there exist a unique $\overline{K_{(p,q)}}$, such that L is presented in $S_{2,\infty}^{\mathbf{Z}_2}(T^2 \times I)$ as

$$L = \sum_i f_i(A) \sqcup_{k=1}^{n_i} \overline{K_{(p,q)}}.$$

We finally remark that $\overline{K_{(p,q)}}$ in this expression is an ambient isotopic invariant of L due to

$$L = \sum_i f_i(A) \sqcup_{k=1}^{n_i} \overline{K_{(p,q)}}$$

being an ambient isotopic invariant of L by Proposition 2.1.

We complete the proof.

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