COMMUNICATIONS IN MATHEMATICAL RESEARCH
35(1)(2019), 1–9

On the Reducibility of a Class of Linear Almost Periodic Differential Equations

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Communicated by Li Yong

Abstract: In this paper, we use KAM methods to prove that there are positive measure Cantor sets such that for small perturbation parameters in these Cantor sets a class of almost periodic linear differential equations are reducible.

Key words: almost periodic, reducibility, KAM iteration
2010 MR subject classification: 37C10, 70H08
Document code: A
Article ID: 1674-5647(2019)01-0001-09
DOI: 10.13447/j.1674-5647.2019.01.01

1 Introduction and the Main Result

This paper considers the reducibility of the following system

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = [\boldsymbol{A} + \varepsilon \boldsymbol{Q}(t)]\, \boldsymbol{x}, \tag{1.1}$$

where $\boldsymbol{A}$ is an $r \times r$ constant matrix, $\boldsymbol{Q}(t)$ is an $r \times r$ almost periodic matrix with respect to t, and ε is a small perturbation parameter.

We say that a function f is a quasiperiodic function of time t with basic frequencies $\boldsymbol{\omega} = (\omega_1, \omega_2, \cdots, \omega_d)$, if $f(t) = F(\theta_1, \theta_2, \cdots, \theta_d)$, where F is 2π periodic in all its arguments and $\theta_n = \omega_n t$ for $n = 1, 2, \cdots, d$. f is called analytic quasiperiodic in a strip of width ρ if F is analytical on

$$D_\rho = \{\boldsymbol{\theta} \mid |\Im\theta_m| \leq \rho,\, m = 1, 2, \cdots, r\}.$$

In this case we denote the norm by

$$\|f\|_\rho = \sum_{k\in\mathbf{Z}^d} |F_k| \mathrm{e}^{\rho|k|}.$$

A function f is almost periodic, if $f(t) = \sum\limits_{n=1}^{\infty} f_n(t)$, where $f_n(t)$ are all quasiperiodic for $n = 1, 2, \cdots$.

Received date: Nov. 15, 2017.
Foundation item: The NSF (11571327) of China and NSF (ZR2013AM026) of Shandong Province.
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A change of variables $\boldsymbol{x} = \boldsymbol{P}(t)\boldsymbol{y}$ is a Lyapunov-Perron (LP) transform if $\boldsymbol{P}$ is non-singular, and $\boldsymbol{P}$, $\boldsymbol{P}^{-1}$ and $\dot{\boldsymbol{P}}$ are bounded. Moreover, if $\boldsymbol{P}$, $\boldsymbol{P}^{-1}$ and $\dot{\boldsymbol{P}}$ are almost periodic, the change $\boldsymbol{x} = \boldsymbol{P}(t)\boldsymbol{y}$ is called almost periodic LP transformation. If there is an almost periodic LP transformation changing the equation (1.1) into $\dot{\boldsymbol{y}} = \boldsymbol{B}\boldsymbol{y}$, the equation (1.1) is called reducible.

If $\boldsymbol{Q} = (q_{mn})$ is periodic the reducibility in all cases is given by the classical Floquet theory. If $\boldsymbol{Q} = (q_{mn})$ is quasiperiodic and the eigenvalues of $\boldsymbol{A}$ are all different, Jorba-Simó[1] proved that if the eigenvalues of $\boldsymbol{A}$ and the frequencies of $\boldsymbol{Q} = (q_{mn})$ satisfy some non-resonant conditions and non-degeneracy conditions, there is a positive measure Cantor set E such that for $\varepsilon \in E$ the equation (1.1) is reducible. Xu[2] proved the similar result when $\boldsymbol{Q} = (q_{mn})$ is quasiperiodic and the eigenvalues of $\boldsymbol{A}$ are multiple. If $\boldsymbol{Q} = (q_{mn})$ is almost periodic, the reducible problem seems difficult to study. The difficulty comes from the description of related "non-resonant condition" for the infinitely many frequencies. Xu and You[3], under the " spacial structure" described in [4] and some non-resonant conditions, obtained reducible result for (1.1) by KAM method when the eigenvalues of $\boldsymbol{A}$ are all different. In this paper, we are going to study the reducibility for the system (1.1) when $\boldsymbol{Q} = (q_{mn})$ is almost periodic and the eigenvalues of $\boldsymbol{A}$ are multiple.

Now let us introduce the "space structure" and "approximation function" and some related definitions.

Definition 1.1[4] *Let τ consist of the subsets of natural numbers set $\mathbf{N}$. $(\tau, [\cdot])$ is called finite spacial structure in $\mathbf{N}$, if τ satisfies*

(1) $\emptyset \in \tau$;

(2) *if* $\Lambda_1, \Lambda_2 \in \tau$, *then* $[\Lambda_1 \cup \Lambda_2] \le [\tau]$;

(3) $\bigcup\limits_{\Lambda \in \tau} \Lambda = \mathbf{N}$.

And $[\cdot]$ is a weight function, i.e., $[\emptyset] = 0$, $[\Lambda_1 \cup \Lambda_2] \le [\Lambda_1] + [\Lambda_2]$.

Definition 1.2 *Let $\boldsymbol{k} \in \mathbf{Z}^{\mathbf{N}}$. Denote the support set of $\boldsymbol{k}$ by*

$$\mathrm{supp}\boldsymbol{k} = \{(n_1, n_2, \cdots, n_l) \mid k_m \ne 0,\ m = n_1, n_2, \cdots, n_l,\ k_m = 0,\ m = \textit{other number}\}.$$

Denote the weight value by

$$[\boldsymbol{k}] = \inf_{\mathrm{supp}\boldsymbol{k} \subset \Lambda, \Lambda \in \tau} [\Lambda].$$

Write $|\boldsymbol{k}| = \sum\limits_{i=1}^{\infty} |k_i|$.

Assume that $\boldsymbol{Q}(t) = (q_{mn}(t))$ is a quasiperiodic $r \times r$ matrix. If for all $m, n = 1, 2, \cdots, r$, $q_{mn}(t)$ are analytic on

$$D_\rho = \{\boldsymbol{\theta} \mid |\Im \theta_m| \le \rho,\ m = 1, 2, \cdots, r\},$$

then $\boldsymbol{Q}(t)$ is called analytic on the strip D_ρ. Denote the norm by

$$\|\boldsymbol{Q}(t)\|_\rho = r \times \max_{1 \le m,n \le r} \|q_{mn}(t)\|_\rho.$$

If $\boldsymbol{Q}(t) = \sum\limits_{\Lambda \in \tau} \boldsymbol{Q}_\Lambda(t)$, where $\boldsymbol{Q}_\Lambda(t)$ are quasiperiodic matrices with basic frequencies $\boldsymbol{\omega}_\Lambda = \{\omega_i \mid i \in \Lambda\}$, then $\boldsymbol{Q}(t)$ is called almost periodic matrix with spatial structure $(\tau, [\cdot])$ and basic frequencies $\boldsymbol{\omega}$, which is the maximum subset of $\bigcup \boldsymbol{\omega}_{\Lambda}$ in the sense of integer modular. Denote the average of $\boldsymbol{Q}(t)$ by $\bar{\boldsymbol{Q}} = (\bar{q}_{mn})$, where

$$\bar{q}_{mn} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} q_{mn}(t)\mathrm{d}t.$$

Definition 1.3 *Let* $\boldsymbol{Q}(t) = \sum\limits_{\Lambda\in\tau} \boldsymbol{Q}_{\Lambda}(t)$. *For* $z > 0,\ \rho > 0$,

$$|||\boldsymbol{Q}(t)|||_{z,\rho} = \sum_{\Lambda\in\tau} \mathrm{e}^{z[\Lambda]} \|\boldsymbol{Q}_{\Lambda}(t)\|_{\rho}$$

is called weight norm with finite spatial structure $(\tau,\ [\,\cdot\,])$.

Definition 1.4 [5] *Δ is called an approximation function, if*

(1) $\Delta : [0,\infty) \to [1,\ \infty)$ *is an increasing function and satisfies* $\Delta(0) = 1$;

(2) $\dfrac{\log \Delta(t)}{t}$ *is decreasing on* $[0,\ \infty)$;

(3) $\displaystyle\int_0^{\infty} \frac{\log \Delta(t)}{t^2}\mathrm{d}t < \infty$.

Remark 1.1 If Δ is an approximation function, then so is Δ^4.

The non-resonant conditions that we use are

$$|\lambda_m - \lambda_n - \mathrm{i}\langle \boldsymbol{k},\ \boldsymbol{\omega}\rangle| \geq \frac{\alpha}{\Delta^4(|\boldsymbol{k}|)\Delta^4([\boldsymbol{k}])},$$

for all $1 \leq m \neq n \leq r$ and $\boldsymbol{k} \in \mathbf{Z}^{\mathbf{N}}\backslash\{0\}$, where $\lambda_1, \lambda_2, \cdots, \lambda_r$ are the eigenvalues of $\boldsymbol{A}$, $\boldsymbol{\omega}$ is the basic frequencies of $\boldsymbol{Q}(t)$, $\Delta(t)$ is an approximation function satisfying

$$\sum_{\boldsymbol{k}\in\mathbf{Z}^{\mathbf{N}}} \frac{1}{\Delta(|\boldsymbol{k}|)\Delta([\boldsymbol{k}])} < \infty,$$

and α is a small positive constant. From [3] and [4], we can choose the weight function

$$[\Lambda] = 1 + \sum_{i\in\Lambda} \log^p(1 + |i|), \qquad p > 2$$

and $\Delta(t)$ such that there exist $\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_r$ and $\boldsymbol{\omega} = (\omega_1,\ \omega_2,\ \cdots,\ \omega_d,\ \cdots)$ satisfying the small divisor conditions.

Here the main theorem is given.

Theorem 1.1 *Assume that* $\boldsymbol{A}$ *has* r *eigenvalues, i.e.,* $\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_r$, *where* $\lambda_m \leq \lambda_n$ *for* $1 \leq m < n \leq r-2$, *and* $\lambda_{r-1} = \lambda_r$. *Assume also that* $\boldsymbol{Q}(t)$ *is an analytic almost periodic matrix on* D_{ρ} *with frequencies* $\boldsymbol{\omega} = (\omega_1,\ \omega_2,\ \cdots,\ \omega_d,\ \cdots)$ *and depends continuously on the perturbation parameter* ε. *If*

(1) *there exists* z_0 *such that* $|||\boldsymbol{Q}(t)|||_{z_0,\rho_o} < \infty$;

(2) (*non-resonant conditions*)

$$|\lambda_m - \lambda_n - \mathrm{i}\langle \boldsymbol{k},\ \boldsymbol{\omega}\rangle| \geq \frac{\alpha}{\Delta(|\boldsymbol{k}|)\Delta([\boldsymbol{k}])}$$

for all $m,\ n = 1, 2, \cdots, r$ *and* $\boldsymbol{k} \in \mathbf{Z}^{\mathbf{N}}\backslash\{0\}$;

(3) (*non-degeneracy conditions*)

$$a_m = \bar{q}_{mm}$$

for $1 \le m \le r-2$ *and the matrix*

$$\begin{pmatrix} \bar{q}_{r-1,r-1} & \bar{q}_{r-1,r} \\ \bar{q}_{r,r-1} & \bar{q}_{r,r} \end{pmatrix}$$

has two different eigenvalues q_{r-1} *and* q_r *such that*

$$a_m = q_m,$$

for $m = r-1, r$ *and*

$$|a_m - a_n| \ge 2\delta,$$

for all $1 \le m < n \le r$*, then there exists a* $\varepsilon_0 > 0$ *and Cantor subset* $E \subset (0, \varepsilon_0)$ *with positive Lebesgue measure, such that* (1.1) *is reducible for* $\varepsilon \in E$.

Moreover, the almost periodic LP transformation has the same basic frequencies and spatial structure as $\boldsymbol{Q}(t)$*, and if* ε *is sufficiently small, the relative measure of* E *is close to* 1.

In Section 2, we give some notations and some lemmas to make the iteration efficient. In Section 3, we give the iteration lemma to ensure the iteration continuing and prove the Theorem 1.1.

2 Notations and Auxiliary Lemma

Let $\boldsymbol{Q}(t) = \widetilde{\boldsymbol{Q}}(t) + \boldsymbol{R}$, $\bar{\boldsymbol{A}} = \boldsymbol{A} + \varepsilon\boldsymbol{R}$, where the matrix is

$$\boldsymbol{R} = \begin{pmatrix} \bar{q}_{11} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \bar{q}_{22} & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \bar{q}_{r-2,r-2} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \bar{q}_{r-1,r-1} & \bar{q}_{r-1,r} \\ 0 & 0 & 0 & \cdots & 0 & \bar{q}_{r,r-1} & \bar{q}_{r,r} \end{pmatrix}.$$

Due to the non-degeneracy conditions of the Theorem 1.1, the matrix $\bar{\boldsymbol{A}}$ has r different eigenvalues. So we can find a matrix $\boldsymbol{B}$ such that

$$\boldsymbol{B}^{-1}\bar{\boldsymbol{A}}\boldsymbol{B} = \boldsymbol{A}^*,$$

where $\boldsymbol{A}^*$ is a diagonal matrix. The eigenvalues of $\boldsymbol{A}^*$ are $\lambda_1^*, \lambda_2^*, \cdots, \lambda_r^*$, where $\lambda_m^* = \lambda_m + \varepsilon\bar{q}_{mm}$ for $m = 1, 2, \cdots, r-2$, $\lambda_{r-1}^* = \lambda_{r-1} + \varepsilon q_{r-1}$ and $\lambda_r^* = \lambda_r + \varepsilon q_r$.

Let $\boldsymbol{x} = \boldsymbol{B}\boldsymbol{z}$, and then the equation (1.1) can be written as

$$\dot{\boldsymbol{z}} = \left[\boldsymbol{A}^* + \varepsilon\boldsymbol{Q}'(t)\right]\boldsymbol{z},$$

where

$$\boldsymbol{Q}'(t) = \boldsymbol{B}^{-1}\widetilde{\boldsymbol{Q}}(t)\boldsymbol{B}.$$

Let

$$\boldsymbol{z} = (\boldsymbol{I} + \varepsilon\boldsymbol{P}(t))\,\boldsymbol{y},$$

where $\boldsymbol{I}$ is the identity matrix. The equation is changed into

$$\dot{\boldsymbol{y}} = \left[(\boldsymbol{I}+\varepsilon\boldsymbol{P})^{-1}(\boldsymbol{A}^* + \varepsilon(\boldsymbol{A}^*\boldsymbol{P} - \dot{\boldsymbol{P}} + \boldsymbol{Q}'(t))) + \varepsilon^2(\boldsymbol{I}+\varepsilon\boldsymbol{P})^{-1}\boldsymbol{Q}'(t)\boldsymbol{P}\right]\boldsymbol{y}.$$

We hope to obtain the form of the equation like

$$\dot{\boldsymbol{y}} = \left[\boldsymbol{A}^* + \varepsilon^2 \boldsymbol{Q}''(t)\right] \boldsymbol{y},$$

where

$$\boldsymbol{Q}''(t) = (\boldsymbol{I} + \varepsilon \boldsymbol{P})^{-1} \boldsymbol{B}^{-1} \boldsymbol{Q}'(t) \boldsymbol{B} \boldsymbol{P}.$$

So we have to ensure that

$$(\boldsymbol{I} + \varepsilon \boldsymbol{P})^{-1} \left[\boldsymbol{A}^* + \varepsilon(\boldsymbol{A}^* \boldsymbol{P} - \dot{\boldsymbol{P}} + \boldsymbol{Q}'(t))\right] = \boldsymbol{A}^*,$$

i.e., the equation

$$\dot{\boldsymbol{P}} = \boldsymbol{A}^* \boldsymbol{P} - \boldsymbol{P} \boldsymbol{A}^* + \boldsymbol{Q}'' \tag{2.1}$$

has a solution $\boldsymbol{P}(t)$.

Write $\boldsymbol{Q}''(t) = (q''_{mn}(t))$, and the average of $\boldsymbol{Q}''(t)$ is denoted by $\bar{\boldsymbol{Q}}''(t) = (\bar{q}''_{mn}(t))$, where $\bar{q}''_{mn}(t)$ is the average value of $q''_{mn}(t)$. Due to the above calculation, there are

$$\bar{q}''_{mm} = 0, \quad \text{for } m = 1, 2, \cdots, r, \qquad \bar{q}_{r-1,r} = \bar{q}_{r,r-1} = 0.$$

Let

$$\boldsymbol{Q}''_{\Lambda} = (q''_{\Lambda mn}), \qquad (q''_{\Lambda mn}) = \sum_{\mathrm{supp}\boldsymbol{k} \subset \Lambda} q''^{\boldsymbol{k}}_{\Lambda mn} \mathrm{e}^{\mathrm{i}\langle \boldsymbol{k}, \boldsymbol{\theta} \rangle}.$$

$$\boldsymbol{P}_{\Lambda} = (p_{\Lambda mn}), \qquad (p^{\boldsymbol{k}}_{\Lambda mn}) = \sum_{\mathrm{supp}\boldsymbol{k} \subset \Lambda} p^{\boldsymbol{k}}_{\Lambda mn} \mathrm{e}^{\mathrm{i}\langle \boldsymbol{k}, \boldsymbol{\theta} \rangle}.$$

Substituting to the equation (2.1), one can see

$$p^{\boldsymbol{k}}_{\Lambda mn} = \frac{q''^{\boldsymbol{k}}_{\Lambda mn}}{\mathrm{i}\langle \boldsymbol{k}, \boldsymbol{\theta} \rangle + \lambda^*_m - \lambda^*_n}, \qquad \boldsymbol{k} \neq 0,$$

$$p^0_{\Lambda mn} = \begin{cases} \dfrac{q''^0_{\Lambda mn}}{\lambda^*_m - \lambda^*_n}, & m < r-1 \text{ or } n < r-1; \\ 0, & m = n \text{ or } m, n = r-1, r. \end{cases}$$

Assume that

$$|\lambda^*_m - \lambda^*_n - \mathrm{i}\langle \boldsymbol{k}, \boldsymbol{\omega} \rangle| \geq \frac{\alpha}{\Delta^4(|\boldsymbol{k}|)\Delta^4([\boldsymbol{k}])},$$

for $\boldsymbol{k} \in \mathbf{Z}^{\mathbf{N}} \backslash \{0\}$, and

$$|\lambda^*_m - \lambda^*_n| \geq \delta,$$

for $m \neq n$ and $m \leq r-1$ or $n \leq r-1$. Then for $0 < \overline{\rho} < \rho$

$$\begin{aligned} \|p_{\Lambda mn}\|_{\rho - \bar{\rho}} &\leq \sum_{\mathrm{supp}\boldsymbol{k} \subset \Lambda} |p^{\boldsymbol{k}}_{\Lambda mn}| \mathrm{e}^{(\rho - \bar{\rho})|\boldsymbol{k}|} + \frac{|q''^0_{\Lambda}|}{\delta} \\ &\leq \sum_{\mathrm{supp}\boldsymbol{k} \subset \Lambda} \frac{\Delta^4(|\boldsymbol{k}|)\mathrm{e}^{-\bar{\rho}|\boldsymbol{k}|}}{\alpha} \Delta^4([\boldsymbol{k}]) |q^{\boldsymbol{k}}_{\Lambda mn}| \mathrm{e}^{\rho|\boldsymbol{k}|} + \frac{|q''^0_{\Lambda}|}{\delta} \\ &\leq c \frac{\Gamma(\bar{\rho})\Delta^4([\Lambda])}{\alpha} \|q_{\Lambda mn}\|_{\rho}, \end{aligned}$$

where $\Gamma(\rho) = \sup\limits_{t \geq 0} [\Delta^4(t) \mathrm{e}^{-\rho t}]$ and $c > 1$. Thus

$$\|P_{\Lambda}\|_{\rho - \bar{\rho}} \leq c \frac{\Gamma(\bar{\rho})\Delta^4([\Lambda])}{\alpha} \|Q_{\Lambda}\|_{\rho}.$$

Let $\boldsymbol{P} = \sum\limits_{\Lambda\in\tau} \boldsymbol{P}_{\Lambda}$. From the definition of weight norm, we have

$$\begin{aligned}|||\boldsymbol{P}|||_{z-\bar{z},\rho-\bar{\rho}} &= \sum_{\Lambda\in\tau}\|\boldsymbol{P}_{\Lambda}\|_{\rho-\bar{\rho}}\mathrm{e}^{(z-\bar{z})[\Lambda]}\\ &\le c\sum_{\Lambda\in\tau}\frac{\Gamma(\bar{\rho})\Delta^4([\Lambda])}{\alpha}\|\boldsymbol{Q}_{\Lambda}\|_{\rho}\mathrm{e}^{z[\Lambda]-\bar{z}[\Lambda]}\\ &\le c\frac{\Gamma(\bar{\rho})\Gamma(\bar{z})}{\alpha}|||\boldsymbol{Q}|||_{z,\rho}.\end{aligned}$$

So the equation (2.1) has a solution

$$\boldsymbol{P} = \sum_{\Lambda\in\tau}\boldsymbol{P}_{\Lambda}.$$

The later steps are given in the Section 3, and the next lemma ensures that the equations (2.1) in the later steps have solutions.

Lemma 2.1 [3] *Assume that*

(1) $\boldsymbol{A} = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_r)$;

(2) $\boldsymbol{Q}(t) = \sum\limits_{\Lambda\in\tau}\boldsymbol{Q}_{\Lambda}(t)$ *is an almost periodic matrix and has finite spacial structure* $(\tau, [\,\cdot\,])$ *and* $|||\boldsymbol{Q}(t)|||_{z,\rho} < \infty,\ z > 0,\ \rho > 0,\ \bar{\boldsymbol{Q}} = \boldsymbol{0}$.

If for all $m, n = 1, 2, \cdots, r,$ *and* $\boldsymbol{k}\in\mathbf{Z}^{\mathbf{N}}\backslash\{0\}$,

$$|\lambda_m - \lambda_n - \mathrm{i}\langle\boldsymbol{k}, \boldsymbol{\omega}\rangle| \ge \frac{\alpha}{\Delta^4(|\boldsymbol{k}|)\Delta^4([\boldsymbol{k}])},$$

where $\boldsymbol{\omega} = (\omega_1, \omega_2, \cdots, \omega_d, \cdots)$ *are the basic frequencies of* $\boldsymbol{Q}(t)$ *and* $\Delta(t)$ *is approximation function, then there exists almost periodic matrix* $\boldsymbol{P}(t)$ *has the same spatial structure and basic frequencies as* $\boldsymbol{Q}(t)$ *such that*

$$\dot{\boldsymbol{P}}(t) = \boldsymbol{A}\boldsymbol{P}(t) - \boldsymbol{P}(t)\boldsymbol{A} + \boldsymbol{Q}(t).$$

Furthermore, for $0 < \bar{z} < z,\ 0 < \bar{\rho} < \rho$,

$$|||\boldsymbol{P}(t)|||_{z-\bar{z},\rho-\bar{\rho}} \le c\frac{\Gamma(\bar{z})\Gamma(\bar{\rho})}{\alpha}|||\boldsymbol{Q}(t)|||_{z,\rho},$$

where $c > 1$ *and* $\Gamma(\rho) = \sup\limits_{t\ge 0}[\Delta^4(t)\mathrm{e}^{-\rho t}]$.

According to the definitions of $\boldsymbol{A}^*$, $\boldsymbol{Q}''$ and Lemma 3.2, we can see that the conditions in the above lemma are satisfied to obtain a solution to the equation (2.1).

3 Iteration and Proof of the Main Result

In this section, our goal is to repeat the process in Section 2 for several times to obtain a series of transformations $\boldsymbol{P}^l$ and equations like

$$\dot{\boldsymbol{y}} = \left[\boldsymbol{A}_{l+1} + \varepsilon^{2^{l+1}}\boldsymbol{Q}_{l+1}(t)\right]\boldsymbol{y},$$

and to prove that $\boldsymbol{P}^l \to \boldsymbol{P}$, $\varepsilon^{2^{l+1}}\boldsymbol{Q}_l(t) \to 0$, $\boldsymbol{A}_l \to \boldsymbol{B}$, where $\boldsymbol{P}$ is an almost periodic matrix and $\boldsymbol{B}$ is a constant matrix, when $l \to \infty$.

Now, we start to build the iteration. Let $\bar{z}_\nu \downarrow 0$ and $\bar{\rho}_\nu \downarrow 0$ satisfy

$$\sum_{\nu=0}^{\infty}\bar{z}_\nu = \frac{1}{2}z_0, \qquad \sum_{\nu=0}^{\infty}\bar{\rho}_\nu = \frac{1}{2}\rho_0.$$

And set $z_l = z_0 - \sum_{\nu=1}^{l} \bar{z}_\nu$, $\rho_l = \rho_0 - \sum_{\nu=1}^{l} \bar{\rho}_\nu$ and $|||\cdot|||_l = |||\cdot|||_{z_l,\rho_l}$.

Assume that

$$\varphi(\rho) = \inf_{\rho_1+\rho_2+\cdots<\rho} \prod_{\nu=1}^{\infty} [\Gamma(\rho_\nu)]^{2^{-\nu-1}},$$

then

$$\varphi\left(\frac{1}{2}z_0\right) = \prod_{\nu=1}^{\infty} [\Gamma(\bar{z}_\nu)]^{2^{-\nu-1}}$$

and

$$\varphi\left(\frac{1}{2}\rho_0\right) = \prod_{\nu=1}^{\infty} [\Gamma(\bar{\rho}_\nu)]^{2^{-\nu-1}}.$$

Let

$$\dot{\boldsymbol{x}}_l = \left[\boldsymbol{A}_l + \varepsilon^{2^l} \boldsymbol{Q}_l(t)\right] \boldsymbol{x}_l, \tag{3.1}$$

where $\boldsymbol{A}_l = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_r)$ and $|||\boldsymbol{Q}_l(t)|||_n < \infty$. Let

$$\bar{\boldsymbol{A}}_l = \boldsymbol{A}_l + \varepsilon^{2^l} \bar{\boldsymbol{Q}}_l, \qquad \boldsymbol{Q}_l(t) = \bar{\boldsymbol{Q}}_l + \widetilde{\boldsymbol{Q}}_l(t),$$

where $\bar{\boldsymbol{Q}}_l$ is the average of $\boldsymbol{Q}_l$, the basic frequencies of $\boldsymbol{Q}_l(t)$ are $\boldsymbol{\omega}$, and the eigenvalues of $\bar{\boldsymbol{A}}_l$ are $\lambda_1^{l+1}, \lambda_2^{l+1}, \cdots, \lambda_r^{l+1}$. So there exists a matrix $\boldsymbol{B}_l$ such that

$$\boldsymbol{B}_l^{-1} \bar{\boldsymbol{A}}_l \boldsymbol{B}_l = \boldsymbol{A}_{l+1} = \mathrm{diag}(\lambda_1^{l+1}, \lambda_2^{l+1}, \cdots, \lambda_r^{l+1}).$$

Let $\boldsymbol{x}_l = \boldsymbol{B}_l \boldsymbol{z}_l$. Then the equation (3.1) is changed into

$$\dot{\boldsymbol{z}}_l = \left[\boldsymbol{A}_{l+1} + \varepsilon^{2^l} \boldsymbol{Q}_l''(t)\right] \boldsymbol{z}_l.$$

Assume that

$$|\lambda_m^{l+1} - \lambda_n^{l+1} - \mathrm{i}\langle \boldsymbol{k}, \boldsymbol{\omega}\rangle| \geq \frac{\alpha_l}{2\Delta^4(|\boldsymbol{k}|)\Delta^4([\boldsymbol{k}])},$$

where $\alpha_l = \dfrac{\alpha}{(l+1)^2}$, for all $k \in \mathbf{Z^N}\backslash\{0\}$ and $m, n = 1, 2, \cdots, r$. Due to Lemma 2.1, there exists transformation $\boldsymbol{z}_l = (\boldsymbol{I} + \varepsilon^{2^l} \boldsymbol{P}_l(t)) \boldsymbol{x}_{l+1}$ which changes the above equation into

$$\dot{\boldsymbol{x}}_{l+1} = \left[\boldsymbol{A}_{l+1} + \varepsilon^{2^{l+1}} \boldsymbol{Q}_{l+1}(t)\right] \boldsymbol{x}_{l+1}, \tag{3.2}$$

where

$$\boldsymbol{Q}_{l+1}(t) = (\boldsymbol{I} + \varepsilon^{2^l} \boldsymbol{P}_l(t))^{-1} \boldsymbol{B}_l^{-1} \widetilde{\boldsymbol{Q}}_l(t) \boldsymbol{B}_l P_l(t),$$

$\boldsymbol{P}_l(t) = \sum_{\Lambda\in\tau} \boldsymbol{P}_{l\Lambda}(t)$ satisfies

$$|||\boldsymbol{P}_l(t)|||_{l+1} \leq \frac{2c\Gamma(\bar{z}_{l+1})\Gamma(\bar{\rho}_{l+1})}{\alpha_l} |||\boldsymbol{B}_l^{-1} \widetilde{\boldsymbol{Q}}_l(t) \boldsymbol{B}_l|||_l, \tag{3.3}$$

Set

$$c_1 = \max\left\{c, \frac{16c}{\alpha}\right\}, \qquad c_l = \left[(l+1)^{2^{-(l+1)}} l^{2^{-l}} \cdots 2^{2^{-2}} \cdot 1^{2^{-1}}\right]^2,$$

$$\Phi_l(z) = \prod_{\nu=1}^{l+1} \left[\Gamma(\bar{z}_\nu)\right]^{2^{-\nu}}, \qquad \Phi_l(\rho) = \prod_{\nu=1}^{l+1} \left[\Gamma(\bar{\rho}_\nu)\right]^{2^{-\nu}}.$$

From [4], c_l, $\Phi_l(z)$, $\Phi_l(\rho)$ are all convergent when l goes to infinity. Let

$$M = \max\left\{1, \sup_l \{c_1 c_l \Phi_l(z) \Phi_l(\rho)\}\right\} |||\boldsymbol{Q}(t)|||_{z_0,\rho_0}.$$

Lemma 3.1 [3] *There exists a* $\varepsilon_3 > 0$ *such that when* $\varepsilon \in (0,\, \varepsilon_3)$, *if*

(1) $\min\limits_{m\neq n} |\lambda_m^l - \lambda_n^l| > \dfrac{1}{2}\lambda$;

(2) $|||\boldsymbol{Q}_l(t)|||_l \le M^{2^l}$,

then there exist $\boldsymbol{B}_l$, $\boldsymbol{P}_l(t)$ *such that the transformation*

$$\boldsymbol{x}_l = \boldsymbol{T}_l(t)\boldsymbol{x}_{l+1},$$

where

$$\boldsymbol{T}_l(t) = (\boldsymbol{I} + \varepsilon^{2^l}\boldsymbol{P}_l)\boldsymbol{B}_l$$

can change the equation (3.1) *into* (3.2).

Moreover, the following conclusions hold:

(1) $\min\limits_{m\neq n} |\lambda_m^{l+1} - \lambda_n^{l+1}| > \min\limits_{m\neq n} |\lambda_m^l - \lambda_n^l| - 2(\varepsilon M)^{2^l}$;

(2) $|||\boldsymbol{B}_l^{-1}|||\,|||\boldsymbol{B}_l||| \le 2$ *and* $|||(\boldsymbol{I} + \varepsilon^{2^l}\boldsymbol{P}_l)^{-1}||| \le 2$;

(3) $\boldsymbol{Q}_{l+1} = (\mathrm{I} + \varepsilon^{2^l}\boldsymbol{P}_l)^{-1}\boldsymbol{B}_l^{-1}\widetilde{\boldsymbol{Q}}_l(t)\boldsymbol{B}_l\boldsymbol{P}_l$ *and* $|||\boldsymbol{Q}_{l+1}|||_{l+1} \le M^{2^{l+1}}$.

Remark 3.1 There exists a ε_3 such that if $\varepsilon \in (0, \varepsilon_3)$ the above iteration can keep going.

Before proving Theorem 1.1, an important lemma is introduced. It is the restatement of Theorem B in [3].

Lemma 3.2 [3] *Assume that the eigenvalues* $\lambda_m^0(\varepsilon)(m = 1, 2, \cdots, r)$ *of*

$$\bar{\boldsymbol{A}} = \boldsymbol{A} + \varepsilon\bar{\boldsymbol{Q}}$$

satisfy

$$\left|\frac{\mathrm{d}}{\mathrm{d}\varepsilon}(\lambda_m^0(\varepsilon) - \lambda_n^0(\varepsilon))|_{\varepsilon=0}\right| \ge 2\delta > 0, \qquad 1 \le m < n \le r.$$

Let

$$\phi_{mn}^l = \lambda_m^l - \lambda_n^l = \lambda_m^0(\varepsilon) - \lambda_n^0(\varepsilon) + \delta_{mn}^l(\varepsilon)\varepsilon^2.$$

If there exists a $\varepsilon_1 > 0$ *and an* $M > 0$ *such that the condition* (2) *of Theorem* 1.1 *holds and* $|\delta_{mn}^l(\varepsilon)| \le M$, *for* $\varepsilon \in (0,\, \varepsilon_1)$, *then there exists* $\varepsilon_2 \le \varepsilon_1$ *and non-empty subset* $E \subset (0, \varepsilon_2)$ *such that*

$$|\phi_{mn}^l(\varepsilon) - \mathrm{i}\langle \boldsymbol{k}, \boldsymbol{\omega}\rangle| \ge \frac{\alpha_l}{2\Delta^4(|\boldsymbol{k}|)\Delta^4([\boldsymbol{k}])},$$

for all $\varepsilon \in E$, $m, n = 1, 2, \cdots, r$, $\boldsymbol{k} \in \mathbf{Z}^{\mathbf{N}}\backslash\{0\}$ *and* $l \ge 1$, *where*

$$\alpha_l = \frac{\alpha}{(l+1)^2}.$$

Furthermore, if ε_2 *is sufficiently small, the relative measure of* $E \in (0,\, \varepsilon_2)$ *is close to* 1.

Proof of Theorem 1.1 Set

$$\varepsilon_0 = \min\{\varepsilon_2,\, \varepsilon_3\}, \qquad \phi_{mn}^l(\varepsilon) = \lambda_m^l(\varepsilon) - \lambda_n^l(\varepsilon).$$

Due to Lemma 3.2, there exists non-empty Cantor subset $E \subset (0,\, \varepsilon_0)$ such that

$$|\phi_{mn}^l(\varepsilon) - \mathrm{i}\langle \boldsymbol{k}, \boldsymbol{\omega}\rangle| \ge \frac{\alpha_l}{2\Delta^4(|\boldsymbol{k}|)\Delta^4([\boldsymbol{k}])}$$

for all $\varepsilon \in E$, m, $n = 1, 2, \cdots, r$, $\boldsymbol{k} \in \mathbf{Z}^{\mathbf{N}} \backslash \{0\}$ and $l \geq 1$, where

$$\alpha_l = \frac{\alpha}{(l+1)^2}.$$

So there exists a sequence of matrices

$$\boldsymbol{T}_l = (\boldsymbol{I} + \varepsilon^{2^l} \boldsymbol{P}_l)\boldsymbol{B}_l, \qquad l \geq 1.$$

One goal is to prove that the composition of all the transformations $\boldsymbol{T}_l$ is convergent. From (3.3) and a direct calculation,

$$|||\varepsilon^{2^l} \boldsymbol{P}_l|||_{l+1} \leq (\varepsilon c_2 M)^{2^l}.$$

Due to Lemma 7 in [1],

$$\begin{aligned} \|\boldsymbol{T}_l\|_l &\leq \left[1 + (\varepsilon c_2 M)^{2^l}\right] \left[1 + \frac{(r-1)\|\varepsilon^{2^l} \bar{\boldsymbol{Q}}_l\|}{\lambda - 2\|\varepsilon^{2^l} \bar{\boldsymbol{Q}}_l\|}\right] \\ &\leq (1 + a_l)(1 + b_l). \end{aligned}$$

It is obvious that a_l and b_l go to zero when l goes to infinity and that the series

$$\sum_{l=0}^{\infty} a_l, \quad \sum_{l=0}^{\infty} b_l$$

are convergent in the sense of norm $\|\cdot\|_{\frac{1}{2}z_0, \frac{1}{2}\rho_0}$ if $\varepsilon < \varepsilon_0$. Thus, the limit

$$\lim_{l \to \infty} \boldsymbol{P}^l = \lim_{l \to \infty} \boldsymbol{T}_l \boldsymbol{T}_{l-1} \cdots \boldsymbol{T}_0$$

exists. Suppose that the limit is $\boldsymbol{P}$, then it is to see that $\boldsymbol{P}$ is almost periodic matrix.

From the definition of m_l, the following inequalities hold:

$$|||\varepsilon^{2^{l+1}} \boldsymbol{Q}_{l+1}(t)|||_{\frac{1}{2}z_0, \frac{1}{2}\rho_0} \leq |||\varepsilon^{2^{l+1}} \boldsymbol{Q}_{l+1}(t)|||_{l+1} \leq (\varepsilon M)^{2^{l+1}} \to 0,$$

when l goes to ∞.

So Theorem 1.1 is proved.

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