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The Direct Sum Decomposition of Type G_2 Lie Algebra

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Abstract: This article mainly discusses the direct sum decomposition of type G_2 Lie algebra, which, under such decomposition, is decomposed into a type A_1 simple Lie algebra and one of its modules. Four theorems are given to describe this module, which could be the direct sum of two or three irreducible modules, or the direct sum of weight modules and trivial modules, or the highest weight module.

Key words: simple Lie algebra G_2, simple Lie algebra A_1, direct sum decomposition, the highest weight module

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1 Introduction and Main Results

The classification of Lie algebra is one of traditional fields of research in [1]–[3]. However, the detailed description of structures of basic 9 types of simple Lie algebra is not easy to achieve in [1]. This article tries to analysis the structure of type G_2 Lie algebra with the help of direct sum decomposition. This method involves the simple type A_1 Lie algebra, the irreducible modules, the trivial modules, the weight modules and the highest weight module. Generally, it is difficult to analysis the structure of type G_2 Lie algebra precisely only by definitions, properties and theorems of Lie algebra. But using the direct sum decomposition enables us to get the detailed description of the structure of type G_2 Lie algebra. The main result is given by the following theorem.

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Theorem 1.1 *The simple type G_2 Lie algebra can be direct sum decomposed as $G_2 = A_1 \oplus M$. There are four possibilities for M:*

(1) *M is the direct sum of four two-dimensional weight modules and three trivial modules*;

(2) *M is the highest weight module and the highest weight is* 10;

(3) *M is the direct sum of two irreducible modules*;

(4) *M is the direct sum of three irreducible modules.*

This article begins with the discussion of the background in [1], [4]–[6] and detailed information of type G_2 Lie algebra in [7]–[11], which could help to understand our work.

2 Basic Description

We begin this section by giving some useful result from [1].

Assume that the standard orthogonal basis of $\mathbf{R}^3$ is $\boldsymbol{\varepsilon}_1$, $\boldsymbol{\varepsilon}_2$, $\boldsymbol{\varepsilon}_3$, and E is the plane in $\mathbf{R}^3$ which goes through the origin and is orthogonal to the vector $\boldsymbol{\varepsilon}_1+\boldsymbol{\varepsilon}_2+\boldsymbol{\varepsilon}_3$. Note I as the glen Z-span by $\boldsymbol{\varepsilon}_1$, $\boldsymbol{\varepsilon}_2$, $\boldsymbol{\varepsilon}_3$, where $I' = I \bigcap E$, and root system

$$\Phi = \pm\{\boldsymbol{\alpha} \in I',\ (\boldsymbol{\alpha}, \boldsymbol{\alpha}) = 2 \text{ or } 6\}.$$

Then the irreducible root system of a simple type G_2 Lie algebras is

$$\Phi = \pm\{\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2,\ \boldsymbol{\varepsilon}_2-\boldsymbol{\varepsilon}_3,\ \boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_3,\ 2\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2-\boldsymbol{\varepsilon}_3,\ 2\boldsymbol{\varepsilon}_2-\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_3,\ 3\boldsymbol{\varepsilon}_3-\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\}.$$

The simple root for simple type G_2 Lie algebra can be chosen for $\boldsymbol{\alpha}_1 = \boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2$, $\boldsymbol{\alpha}_2 = -2\boldsymbol{\varepsilon}_1+\boldsymbol{\varepsilon}_2+\boldsymbol{\varepsilon}_3$. The highest root is $3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2$, and the shortest root is $2\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2$.

If the base is fix to be $\Delta = \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2\}$, then the strongly dominant is: $\boldsymbol{\lambda}_1 = 2\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2$, $\boldsymbol{\lambda}_2 = 3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2$. This has a 1-1 irreducible representation of a 7×7 matrix. And $\dim G_2 = 14$, $\boldsymbol{d}_i = \boldsymbol{e}_{i+1,i+1}-\boldsymbol{e}_{i+4,i+4}$ $(i = 1, 2, 3)$. The Cartan Subalgebra (CSA for short) of G_2 is H, $H = \{\boldsymbol{h}_1, \boldsymbol{h}_2 \mid \boldsymbol{h}_1 = \boldsymbol{d}_1-\boldsymbol{d}_2,\ \boldsymbol{h}_2 = \boldsymbol{d}_2-\boldsymbol{d}_3\}$, $\dim H = 2$.

The six long roots about H is $\boldsymbol{g}_{i,-j}$ $(i \neq j,\ i, j = 1, 2, 3)$,

$$\boldsymbol{g}_{1,-2} = \boldsymbol{g}_{2,-1} = \boldsymbol{e}_{23}-\boldsymbol{e}_{65},$$
$$\boldsymbol{g}_{1,-3} = \boldsymbol{g}_{3,-1} = \boldsymbol{e}_{24}-\boldsymbol{e}_{75},$$
$$\boldsymbol{g}_{2,-3} = \boldsymbol{g}_{3,-2} = \boldsymbol{e}_{34}-\boldsymbol{e}_{76}.$$

The six short roots about H is $\boldsymbol{g}_{\pm i}$ $(i = 1, 2, 3)$,

$$\boldsymbol{g}_1 = -\boldsymbol{g}_{-1}^t = \sqrt{\boldsymbol{e}_{12}-\boldsymbol{e}_{51}}-\sqrt{\boldsymbol{e}_{37}-\boldsymbol{e}_{46}},$$
$$\boldsymbol{g}_2 = -\boldsymbol{g}_{-2}^t = \sqrt{\boldsymbol{e}_{13}-\boldsymbol{e}_{61}}-\sqrt{\boldsymbol{e}_{27}-\boldsymbol{e}_{45}},$$
$$\boldsymbol{g}_3 = -\boldsymbol{g}_{-3}^t = \sqrt{\boldsymbol{e}_{14}-\boldsymbol{e}_{71}}-\sqrt{\boldsymbol{e}_{26}-\boldsymbol{e}_{35}}.$$

The 12 roots of the above are the common feature vectors of ad$H$.

The operations between the bases of the G_2

$$[\boldsymbol{g}_{i,-j},\ \boldsymbol{g}_{k,-l}] = \delta_{jk}\boldsymbol{g}_{i,-l}-\delta_{il}\boldsymbol{g}_{k,-j},$$
$$[\boldsymbol{g}_i,\ \boldsymbol{g}_{-i}] = 3\boldsymbol{d}_i-(\boldsymbol{d}_1+\boldsymbol{d}_2+\boldsymbol{d}_3),$$
$$[\boldsymbol{g}_{i,-j},\ \boldsymbol{g}_{-k}] = -\delta_{jk}\boldsymbol{g}_{-i},$$
$$[\boldsymbol{g}_{i,-j},\ \boldsymbol{g}_k] = -\delta_{ik}\boldsymbol{g}_j,$$

$$[\boldsymbol{g}_i,\ \boldsymbol{g}_{-j}] = 3\boldsymbol{g}_{j,-i} \quad (i \neq j),$$
$$[\boldsymbol{g}_i,\ \boldsymbol{g}_j] = \pm 2\boldsymbol{g}_{-k} \qquad (i \neq j,\ j \neq k,\ k \neq i),$$
$$[\boldsymbol{g}_{-i},\ \boldsymbol{g}_{-j}] = \pm 2\boldsymbol{g}_k \qquad (i \neq j,\ j \neq k,\ k \neq i).$$

3 Decomposition

3.1 Decomposition One

Let simple root of G_2 be $\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2\}$, $\boldsymbol{\alpha}_1$ be long root, $\boldsymbol{\alpha}_2$ be short root. Then the integrality properties is

$$G_2 = H + \coprod_{\boldsymbol{\alpha} \in \Phi} L_{\boldsymbol{\alpha}},$$

where $\dim H = 2$, $\Phi = \pm\{\boldsymbol{\alpha}_1,\ \boldsymbol{\alpha}_2,\ \boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2,\ 2\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2,\ 3\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2,\ 3\boldsymbol{\alpha}_1 + 2\boldsymbol{\alpha}_2,\}$.

In the matrix representation of G_2, there are

$$[\boldsymbol{h}_1,\ \boldsymbol{g}_{1,-2}] = 2\boldsymbol{g}_{1,-2}, \quad [\boldsymbol{h}_1,\ \boldsymbol{g}_{2,-1}] = 2\boldsymbol{g}_{2,-1}, \quad [\boldsymbol{g}_{1,-2},\ \boldsymbol{g}_{2,-1}] = 2\boldsymbol{h}_1.$$

So $\mathrm{span}\{\boldsymbol{h}_1,\ \boldsymbol{g}_{1,-2},\ \boldsymbol{g}_{2,-1}\}$ is isomorphic to $\mathrm{sl}(2,\ F)$. Let

$$A_1 = \mathrm{span}\{\boldsymbol{h}_1,\ \boldsymbol{g}_{1,-2},\ \boldsymbol{g}_{2,-1}\}.$$

So A_1 is subalgebra of G_2. By $\dim G_2 = 14$, let the complement space of A_1 on G_2 be M. Then M is adjoint action by A_1,

$$\begin{aligned}
&\mathrm{ad}\boldsymbol{g}_{1,-2}(\boldsymbol{g}_{1,-3}) = \boldsymbol{0}, && \mathrm{ad}\boldsymbol{h}_1(\boldsymbol{g}_{1,-3}) = \boldsymbol{g}_{1,-3}, && \mathrm{ad}\boldsymbol{g}_{2,-1}(\boldsymbol{g}_{1,-3}) = \boldsymbol{g}_{2,-3},\\
&\mathrm{ad}\boldsymbol{g}_{1,-2}(\boldsymbol{g}_{3,-1}) = -\boldsymbol{g}_{3,-2}, && \mathrm{ad}\boldsymbol{h}_1(\boldsymbol{g}_{3,-1}) = -\boldsymbol{g}_{3,-1}, && \mathrm{ad}\boldsymbol{g}_{2,-1}(\boldsymbol{g}_{3,-1}) = \boldsymbol{0},\\
&\mathrm{ad}\boldsymbol{g}_{1,-2}(\boldsymbol{g}_{2,-3}) = \boldsymbol{g}_{1,-3}, && \mathrm{ad}\boldsymbol{h}_1(\boldsymbol{g}_{2,-3}) = -\boldsymbol{g}_{2,-3}, && \mathrm{ad}\boldsymbol{g}_{2,-1}(\boldsymbol{g}_{2,-3}) = \boldsymbol{0},\\
&\mathrm{ad}\boldsymbol{g}_{1,-2}(\boldsymbol{g}_{3,-2}) = \boldsymbol{0}, && \mathrm{ad}\boldsymbol{h}_1(\boldsymbol{g}_{3,-2}) = -\boldsymbol{g}_{3,-2}, && \mathrm{ad}\boldsymbol{g}_{2,-1}(\boldsymbol{g}_{3,-2}) = \boldsymbol{0},\\
&\mathrm{ad}\boldsymbol{g}_{1,-2}(\boldsymbol{g}_1) = -\boldsymbol{g}_{-2}, && \mathrm{ad}\boldsymbol{h}_1(\boldsymbol{g}_1) = -\boldsymbol{g}_1, && \mathrm{ad}\boldsymbol{g}_{2,-1}(\boldsymbol{g}_1) = \boldsymbol{0},\\
&\mathrm{ad}\boldsymbol{g}_{1,-2}(\boldsymbol{g}_{-1}) = \boldsymbol{0}, && \mathrm{ad}\boldsymbol{h}_1(\boldsymbol{g}_{-1}) = \boldsymbol{g}_1, && \mathrm{ad}\boldsymbol{g}_{2,-1}(\boldsymbol{g}_{-1}) = \boldsymbol{g}_{-2},\\
&\mathrm{ad}\boldsymbol{g}_{1,-2}(\boldsymbol{g}_2) = \boldsymbol{0}, && \mathrm{ad}\boldsymbol{h}_1(\boldsymbol{g}_2) = -\boldsymbol{g}_2, && \mathrm{ad}\boldsymbol{g}_{2,-1}(\boldsymbol{g}_2) = -\boldsymbol{g}_1,\\
&\mathrm{ad}\boldsymbol{g}_{1,-2}(\boldsymbol{g}_{-2}) = -\boldsymbol{g}_{-1}, && \mathrm{ad}\boldsymbol{h}_1(\boldsymbol{g}_{-2}) = -\boldsymbol{g}_2, && \mathrm{ad}\boldsymbol{g}_{2,-1}(\boldsymbol{g}_{-2}) = \boldsymbol{0},\\
&\mathrm{ad}\boldsymbol{g}_{1,-2}(\boldsymbol{g}_3) = \boldsymbol{0}, && \mathrm{ad}\boldsymbol{h}_1(\boldsymbol{g}_3) = \boldsymbol{0}, && \mathrm{ad}\boldsymbol{g}_{2,-1}(\boldsymbol{g}_3) = \boldsymbol{0},\\
&\mathrm{ad}\boldsymbol{g}_{1,-2}(\boldsymbol{g}_{-3}) = \boldsymbol{0} && \mathrm{ad}\boldsymbol{h}_1(\boldsymbol{g}_{-3}) = \boldsymbol{0}, && \mathrm{ad}\boldsymbol{g}_{2,-1}(\boldsymbol{g}_{-3}) = \boldsymbol{0},\\
&\mathrm{ad}\boldsymbol{g}_{1,-2}(\boldsymbol{h}_2) = \boldsymbol{g}_{1,-2}, && \mathrm{ad}\boldsymbol{h}_1(\boldsymbol{h}_2) = \boldsymbol{0}, && \mathrm{ad}\boldsymbol{g}_{2,-1}(\boldsymbol{h}_2) = -\boldsymbol{g}_{2,-1}.
\end{aligned}$$

Then relational formulas ($\xrightarrow{\boldsymbol{g}_{i,-j}}$ is adjoint action)

$$\boldsymbol{g}_{2,-3} \xrightarrow{\boldsymbol{g}_{1,-2}} \boldsymbol{g}_{1,-3} \xrightarrow{\boldsymbol{g}_{1,-2}} \boldsymbol{0},$$
$$\boldsymbol{g}_{1,-3} \xrightarrow{\boldsymbol{g}_{2,-1}} \boldsymbol{g}_{2,-3} \xrightarrow{\boldsymbol{g}_{2,-1}} \boldsymbol{0},$$
$$\boldsymbol{g}_{3,-1} \xrightarrow{\boldsymbol{g}_{1,-2}} \boldsymbol{g}_{3,-2} \xrightarrow{\boldsymbol{g}_{1,-2}} \boldsymbol{0},$$
$$\boldsymbol{g}_{3,-1} \xrightarrow{\boldsymbol{g}_{2,-1}} \boldsymbol{g}_{3,-2} \xrightarrow{\boldsymbol{g}_{2,-1}} \boldsymbol{0},$$
$$\boldsymbol{g}_1 \xrightarrow{\boldsymbol{g}_{1,-2}} \boldsymbol{g}_2 \xrightarrow{\boldsymbol{g}_{1,-2}} \boldsymbol{0},$$
$$\boldsymbol{g}_2 \xrightarrow{\boldsymbol{g}_{2,-1}} \boldsymbol{g}_1 \xrightarrow{\boldsymbol{g}_{2,-1}} \boldsymbol{0},$$

$$\boldsymbol{g}_{-2} \xrightarrow{\boldsymbol{g}_{1,-2}} \boldsymbol{g}_{-1} \xrightarrow{\boldsymbol{g}_{1,-2}} \mathbf{0},$$
$$\boldsymbol{g}_{-1} \xrightarrow{\boldsymbol{g}_{2,-1}} \boldsymbol{g}_{-2} \xrightarrow{\boldsymbol{g}_{2,-1}} \mathbf{0}.$$

We can select a corresponding

$$\begin{aligned}
&\boldsymbol{g}_{1,-2} \to L_{\boldsymbol{\alpha}_2}, && \boldsymbol{g}_{2,-1} \to L_{-\boldsymbol{\alpha}_2},\\
&\boldsymbol{g}_{1,-3} \to L_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2}, && \boldsymbol{g}_{3,-1} \to L_{-3\boldsymbol{\alpha}_1-2\boldsymbol{\alpha}_2},\\
&\boldsymbol{g}_{2,-3} \to L_{3\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}, && \boldsymbol{g}_{3,-2} \to L_{-3\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2},\\
&\boldsymbol{g}_{1} \to L_{\boldsymbol{\alpha}_1}, && \boldsymbol{g}_{-1} \to L_{-\boldsymbol{\alpha}_1},\\
&\boldsymbol{g}_{2} \to L_{\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}, && \boldsymbol{g}_{-2} \to L_{-\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2},\\
&\boldsymbol{g}_{3} \to L_{2\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}, && \boldsymbol{g}_{-3} \to L_{-2\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2},\\
&\boldsymbol{h}_1 \to H_{\boldsymbol{\alpha}_2},\\
&\boldsymbol{h}_2 \to H_{3\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2},
\end{aligned}$$

where

$$\begin{aligned}
&\boldsymbol{h}_1 \to H_{\boldsymbol{\alpha}_2} && \text{as } [\boldsymbol{g}_{1,-2},\ \boldsymbol{g}_{2,-1}] = \boldsymbol{h}_1 \in [L_{\boldsymbol{\alpha}_2},\ L_{\boldsymbol{\alpha}_{-2}}] = H_{\boldsymbol{\alpha}_2},\\
&\boldsymbol{h}_2 \to H_{3\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2} && \text{as } [\boldsymbol{g}_{2,-3},\ \boldsymbol{g}_{3,-2}] = \boldsymbol{h}_2 \in [L_{3\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2},\ L_{-3\boldsymbol{\alpha}_{-1}-\boldsymbol{\alpha}_2}] = H_{3\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}.
\end{aligned}$$

Theorem 3.1 *The simple type G_2 Lie algebra can be direct sum decomposed as $G_2 = A_1 \oplus M$, where M is the direct sum of four two-dimensional weight modules and three trivial modules.*

Proof. As mentioned above A_1 and eleven-dimensional space M, if $\boldsymbol{m} \in M$, then from A_1 adjoint action $\boldsymbol{m}$: $A_1.\boldsymbol{m} = M$.

The three weight spaces are

$$\begin{aligned}
&\text{Weight } 1: && V_1 = \operatorname{span}\{\boldsymbol{g}_{1,-3},\ \boldsymbol{g}_{3,-2},\ \boldsymbol{g}_{-1},\ \boldsymbol{g}_2\},\\
&\text{Weight } 0: && V_0 = \operatorname{span}\{\boldsymbol{g}_3,\ \boldsymbol{g}_{-3},\ \boldsymbol{h}_2\},\\
&\text{Weight } -1: && V_{-1} = \operatorname{span}\{\boldsymbol{g}_{3,-1},\ \boldsymbol{g}_{2,-3},\ \boldsymbol{g}_1,\ \boldsymbol{g}_{-2}\}.
\end{aligned}$$

The four two-dimensional weight modules are

$$\begin{aligned}
M_1 &= \operatorname{span}\{\boldsymbol{g}_{1,-3},\ \boldsymbol{g}_{2,-3}\} = L_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2} + L_{3\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2},\\
M_2 &= \operatorname{span}\{\boldsymbol{g}_{3,-2},\ \boldsymbol{g}_{3,-1}\} = L_{-3\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2} + L_{-3\boldsymbol{\alpha}_1-2\boldsymbol{\alpha}_2},\\
M_3 &= \operatorname{span}\{\boldsymbol{g}_1,\ \boldsymbol{g}_2\} = L_{\boldsymbol{\alpha}_1} + L_{\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2},\\
M_4 &= \operatorname{span}\{\boldsymbol{g}_{-1},\ \boldsymbol{g}_{-2}\} = L_{-\boldsymbol{\alpha}_1} + L_{-\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2}.
\end{aligned}$$

The three trivial modules are

$$\begin{aligned}
M_5 &= \operatorname{span}\{\boldsymbol{g}_3\} = L_{2\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2},\\
M_6 &= \operatorname{span}\{\boldsymbol{g}_{-3}\} = L_{-2\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2},\\
M_7 &= \operatorname{span}\{\boldsymbol{h}_2\} = L_{3\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}.
\end{aligned}$$

Corollary 3.1 *G_2 can be normal decomposed as $G_2 = A_1 \propto_{A_1} M$, where M is the direct sum of four two-dimensional weight modules and three trivial modules.*

Proof. Obviously, M is normal subgroup, so the corollary is proved.

3.2 Decomposition Two

Lemma 3.1 *If* $\mathbf{0} \neq \boldsymbol{x}_{\boldsymbol{\alpha}_1} \in L_{\boldsymbol{\alpha}_1}$ *and* $\mathbf{0} \neq \boldsymbol{x}_{\boldsymbol{\alpha}_2} \in L_{\boldsymbol{\alpha}_2}$, *let* $\boldsymbol{x}_{\boldsymbol{\alpha}_1} + \boldsymbol{x}_{\boldsymbol{\alpha}_2} = \boldsymbol{X}$, *then there exists a* $\boldsymbol{Y} \in L_{-\boldsymbol{\alpha}_1} + L_{-\boldsymbol{\alpha}_2}$ *such that* $\boldsymbol{X},\ \boldsymbol{Y},\ \boldsymbol{H} = [\boldsymbol{X},\ \boldsymbol{Y}]$ *span a three dimensional simple subalgebra of* G_2 *isomorphic to* $sl(2,\ F)$ *via*

$$\boldsymbol{X} \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \boldsymbol{Y} \mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad \boldsymbol{H} \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Proof. From the literature (see [1]), for $\boldsymbol{\alpha} \in \Phi$, if $\boldsymbol{x}_{\boldsymbol{\alpha}} \in L_{\boldsymbol{\alpha}}$, $\boldsymbol{x}_{\boldsymbol{\alpha}} \neq \mathbf{0}$ such that $\boldsymbol{x}_{\boldsymbol{\alpha}},\ \boldsymbol{y}_{\boldsymbol{\alpha}},\ \boldsymbol{h}_{\boldsymbol{\alpha}} = [\boldsymbol{x}_{\boldsymbol{\alpha}},\ \boldsymbol{y}_{\boldsymbol{\alpha}}]$ span a three dimensional simple subalgebra of G_2 isomorphic to $\mathrm{sl}(2,\ F)$ via

$$\boldsymbol{x}_{\boldsymbol{\alpha}} \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \boldsymbol{y}_{\boldsymbol{\alpha}} \mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad \boldsymbol{h}_{\boldsymbol{\alpha}} \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

So, if $\boldsymbol{\alpha}_1,\ \boldsymbol{\alpha}_2 \in \Phi$ and $\mathbf{0} \neq \boldsymbol{x}_{\boldsymbol{\alpha}_1} \in L_{\boldsymbol{\alpha}_1}$, $\mathbf{0} \neq \boldsymbol{x}_{\boldsymbol{\alpha}_2} \in L_{\boldsymbol{\alpha}_2}$, then there exist $\boldsymbol{y}_{\boldsymbol{\alpha}_1} \in L_{-\boldsymbol{\alpha}_1}$, $\boldsymbol{y}_{\boldsymbol{\alpha}_2} \in L_{-\boldsymbol{\alpha}_2}$ such that $\boldsymbol{x}_{\boldsymbol{\alpha}_1},\ \boldsymbol{y}_{\boldsymbol{\alpha}_1},\ \boldsymbol{h}_{\boldsymbol{\alpha}_1} = [\boldsymbol{x}_{\boldsymbol{\alpha}_1},\ \boldsymbol{y}_{\boldsymbol{\alpha}_1}]$ and $\boldsymbol{x}_{\boldsymbol{\alpha}_2},\ \boldsymbol{y}_{\boldsymbol{\alpha}_2},\ \boldsymbol{h}_{\boldsymbol{\alpha}_2} = [\boldsymbol{x}_{\boldsymbol{\alpha}_2},\ \boldsymbol{y}_{\boldsymbol{\alpha}_2}]$ span respectively a three dimensional simple subalgebra of G_2 isomorphic to $\mathrm{sl}(2,\ F)$.

Let $\boldsymbol{X} = \boldsymbol{x}_{\boldsymbol{\alpha}_1} + \boldsymbol{y}_{\boldsymbol{\alpha}_2}$, $\boldsymbol{Y} = 6\boldsymbol{y}_{\boldsymbol{\alpha}_1} + 10\boldsymbol{y}_{\boldsymbol{\alpha}_2}$, $\boldsymbol{H} = 6\boldsymbol{h}_{\boldsymbol{\alpha}_1} + 10\boldsymbol{h}_{\boldsymbol{\alpha}_2}$. Then

$$\begin{aligned} [\boldsymbol{H},\ \boldsymbol{X}] &= [6\boldsymbol{h}_{\boldsymbol{\alpha}_1} + 10\boldsymbol{h}_{\boldsymbol{\alpha}_2},\ \boldsymbol{x}_{\boldsymbol{\alpha}_1} + \boldsymbol{y}_{\boldsymbol{\alpha}_2}] \\ &= 6[\boldsymbol{h}_{\boldsymbol{\alpha}_1}\boldsymbol{x}_{\boldsymbol{\alpha}_1}] + 6[\boldsymbol{h}_{\boldsymbol{\alpha}_1}\boldsymbol{x}_{\boldsymbol{\alpha}_2}] + 10[\boldsymbol{h}_{\boldsymbol{\alpha}_2}\boldsymbol{x}_{\boldsymbol{\alpha}_1}] + 10[\boldsymbol{h}_{\boldsymbol{\alpha}_2}\boldsymbol{x}_{\boldsymbol{\alpha}_2}] \\ &= 12\boldsymbol{x}_{\boldsymbol{\alpha}_1} + 6\langle \boldsymbol{\alpha}_2,\ \boldsymbol{\alpha}_1 \rangle \boldsymbol{x}_{\boldsymbol{\alpha}_2} + 10\langle \boldsymbol{\alpha}_1,\ \boldsymbol{\alpha}_2 \rangle \boldsymbol{x}_{\boldsymbol{\alpha}_1} + 20\boldsymbol{x}_{\boldsymbol{\alpha}_2} \\ &= 12\boldsymbol{x}_{\boldsymbol{\alpha}_1} - 18\boldsymbol{x}_{\boldsymbol{\alpha}_2} - 10\boldsymbol{x}_{\boldsymbol{\alpha}_1} + 20\boldsymbol{x}_{\boldsymbol{\alpha}_2} \\ &= 2(\boldsymbol{x}_{\boldsymbol{\alpha}_1} + \boldsymbol{x}_{\boldsymbol{\alpha}_2}) \\ &= 2\boldsymbol{X}. \end{aligned}$$

In the same way, we can get

$$[\boldsymbol{H},\ \boldsymbol{Y}] = 12\boldsymbol{y}_{\boldsymbol{\alpha}_1} + 20\boldsymbol{y}_{\boldsymbol{\alpha}_2} = 2\boldsymbol{Y}.$$

So,

$$\begin{aligned} [\boldsymbol{X},\ \boldsymbol{Y}] &= [\boldsymbol{x}_{\boldsymbol{\alpha}_1} + \boldsymbol{x}_{\boldsymbol{\alpha}_2},\ 6\boldsymbol{y}_{\boldsymbol{\alpha}_1} + 10\boldsymbol{y}_{\boldsymbol{\alpha}_2}] \\ &= 6[\boldsymbol{x}_{\boldsymbol{\alpha}_1}\boldsymbol{y}_{\boldsymbol{\alpha}_1}] + 6[\boldsymbol{x}_{\boldsymbol{\alpha}_1}\boldsymbol{y}_{\boldsymbol{\alpha}_2}] + 10[\boldsymbol{x}_{\boldsymbol{\alpha}_2}\boldsymbol{y}_{\boldsymbol{\alpha}_1}] + 10[\boldsymbol{x}_{\boldsymbol{\alpha}_2}\boldsymbol{y}_{\boldsymbol{\alpha}_2}] \\ &= 6\boldsymbol{h}_{\boldsymbol{\alpha}_1} + 10\boldsymbol{h}_{\boldsymbol{\alpha}_1} \\ &= \boldsymbol{H}. \end{aligned}$$

So $\boldsymbol{X}$, $\boldsymbol{Y}$, $\boldsymbol{H}$ span a three dimensional simple subalgebra of L isomorphic to $\mathrm{sl}(2,\ F)$. The proof is completed.

From Lemma 3.1, chosen $A_1 \cong \mathrm{sl}(2,\ F)$, $\boldsymbol{X} = \boldsymbol{x}_{\boldsymbol{\alpha}_1} + \boldsymbol{y}_{\boldsymbol{\alpha}_2}$, $\boldsymbol{Y} = 6\boldsymbol{y}_{\boldsymbol{\alpha}_1} + 10\boldsymbol{y}_{\boldsymbol{\alpha}_2}$, $\boldsymbol{H} = 6\boldsymbol{h}_{\boldsymbol{\alpha}_1} + 10\boldsymbol{h}_{\boldsymbol{\alpha}_2}$, where $\boldsymbol{X} \in L_{\boldsymbol{\alpha}_1} + L_{\boldsymbol{\alpha}_2}$, $\boldsymbol{Y} \in L_{-\boldsymbol{\alpha}_1} + L_{-\boldsymbol{\alpha}_2}$, $\boldsymbol{H} \in H_{\boldsymbol{\alpha}_1} + H_{\boldsymbol{\alpha}_2}$.

By $\dim G_2 = 14$, let the complement space of A_1 on G_2 be M. Then the direct sum decomposition

$$M = \bar{H} \bigoplus \coprod_{\boldsymbol{\alpha} \in \Phi'} L_{\boldsymbol{\alpha}} \bigoplus (L_{\boldsymbol{\alpha}_1} - L_{\boldsymbol{\alpha}_2}) \bigoplus (L_{-\boldsymbol{\alpha}_1} - L_{-\boldsymbol{\alpha}_2}),$$

where $\bar{H}$ is complementary space of $H_{\boldsymbol{\alpha}_1} + H_{\boldsymbol{\alpha}_2}$ on CSA,

$$\Phi' = \pm\{\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2,\ 2\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2,\ 3\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2,\ 3\boldsymbol{\alpha}_1 + 2\boldsymbol{\alpha}_2\}.$$

Lemma 3.2 *M in Theorem* 3.2 *is weight module.*

Proof. As mentioned above A_1 and eleven-dimensional space M, we have

$$\begin{aligned}
\mathrm{ad}H(L_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2}) &= [6\boldsymbol{h}_{\boldsymbol{\alpha}_1}+10\boldsymbol{h}_{\boldsymbol{\alpha}_2},\, L_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2}] \\
&= (6\langle 3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2,\, \boldsymbol{\alpha}_1\rangle + 10\langle 3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2,\, \boldsymbol{\alpha}_2\rangle)L_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2} \\
&= 10L_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2}, \\
\mathrm{ad}H(L_{3\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}) &= [6\boldsymbol{h}_{\boldsymbol{\alpha}_1}+10\boldsymbol{h}_{\boldsymbol{\alpha}_2},\, L_{3\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}] \\
&= (6\langle 3\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2,\, \boldsymbol{\alpha}_1\rangle + 10\langle 3\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2,\, \boldsymbol{\alpha}_2\rangle)L_{3\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2} \\
&= 8L_{3\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}, \\
\mathrm{ad}H(L_{2\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}) &= [6\boldsymbol{h}_{\boldsymbol{\alpha}_1}+10\boldsymbol{h}_{\boldsymbol{\alpha}_2},\, L_{2\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}] \\
&= (6\langle 2\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2,\, \boldsymbol{\alpha}_1\rangle + 10\langle 2\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2,\, \boldsymbol{\alpha}_1\rangle)L_{2\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2} \\
&= 6L_{2\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}, \\
\mathrm{ad}H(L_{\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}) &= [6\boldsymbol{h}_{\boldsymbol{\alpha}_1}+10\boldsymbol{h}_{\boldsymbol{\alpha}_2},\, L_{\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}] \\
&= (6\langle \boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2,\, \boldsymbol{\alpha}_1\rangle + 10\langle \boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2,\, \boldsymbol{\alpha}_1\rangle)L_{\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2} \\
&= 4L_{\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}, \\
\mathrm{ad}H(L_{\boldsymbol{\alpha}_1}-L_{\boldsymbol{\alpha}_2}) &= [6\boldsymbol{h}_{\boldsymbol{\alpha}_1}+10\boldsymbol{h}_{\boldsymbol{\alpha}_2},\, L_{\boldsymbol{\alpha}_1}-L_{\boldsymbol{\alpha}_2}] \\
&= 2(L_{\boldsymbol{\alpha}_1}-L_{\boldsymbol{\alpha}_2}), \\
\mathrm{ad}H(\bar{H}) &= [6\boldsymbol{h}_{\boldsymbol{\alpha}_1}+10\boldsymbol{h}_{\boldsymbol{\alpha}_2},\, \bar{H}] \\
&= 0, \\
\mathrm{ad}H(L_{-\boldsymbol{\alpha}_1}-L_{-\boldsymbol{\alpha}_2}) &= [6\boldsymbol{h}_{\boldsymbol{\alpha}_1}+10\boldsymbol{h}_{\boldsymbol{\alpha}_2},\, L_{-\boldsymbol{\alpha}_1}-L_{-\boldsymbol{\alpha}_2}] \\
&= -2(L_{-\boldsymbol{\alpha}_1}-L_{-\boldsymbol{\alpha}_2}), \\
\mathrm{ad}H(L_{-\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2}) &= [6\boldsymbol{h}_{\boldsymbol{\alpha}_1}+10\boldsymbol{h}_{\boldsymbol{\alpha}_2},\, L_{-\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2}] \\
&= (6\langle -\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2,\, \boldsymbol{\alpha}_1\rangle + 10\langle -\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2,\, \boldsymbol{\alpha}_2\rangle)L_{-\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2} \\
&= -4L_{-\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2}, \\
\mathrm{ad}H(L_{-2\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2}) &= [6\boldsymbol{h}_{\boldsymbol{\alpha}_1}+10\boldsymbol{h}_{\boldsymbol{\alpha}_2},\, L_{-2\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2}] \\
&= (6\langle -2\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2,\, \boldsymbol{\alpha}_1\rangle + 10\langle -2\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2,\, \boldsymbol{\alpha}_2\rangle)L_{-2\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2} \\
&= -6L_{-2\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2}, \\
\mathrm{ad}H(L_{-3\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2}) &= [6\boldsymbol{h}_{\boldsymbol{\alpha}_1}+10\boldsymbol{h}_{\boldsymbol{\alpha}_2},\, L_{-3\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2}] \\
&= (6\langle -3\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2,\, \boldsymbol{\alpha}_1\rangle + 10\langle -3\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2,\, \boldsymbol{\alpha}_2\rangle)L_{-3\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2} \\
&= -8L_{-3\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2}, \\
\mathrm{ad}H(L_{-3\boldsymbol{\alpha}_1-2\boldsymbol{\alpha}_2}) &= [6\boldsymbol{h}_{\boldsymbol{\alpha}_1}+10\boldsymbol{h}_{\boldsymbol{\alpha}_2},\, L_{-3\boldsymbol{\alpha}_1-2\boldsymbol{\alpha}_2}] \\
&= (6\langle -3\boldsymbol{\alpha}_1-2\boldsymbol{\alpha}_2,\, \boldsymbol{\alpha}_1\rangle + 10\langle -3\boldsymbol{\alpha}_1-2\boldsymbol{\alpha}_2,\, \boldsymbol{\alpha}_2\rangle)L_{-3\boldsymbol{\alpha}_1-2\boldsymbol{\alpha}_2} \\
&= -10L_{-3\boldsymbol{\alpha}_1-2\boldsymbol{\alpha}_2}.
\end{aligned}$$

From the base of H adjoint action M, we obtain that the weight module M is A_1.

Lemma 3.3 *M in Theorem* 3.2 *is irreducible module.*

Proof. From Lemma 3.2, M is weight module.

Base of M is adjoint action by $\boldsymbol{X}$, by $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \varPhi$, $\boldsymbol{\alpha}+\boldsymbol{\beta} \neq \boldsymbol{0}$, then $0 \neq [L_{\boldsymbol{\alpha}}, L_{\boldsymbol{\beta}}] \subset L_{\boldsymbol{\alpha}+\boldsymbol{\beta}}$ such that

$$
\begin{aligned}
&[\boldsymbol{X}, L_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2}] \subset [L_{\boldsymbol{\alpha}_1}+L_{\boldsymbol{\alpha}_2}, L_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2}] = 0,\\
0 \neq &[\boldsymbol{X}, L_{3\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}] \subset [L_{\boldsymbol{\alpha}_1}+L_{\boldsymbol{\alpha}_2}, L_{3\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}] \subset L_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2},\\
0 \neq &[\boldsymbol{X}, L_{2\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}] \subset [L_{\boldsymbol{\alpha}_1}+L_{\boldsymbol{\alpha}_2}, L_{2\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}] \subset L_{3\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2},\\
0 \neq &[\boldsymbol{X}, L_{\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}] \subset [L_{\boldsymbol{\alpha}_1}+L_{\boldsymbol{\alpha}_2}, L_{\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}] \subset L_{2\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2},\\
0 \neq &[\boldsymbol{X}, L_{\boldsymbol{\alpha}_1}-L_{\boldsymbol{\alpha}_2}] \subset [L_{\boldsymbol{\alpha}_1}+L_{\boldsymbol{\alpha}_2}, L_{\boldsymbol{\alpha}_1}-L_{\boldsymbol{\alpha}_2}] \subset L_{\boldsymbol{\alpha}_1},\\
0 \neq &[\boldsymbol{X}, H^{\perp}] \subset [L_{\boldsymbol{\alpha}_1}+L_{\boldsymbol{\alpha}_2}, \bar{H}] \subset L_{\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2},\\
0 \neq &[\boldsymbol{X}, L_{-\boldsymbol{\alpha}_1}-L_{-\boldsymbol{\alpha}_2}] \subset [L_{\boldsymbol{\alpha}_1}+L_{\boldsymbol{\alpha}_2}, L_{-\boldsymbol{\alpha}_1}-L_{-\boldsymbol{\alpha}_2}] \subset \bar{H},\\
0 \neq &[\boldsymbol{X}, L_{-\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2}] \subset [L_{\boldsymbol{\alpha}_1}+L_{\boldsymbol{\alpha}_2}, L_{-\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2}] \subset L_{-\boldsymbol{\alpha}_1}-L_{-\boldsymbol{\alpha}_2},\\
0 \neq &[\boldsymbol{X}, L_{-2\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2}] \subset [L_{\boldsymbol{\alpha}_1}+L_{\boldsymbol{\alpha}_2}, L_{-2\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2}] \subset L_{-\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2},\\
0 \neq &[\boldsymbol{X}, L_{-3\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2}] \subset [L_{\boldsymbol{\alpha}_1}+L_{\boldsymbol{\alpha}_2}, L_{-3\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2}] \subset L_{-2\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2},\\
0 \neq &[\boldsymbol{X}, L_{-3\boldsymbol{\alpha}_1-2\boldsymbol{\alpha}_2}] \subset [L_{\boldsymbol{\alpha}_1}+L_{\boldsymbol{\alpha}_2}, L_{-3\boldsymbol{\alpha}_1-2\boldsymbol{\alpha}_2}] \subset L_{-3\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2}.
\end{aligned}
$$

Base of M is adjoint action by $\boldsymbol{Y}$, by $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \varPhi$, $\boldsymbol{\alpha}+\boldsymbol{\beta} \neq \boldsymbol{0}$, then $0 \neq [L_{\boldsymbol{\alpha}}, L_{\boldsymbol{\beta}}] \subset L_{\boldsymbol{\alpha}+\boldsymbol{\beta}}$ such that

$$
\begin{aligned}
0 \neq &[\boldsymbol{Y}, L_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2}] \subset [L_{-\boldsymbol{\alpha}_1}+L_{-\boldsymbol{\alpha}_2}, L_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2}] \subset L_{3\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2},\\
0 \neq &[\boldsymbol{Y}, L_{3\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}] \subset [L_{-\boldsymbol{\alpha}_1}+L_{-\boldsymbol{\alpha}_2}, L_{3\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}] \subset L_{2\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2},\\
0 \neq &[\boldsymbol{Y}, L_{2\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}] \subset [L_{-\boldsymbol{\alpha}_1}+L_{-\boldsymbol{\alpha}_2}, L_{2\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}] \subset L_{\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2},\\
0 \neq &[\boldsymbol{Y}, L_{\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}] \subset [L_{-\boldsymbol{\alpha}_1}+L_{-\boldsymbol{\alpha}_2}, L_{\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}] \subset L_{\boldsymbol{\alpha}_1}-L_{\boldsymbol{\alpha}_2},\\
0 \neq &[\boldsymbol{Y}, L_{\boldsymbol{\alpha}_1}-L_{\boldsymbol{\alpha}_2}] \subset [L_{-\boldsymbol{\alpha}_1}+L_{-\boldsymbol{\alpha}_2}, L_{\boldsymbol{\alpha}_1}-L_{\boldsymbol{\alpha}_2}] \subset \bar{H},\\
0 \neq &[\boldsymbol{Y}, \bar{H}] \subset [L_{-\boldsymbol{\alpha}_1}+L_{-\boldsymbol{\alpha}_2}, \bar{H}] \subset L_{-\boldsymbol{\alpha}_1}-L_{-\boldsymbol{\alpha}_2},\\
0 \neq &[\boldsymbol{Y}, L_{-\boldsymbol{\alpha}_1}-L_{-\boldsymbol{\alpha}_2}] \subset [L_{-\boldsymbol{\alpha}_1}+L_{-\boldsymbol{\alpha}_2}, L_{-\boldsymbol{\alpha}_1}-L_{\boldsymbol{\alpha}_2}] \subset L_{-\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2},\\
0 \neq &[\boldsymbol{Y}, L_{-\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2}] \subset [L_{-\boldsymbol{\alpha}_1}+L_{-\boldsymbol{\alpha}_2}, L_{-\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2}] \subset L_{-2\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2},\\
0 \neq &[\boldsymbol{Y}, L_{-2\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2}] \subset [L_{-\boldsymbol{\alpha}_1}+L_{-\boldsymbol{\alpha}_2}, L_{-2\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2}] \subset L_{-3\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2},\\
0 \neq &[\boldsymbol{Y}, L_{-3\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2}] \subset [L_{-\boldsymbol{\alpha}_1}+L_{-\boldsymbol{\alpha}_2}, L_{-3\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2}] \subset L_{-3\boldsymbol{\alpha}_1-2\boldsymbol{\alpha}_2},\\
&[\boldsymbol{Y}, L_{-3\boldsymbol{\alpha}_1-2\boldsymbol{\alpha}_2}] \subset [L_{-\boldsymbol{\alpha}_1}+L_{-\boldsymbol{\alpha}_2}, L_{-3\boldsymbol{\alpha}_1-2\boldsymbol{\alpha}_2}] = 0.
\end{aligned}
$$

If $\boldsymbol{m} \in M$, from A_1 adjoint action $\boldsymbol{m}$ we know $A_1.\boldsymbol{m} = M$. So M is irreducible module.

Theorem 3.2 *The simple type G_2 Lie algebra can be direct sum decomposed as $G_2 = A_1 \oplus M$, where M is the highest weight module and the highest weight is* 10.

Proof. From Lemma 3.3, M is irreducible module. From Lemma 3.2, base of M is adjoint action by H, then M is a highest weight module with the highest power of 10.

Corollary 3.2 *G_2 can be normal decomposed as $G_2 = A_1 \propto_{A_1} M$, where M is the highest weight module and the highest weight is* 10.

Proof. Obviously, M is normal subgroup, so the corollary is proved.

3.3 Decomposition Three

From Lemma 3.1, chosen $A_1 \cong \mathrm{sl}(2, F)$, $\boldsymbol{X} \in L_{-\boldsymbol{\alpha}_1} + L_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2}$, $\boldsymbol{Y} \in L_{\boldsymbol{\alpha}_1} + L_{-3\boldsymbol{\alpha}_1-2\boldsymbol{\alpha}_2}$, $\boldsymbol{H} \in H_{\boldsymbol{\alpha}_1} + H_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2}$. By $\dim G_2 = 14$, let the complement space of A_1 on G_2 be M. Then the direct sum decomposition

$$M = \bar{H} \bigoplus \coprod_{\boldsymbol{\alpha} \in \Phi''} L_{\boldsymbol{\alpha}} \bigoplus (L_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2} - L_{-\boldsymbol{\alpha}_1}) \bigoplus (L_{-3\boldsymbol{\alpha}_1-2\boldsymbol{\alpha}_2} - L_{\boldsymbol{\alpha}_1}),$$

where $\bar{H}$ is complementary space of $H_{\boldsymbol{\alpha}_1} + H_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2}$ on CSA,

$$\Phi'' = \pm\{\boldsymbol{\alpha}_2, \boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2, 2\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2, 3\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2\}.$$

Lemma 3.4 *M in Theorem* 3.3 *is the direct sum of two irreducible module.*

Proof. As mentioned above A_1 and eleven-dimensional space M.

Base of M is adjoint action by $\boldsymbol{X} = \boldsymbol{y}_{\boldsymbol{\alpha}_1} + \boldsymbol{x}_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2}$, by $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \Phi$, $\boldsymbol{\alpha} + \boldsymbol{\beta} \neq \boldsymbol{0}$, then $0 \neq [L_{\boldsymbol{\alpha}}, L_{\boldsymbol{\beta}}] \subset L_{\boldsymbol{\alpha}+\boldsymbol{\beta}}$ such that

$$[\boldsymbol{X}, L_{\boldsymbol{\alpha}_2}] \subset [L_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2} + L_{-\boldsymbol{\alpha}_1}, L_{\boldsymbol{\alpha}_2}] = 0,$$
$$0 \neq [\boldsymbol{X}, L_{\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}] \subset [L_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2} + L_{-\boldsymbol{\alpha}_1}, L_{\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}] \subset L_{\boldsymbol{\alpha}_2},$$
$$0 \neq [\boldsymbol{X}, L_{2\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}] \subset [L_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2} + L_{-\boldsymbol{\alpha}_1}, L_{2\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}] \subset L_{\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2},$$
$$0 \neq [\boldsymbol{X}, L_{3\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}] \subset [L_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2} + L_{-\boldsymbol{\alpha}_1}, L_{3\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}] \subset L_{2\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2},$$
$$0 \neq [\boldsymbol{X}, L_{-\boldsymbol{\alpha}_2}] \subset [L_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2} + L_{-\boldsymbol{\alpha}_1}, L_{-\boldsymbol{\alpha}_2}] \subset L_{3\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2} + L_{-\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2},$$
$$0 \neq [\boldsymbol{X}, L_{-\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2}] \subset [L_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2} + L_{-\boldsymbol{\alpha}_1}, L_{-\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2}] \subset L_{2\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2} + L_{-2\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2},$$
$$0 \neq [\boldsymbol{X}, L_{-2\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2}] \subset L_{\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2} + L_{-3\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2},$$
$$0 \neq [\boldsymbol{X}, L_{-3\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2}] \subset [L_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2} + L_{-\boldsymbol{\alpha}_1}, L_{-3\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2}] \subset L_{\boldsymbol{\alpha}_2},$$
$$[\boldsymbol{X}, L_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2} - L_{-\boldsymbol{\alpha}_1}] \subset [L_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2} + L_{-\boldsymbol{\alpha}_1}, L_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2} - L_{-\boldsymbol{\alpha}_1}] = 0,$$
$$0 \neq [\boldsymbol{X}, \bar{H}] \subset [L_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2} + L_{-\boldsymbol{\alpha}_1}, \bar{H}] \subset L_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2} - L_{-\boldsymbol{\alpha}_2},$$
$$0 \neq [\boldsymbol{X}, L_{-3\boldsymbol{\alpha}_1-2\boldsymbol{\alpha}_2} - L_{\boldsymbol{\alpha}_1}] \subset [L_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2} + L_{-\boldsymbol{\alpha}_1}, L_{-3\boldsymbol{\alpha}_1-2\boldsymbol{\alpha}_2} - L_{\boldsymbol{\alpha}_1}] \subset \bar{H}.$$

Base of M is adjoint action by $\boldsymbol{Y}$, by $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \Phi$, $\boldsymbol{\alpha} + \boldsymbol{\beta} \neq \boldsymbol{0}$, then $0 \neq [L_{\boldsymbol{\alpha}}, L_{\boldsymbol{\beta}}] \subset L_{\boldsymbol{\alpha}+\boldsymbol{\beta}}$ such that

$$0 \neq [\boldsymbol{Y}, L_{\boldsymbol{\alpha}_2}] \subset [L_{-3\boldsymbol{\alpha}_1-2\boldsymbol{\alpha}_2} + L_{\boldsymbol{\alpha}_1}, L_{\boldsymbol{\alpha}_2}] \subset L_{-3\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2} + L_{\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2},$$
$$0 \neq [\boldsymbol{Y}, L_{\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}] \subset [L_{-3\boldsymbol{\alpha}_1-2\boldsymbol{\alpha}_2} + L_{\boldsymbol{\alpha}_1}, L_{\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}] \subset L_{-2\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2} + L_{2\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2},$$
$$0 \neq [\boldsymbol{Y}, L_{2\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}] \subset [L_{-3\boldsymbol{\alpha}_1-2\boldsymbol{\alpha}_2} + L_{\boldsymbol{\alpha}_1}, L_{2\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}] \subset L_{-\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2} + L_{3\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2},$$
$$0 \neq [\boldsymbol{Y}, L_{3\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}] \subset [L_{-3\boldsymbol{\alpha}_1-2\boldsymbol{\alpha}_2} + L_{\boldsymbol{\alpha}_1}, L_{3\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}] \subset L_{-\boldsymbol{\alpha}_2},$$
$$0 \neq [\boldsymbol{Y}, L_{-\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2}] \subset [L_{-3\boldsymbol{\alpha}_1-2\boldsymbol{\alpha}_2} + L_{\boldsymbol{\alpha}_1}, L_{-\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2}] \subset L_{-\boldsymbol{\alpha}_2},$$
$$0 \neq [\boldsymbol{Y}, L_{-2\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2}] \subset [L_{-3\boldsymbol{\alpha}_1-2\boldsymbol{\alpha}_2} + L_{\boldsymbol{\alpha}_1}, L_{-2\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2}] \subset L_{-\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2},$$
$$0 \neq [\boldsymbol{Y}, L_{-3\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2}] \subset [L_{-3\boldsymbol{\alpha}_1-2\boldsymbol{\alpha}_2} + L_{\boldsymbol{\alpha}_1}, L_{-3\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2}] \subset L_{-2\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2},$$
$$[\boldsymbol{Y}, L_{-\boldsymbol{\alpha}_2}] \subset [L_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2} + L_{\boldsymbol{\alpha}_1}, L_{-\boldsymbol{\alpha}_2}] = 0,$$
$$0 \neq [\boldsymbol{Y}, L_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2} - L_{-\boldsymbol{\alpha}_1}] \subset [L_{-3\boldsymbol{\alpha}_1-2\boldsymbol{\alpha}_2} + L_{\boldsymbol{\alpha}_1}, L_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2} - L_{-\boldsymbol{\alpha}_1}] \subset \bar{H},$$
$$0 \neq [\boldsymbol{Y}, \bar{H}] \subset [L_{-3\boldsymbol{\alpha}_1-2\boldsymbol{\alpha}_2} + L_{\boldsymbol{\alpha}_1}, \bar{H}] \subset L_{-3\boldsymbol{\alpha}_1-2\boldsymbol{\alpha}_2} - L_{\boldsymbol{\alpha}_1},$$

$$[\boldsymbol{Y},\, L_{-3\boldsymbol{\alpha}_1-2\boldsymbol{\alpha}_2} - L_{\boldsymbol{\alpha}_1}] \subset [L_{-3\boldsymbol{\alpha}_1-2\boldsymbol{\alpha}_2} + L_{\boldsymbol{\alpha}_1},\, L_{-3\boldsymbol{\alpha}_1-2\boldsymbol{\alpha}_2} - L_{\boldsymbol{\alpha}_1}] = 0.$$

Let

$$M_1 = \coprod_{\boldsymbol{\alpha}\in\Phi''} L_{\boldsymbol{\alpha}} \quad (\text{with } \Phi'' = \pm\{\boldsymbol{\alpha}_2,\, \boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2,\, 2\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2,\, 3\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2\}),$$

$$M_2 = \mathrm{span}\{L_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2} - L_{-\boldsymbol{\alpha}_1},\, \bar{H},\, L_{-3\boldsymbol{\alpha}_1-2\boldsymbol{\alpha}_2} - L_{-\boldsymbol{\alpha}_1}\}.$$

If $\boldsymbol{m} \in M_1$, from A_1 adjoint action $\boldsymbol{m}$ we know that $A_1.\boldsymbol{m} = M_1$. So M_1 is irreducible module.

In the same way, we can know that M_2 is irreducible module.

Theorem 3.3 *The simple type G_2 Lie algebra can be direct sum decomposed as $G_2 = A_1 \oplus M$, where M is the direct sum of two irreducible modules.*

Proof. From Lemma 3.4, M in Theorem 3.3 is the direct sum of two irreducible module.

Corollary 3.3 *G_2 can be normal decomposed as $G_2 = A_1 \propto_{A_1} M$, where M is the direct sum of two irreducible modules.*

Proof. Obviously, M is normal subgroup, so the corollary is proved.

3.4 Decomposition Four

From Lemma 3.1, chosen $A_1 \cong \mathrm{sl}(2, F)$, $\boldsymbol{X} \in L_{-\boldsymbol{\alpha}_2} + L_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2}$, $\boldsymbol{Y} \in L_{\boldsymbol{\alpha}_2} + L_{-3\boldsymbol{\alpha}_1-2\boldsymbol{\alpha}_2}$, $\boldsymbol{H} \in H_{\boldsymbol{\alpha}_2} + H_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2}$.

By $\dim G_2 = 14$, let the complement space of A_1 on G_2 be M. Then the direct sum decomposition

$$M = \bar{H} \bigoplus \coprod_{\boldsymbol{\alpha}\in\Phi'''} L_{\boldsymbol{\alpha}} \bigoplus (L_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2} - L_{-\boldsymbol{\alpha}_2}) \bigoplus (L_{-3\boldsymbol{\alpha}_1-2\boldsymbol{\alpha}_2} - L_{\boldsymbol{\alpha}_2}),$$

where $\bar{H}$ is complementary space of $H_{\boldsymbol{\alpha}_2} + H_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2}$ on CSA,

$$\Phi''' = \pm\{\boldsymbol{\alpha}_1,\, \boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2,\, 2\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2,\, 3\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2\}.$$

Lemma 3.5 *M in Theorem 3.4 is the direct sum of three irreducible module.*

Proof. As mentioned above A_1 and eleven-dimensional space M.

Base of M is adjoint action by $\boldsymbol{X} = \boldsymbol{x}_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2} + \boldsymbol{y}_{\boldsymbol{\alpha}_1}$, by $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \Phi$, $\boldsymbol{\alpha}+\boldsymbol{\beta} \neq \boldsymbol{0}$, then $0 \neq [L_{\boldsymbol{\alpha}}, L_{\boldsymbol{\beta}}] \subset L_{\boldsymbol{\alpha}+\boldsymbol{\beta}}$ such that

$$[\boldsymbol{X},\, L_{\boldsymbol{\alpha}_1}] \subset [L_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2} + L_{-\boldsymbol{\alpha}_2},\, L_{\boldsymbol{\alpha}_1}] = 0,$$

$$0 \neq [\boldsymbol{X},\, L_{\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}] \subset [L_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2} + L_{-\boldsymbol{\alpha}_2},\, L_{\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}] \subset L_{\boldsymbol{\alpha}_1},$$

$$0 \neq [\boldsymbol{X},\, L_{-2\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2}] \subset [L_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2} + L_{-\boldsymbol{\alpha}_2},\, L_{-2\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2}] \subset L_{\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2},$$

$$[\boldsymbol{X},\, L_{2\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}] \subset [L_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2} + L_{-\boldsymbol{\alpha}_2},\, L_{2\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}] = 0,$$

$$0 \neq [\boldsymbol{X},\, L_{-\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2}] \subset [L_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2} + L_{-\boldsymbol{\alpha}_2},\, L_{-\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2}] \subset L_{2\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2},$$

$$0 \neq [\boldsymbol{X},\, L_{-\boldsymbol{\alpha}_1}] \subset [L_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2} + L_{-\boldsymbol{\alpha}_2},\, L_{-\boldsymbol{\alpha}_1}] \subset L_{-\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2},$$

$$[\boldsymbol{X},\, L_{3\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}] \subset [L_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2} + L_{-\boldsymbol{\alpha}_2},\, L_{3\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}] = 0,$$

$$0 \neq [\boldsymbol{X},\, L_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2} - L_{-\boldsymbol{\alpha}_2}] \subset [L_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2} + L_{-\boldsymbol{\alpha}_2},\, L_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2} - L_{-\boldsymbol{\alpha}_2}] \subset L_{3\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2},$$
$$0 \neq [\boldsymbol{X},\, \bar{H}] \subset [L_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2} + L_{-\boldsymbol{\alpha}_2},\, \bar{H}] \subset L_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2} + L_{-\boldsymbol{\alpha}_2},$$
$$0 \neq [\boldsymbol{X},\, L_{-3\boldsymbol{\alpha}_1-2\boldsymbol{\alpha}_2} - L_{\boldsymbol{\alpha}_2}] \subset [L_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2} + L_{-\boldsymbol{\alpha}_2},\, L_{-3\boldsymbol{\alpha}_1-2\boldsymbol{\alpha}_2} - L_{\boldsymbol{\alpha}_2}] \subset \bar{H},$$
$$0 \neq [\boldsymbol{X},\, L_{-3\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2}] \subset [L_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2} + L_{-\boldsymbol{\alpha}_2},\, L_{-3\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2}] \subset L_{-3\boldsymbol{\alpha}_1-2\boldsymbol{\alpha}_2} - L_{\boldsymbol{\alpha}_2}.$$

Base of M is adjoint action by $\boldsymbol{Y}$, by $\boldsymbol{\alpha}$, $\boldsymbol{\beta} \in \Phi$, $\boldsymbol{\alpha}+\boldsymbol{\beta} \neq \mathbf{0}$, then $0 \neq [L_{\boldsymbol{\alpha}},\, L_{\boldsymbol{\beta}}] \subset L_{\boldsymbol{\alpha}+\boldsymbol{\beta}}$ such that

$$0 \neq [\boldsymbol{Y},\, L_{\boldsymbol{\alpha}_1}] \subset [L_{-3\boldsymbol{\alpha}_1-2\boldsymbol{\alpha}_2} + L_{\boldsymbol{\alpha}_2},\, L_{\boldsymbol{\alpha}_1}] \subset L_{\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2},$$
$$0 \neq [\boldsymbol{Y},\, L_{\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}] \subset [L_{-3\boldsymbol{\alpha}_1-2\boldsymbol{\alpha}_2} + L_{\boldsymbol{\alpha}_2},\, L_{\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}] \subset L_{-2\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2},$$
$$[\boldsymbol{Y},\, L_{-2\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2}] \subset [L_{-3\boldsymbol{\alpha}_1-2\boldsymbol{\alpha}_2} + L_{\boldsymbol{\alpha}_2},\, L_{-2\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2}] = 0,$$
$$0 \neq [\boldsymbol{Y},\, L_{2\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}] \subset [L_{-3\boldsymbol{\alpha}_1-2\boldsymbol{\alpha}_2} + L_{\boldsymbol{\alpha}_2},\, L_{2\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}] \subset L_{-\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2},$$
$$0 \neq [\boldsymbol{Y},\, L_{-\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2}] \subset [L_{-3\boldsymbol{\alpha}_1-2\boldsymbol{\alpha}_2} + L_{\boldsymbol{\alpha}_2},\, L_{-\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2}] \subset L_{-\boldsymbol{\alpha}_1},$$
$$[\boldsymbol{Y},\, L_{-\boldsymbol{\alpha}_1}] \subset [L_{-3\boldsymbol{\alpha}_1-2\boldsymbol{\alpha}_2} + L_{\boldsymbol{\alpha}_2},\, L_{-\boldsymbol{\alpha}_1}] = 0,$$
$$0 \neq [\boldsymbol{Y},\, L_{3\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}] \subset [L_{-3\boldsymbol{\alpha}_1-2\boldsymbol{\alpha}_2} + L_{\boldsymbol{\alpha}_2},\, L_{3\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}] \subset L_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2} - L_{-\boldsymbol{\alpha}_2},$$
$$0 \neq [\boldsymbol{Y},\, L_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2} - L_{-\boldsymbol{\alpha}_2}] \subset [L_{-3\boldsymbol{\alpha}_1-2\boldsymbol{\alpha}_2} + L_{\boldsymbol{\alpha}_2},\, L_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2} - L_{-\boldsymbol{\alpha}_2}] \subset \bar{H},$$
$$0 \neq [\boldsymbol{Y},\, \bar{H}] \subset [L_{-3\boldsymbol{\alpha}_1-2\boldsymbol{\alpha}_2} + L_{\boldsymbol{\alpha}_2},\, \bar{H}] \subset L_{-3\boldsymbol{\alpha}_1-2\boldsymbol{\alpha}_2} + L_{\boldsymbol{\alpha}_2},$$
$$0 \neq [\boldsymbol{Y},\, L_{-3\boldsymbol{\alpha}_1-2\boldsymbol{\alpha}_2} + L_{\boldsymbol{\alpha}_2}] \subset [L_{-3\boldsymbol{\alpha}_1-2\boldsymbol{\alpha}_2} + L_{\boldsymbol{\alpha}_2},\, L_{-3\boldsymbol{\alpha}_1-2\boldsymbol{\alpha}_2} + L_{\boldsymbol{\alpha}_2}] \subset L_{-3\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2},$$
$$0 \neq [\boldsymbol{Y},\, L_{-3\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2}] \subset [L_{-3\boldsymbol{\alpha}_1-2\boldsymbol{\alpha}_2} + L_{\boldsymbol{\alpha}_2},\, L_{-3\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2}] = 0.$$

Let

$$M_1 = \operatorname{span}\{L_{\boldsymbol{\alpha}_1},\, L_{\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2},\, L_{-2\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2}\},$$
$$M_2 = \operatorname{span}\{L_{2\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2},\, L_{\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2},\, L_{-\boldsymbol{\alpha}_1}\},$$
$$M_3 = \operatorname{span}\{L_{3\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2},\, L_{3\boldsymbol{\alpha}_1+2\boldsymbol{\alpha}_2},\, \bar{H},\, L_{-3\boldsymbol{\alpha}_1-2\boldsymbol{\alpha}_2} - L_{-3\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2}\}.$$

If $\boldsymbol{m} \in M_1$, from A_1 adjoint action $\boldsymbol{m}$ we know that $A_1.\boldsymbol{m} = M_1$. So M_1 is irreducible module.

In the same way, we can know that M_2 and M_3 are irreducible modules.

Theorem 3.4 *The simple type G_2 Lie algebra can be direct sum decomposed as $G_2 = A_1 \oplus M$, where M is the direct sum of three irreducible modules.*

Proof. From Lemma 3.5, M in Theorem 3.4 is the direct sum of three irreducible module.

Corollary 3.4 *G_2 can be normal decomposed as $G_2 = A_1 \propto_{A_1} M$, where M is the direct sum of three irreducible modules.*

Proof. M obviously is normal subgroup, so the corollary is proved.

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