Communications
in
Mathematical
Research
35(1)(2019), 21–26

Fundamental Solution of Dirichlet Boundary Value Problem of Axisymmetric Helmholtz Equation

Zhang Kang-qun
(*Department of Mathematics and Physics, Nanjing Institute of Technology, Nanjing, 211167*)

Communicated by Yuan Hong-jun

Abstract: Fundamental solution of Dirichlet boundary value problem of axisymmetric Helmholtz equation is constructed via modified Bessel function of the second kind, which unified the formulas of fundamental solution of Helmholtz equation, elliptic type Euler-Poisson-Darboux equation and Laplace equation in any dimensional space.

Key words: Axisymmetic Helmholtz equation, fundamental solution, Dirichlet boundary value problem, similarity method

2010 MR subject classification: 35A08, 35J05, 35J25

Document code: A

Article ID: 1674-5647(2019)01-0021-06

DOI: 10.13447/j.1674-5647.2019.01.03

1 Introduction

In this paper, we study fundamental solution of Dirichlet boundary value problem of axisymmetric Helmholtz equation in the upper half space

$$\begin{cases} \partial_t^2 u + \Delta_x u + \dfrac{\alpha}{t}\partial_t u + \lambda^2 u = 0 & \text{in } \mathbf{R}_+^{n+1}, \\ u(0, x) = \delta(x) & \text{in } \mathbf{R}^n, \\ u(+\infty, x) \ \text{ is bounded}, \end{cases} \tag{1.1}$$

where

$$\mathbf{R}_+^{n+1} = \{(t, x) : t > 0, x \in \mathbf{R}^n\},$$

and real-valued parameters $\alpha < 1$ and $\lambda > 0$.

Received date: Sept. 18, 2017.

Foundation item: The NSF (11326152) of China, the NSF (BK20130736) of Jiangsu Province of China and the NSF (CKJB201709) of Nanjing Institute of Technology.

E-mail address: chkqnju@hotmail.com (Zhang K Q).

This problem is closely connected with the study of electromagnetic scattering (see [1]–[2]). During considering the processes taking place in some inhomogeneous media with fractal structure one also must take into account fluctuations of the parameter's value. For example, the parameter α in (1.1) has the sense of Hausdorff dimension when studying the probability density function in the capacity of unknown (see [3]).

For the case $\alpha = 0$ and $\lambda = 0$, (1.1) is the classic Laplace equation. When $\alpha = 0$ and $\lambda > 0$, (1.1) becomes the classic Helmholtz equation, this is the reason that it is called axisymmetric Helmholtz equation.

The method of fundamental solutions plays an important role in study of partial differential equations. In [4], the potentials were constructed via fundamental solutions of Helmholtz equation. Analogously to the potential theory for Helmholtz equation, one can construct a potential for axisymmetric Helmholtz equation whose kernels are written via fundamental solutions of axisymmetric Helmholtz equation. Various regularization approaches for the study of the method of fundamental solutions were studied in [5]. In this scope, the method of fundamental solutions generated by classic Helmholtz operator and axisymmetric Helmholtz equation with $\alpha > 0$ in different dimensional space were also investigated, see [6]–[10] and the references therein. Thus, the case of $\alpha < 0$ and $\lambda > 0$ might become an object of a new research.

When $\lambda = 0$, (1.1) is elliptic type generalized Euler-Poisson-Darboux equation whose fundamental solutions was established by similarity method in [11]. Motivated by this, we construct fundamental solution

$$P(t,\, x,\, \alpha,\, \lambda) = \frac{(\mathrm{i}\lambda)^{\frac{1+n-\alpha}{2}}}{2^{\frac{n-\alpha-1}{2}}\pi^{\frac{n}{2}}\Gamma\Big(\dfrac{1-\alpha}{2}\Big)} \cdot \frac{t^{1-\alpha}K_{\frac{1+n-\alpha}{2}}(\mathrm{i}\lambda\sqrt{t^2+|x|^2})}{(\sqrt{t^2+|x|^2})^{\frac{1+n-\alpha}{2}}} \tag{1.2}$$

and then solves (1.1) in general sense in the upper half space. In particular, the explicit formula of $P(t,\, x,\, \alpha,\, \lambda)$ is not restricted by dimensional numbers.

2 Construction of Fundamental Solution

In [11], the fundamental solution $P(x,\, y,\, \alpha,\, 0)$ of (1.1) was constructed by use of similarity method

$$P(t,x,\alpha,0) = \frac{\Gamma\Big(\dfrac{1+n-\alpha}{2}\Big)}{\pi^{\frac{n}{2}}\Gamma\Big(\dfrac{1-\alpha}{2}\Big)}\frac{t^{1-\alpha}}{(t^2+|x|^2)^{\frac{1+n-\alpha}{2}}}, \tag{2.1}$$

where $\Gamma(z)$ is Gamma function. In this section, we seek a fundamental solution in the same form

$$P(x,\, y,\, \alpha,\, \lambda) = C(\alpha,\, \lambda)t^{1-\alpha}p(t^2+|x|^2)$$

for the case $\lambda > 0$, where the constant $C(\alpha,\, \lambda)$ will be determined in the following.

Set

$$r = |x|, \qquad s = t^2 + r^2, \qquad u(t,x) = t^{1-\alpha}v(s).$$

Then (1.1) is changed into

$$4sv''(s) + (6 + 2n - 2\alpha)v'(s) + \lambda^2 v(s) = 0.$$

Let

$$v(s) = s^{\frac{\alpha-1-n}{4}} w(z), \qquad z = \mathrm{i}\lambda s^{\frac{1}{2}}.$$

Then we derive

$$w''(z) + \frac{1}{z}w'(z) - \left(1 + \frac{(1+n-\alpha)^2}{4z^2}\right) w(z) = 0.$$

This is a special Bessel equation (see [10]) which has a solutions K_ν (modified Bessel function of the second kind) with

$$\nu = \frac{1+n-\alpha}{2}.$$

In order to derive the fundamental solution of (1.1), we recite some results of Bessel functions given in [12] as our Lemmas 2.1 and 2.2.

Lemma 2.1 *For $\nu > 0$, the asymptotic behavior for a small z,*

$$J_\nu(z) \sim \frac{1}{\Gamma(\mu+1)} \left(\frac{z}{2}\right)^\nu, \qquad z \to 0, \tag{2.2}$$

$$K_\nu(z) \sim \frac{1}{2}\Gamma(\mu) \left(\frac{z}{2}\right)^{-\nu}, \qquad z \to 0, \tag{2.3}$$

where J_ν is a Bessel function of the first kind.

Lemma 2.2 *For $\nu > 0$ and $-\pi < \operatorname{Arg} z \le \pi$, the asymptotic behavior for large z satisfies*

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}} \mathrm{e}^{-z} \sum_{r=1}^{\infty} \frac{(4\nu^2 - 1^2)\cdots(4\nu^2 - (2r-3)^2)}{(r-1)!(8z)^{r-1}}.$$

Lemma 2.3 *For $\nu = \dfrac{1+n-\alpha}{2}$, there exists*

$$\int_0^\infty \frac{K_\nu(\mathrm{i}\lambda\sqrt{t^2+r^2})}{(\sqrt{t^2+r^2})^\nu} r^{2(\frac{n}{2}-1)+1} \mathrm{d}r = \frac{2^{\frac{n-2}{2}}\Gamma\left(\frac{n}{2}\right)}{(\mathrm{i}\lambda)^{\frac{n}{2}} t^{\nu-\frac{n}{2}}} K_{\nu-\frac{n}{2}}(\mathrm{i}\lambda t).$$

Proof. The integral given in page 416 of [13]

$$\int_0^\infty J_\mu(br) \frac{K_\nu(a\sqrt{t^2+r^2})}{(\sqrt{t^2+r^2})^\nu} r^{\mu+1} \mathrm{d}r$$
$$= \left(\frac{b}{a}\right)^\mu \left(\frac{\sqrt{a^2+b^2}}{t}\right)^{\mu-\nu-1} K_{\nu-\mu-1}\left(t\sqrt{a^2+b^2}\right) \tag{2.4}$$

is uniformly convergent for $\mu = \dfrac{n}{2} - 1$, $b > 0$ and $a = \mathrm{i}\lambda$. Then, in terms of (2.2), we divide by $b^{\frac{n}{2}-1}$ in integral (2.4) and make b tending to zero, we obtain Lemma 2.3.

According to Lemma 2.2, $K_\nu(z)$ tends to zero at infinity. Then

$$u(t, x) = C(\alpha, \lambda) \frac{t^{1-\alpha} K_\nu(\mathrm{i}\lambda\sqrt{t^2+|x|^2})}{(\sqrt{t^2+|x|^2})^\nu}$$

is bounded at infinity and satisfies (1.1), where $C(\alpha,\,\lambda)$ is a constant. In order to determine $C(\alpha,\,\lambda)$, integrate $u(t,\,x)$ in spherical coordinates, then take the value equal to one when t tends to zero, we arrive at

$$\begin{aligned}1 &= C(\alpha,\,\lambda)\sigma_{n-1}\lim_{t\to 0} t^{1-\alpha}\int_0^\infty \frac{r^{n-1}K_\nu\left(\mathrm{i}\lambda\sqrt{t^2+r^2}\right)}{\left(\sqrt{t^2+|r|^2}\right)^\nu}\mathrm{d}r\\ &= C(\alpha,\,\lambda)\left(\frac{2\pi}{\mathrm{i}\lambda}\right)^{\frac{n}{2}}\lim_{t\to 0} t^{\frac{1-\alpha}{2}}K_{\nu-\frac{n}{2}}(\mathrm{i}\lambda t). \qquad (2.5)\end{aligned}$$

In the second equality we have used Lemma 2.3. Furthermore, by use of (2.5) and the asymptotic behavior of $K_\mu(z)$ given in (2.3), we obtain

$$C(\alpha,\,\lambda) = \frac{(\mathrm{i}\lambda)^{\frac{1+n-\alpha}{2}}}{2^{\frac{n-\alpha-1}{2}}\pi^{\frac{n}{2}}\Gamma\Big(\dfrac{1-\alpha}{2}\Big)}.$$

This yields

$$u(t,x) = \frac{(\mathrm{i}\lambda)^{\frac{1+n-\alpha}{2}}}{2^{\frac{n-\alpha-1}{2}}\pi^{\frac{n}{2}}\Gamma\Big(\dfrac{1-\alpha}{2}\Big)}\cdot\frac{t^{1-\alpha}K_{\frac{1+n-\alpha}{2}}(\mathrm{i}\lambda\sqrt{t^2+|x|^2})}{(\sqrt{t^2+|x|^2})^{\frac{1+n-\alpha}{2}}}. \qquad (2.6)$$

Then, we conclude

Theorem 2.1 *For $\alpha<1$ and $\lambda>0$, $u(t,\,x)$ defined by* (2.6) *is the fundamental solution $P(t,\,x,\,\alpha,\,\lambda)$ of Dirichlet boundary value problem* (1.1) *which satisfies*

$$\lim_{t\to 0}\int_{\mathbf{R}^n} P(t,\,x,\,\alpha,\,\lambda)\mathrm{d}x = 1.$$

Proof. For any infinitely differentiable finite function $\phi(x)$ from the space $\mathfrak{D}(\mathbf{R}^n)$,

$$\begin{aligned}&\Big|\phi(x)*P(t,x,\,\alpha,\,\lambda) - \int_{\mathbf{R}^n}\phi(x)P(t,\,y,\,\alpha,\lambda)\mathrm{d}y\Big|\\ =&\Big|\int_{\mathbf{R}^n}\phi(x-y)P(t,\,y,\,\alpha,\,\lambda)\mathrm{d}y - \phi(x)\int_{\mathbf{R}^n}P(t,\,y,\,\alpha,\,\lambda)\mathrm{d}y\Big|\\ \le&\int_{\mathbf{R}^n}\big|\phi(x-y)-\phi(x)\big|\big|P(t,\,y,\,\alpha,\,\lambda)\big|\mathrm{d}y\\ \le&\int_{|x|<\delta}\big|\phi(x-y)-\phi(x)\big|\big|P(t,\,y,\,\alpha,\,\lambda)\big|\mathrm{d}y\\ &\quad+\int_{|x|\ge\delta}\big|\phi(x-y)-\phi(x)\big|\big|P(t,\,y,\,\alpha,\,\lambda)\big|\mathrm{d}y\\ \le&\frac{\varepsilon}{M}\int_{\mathbf{R}^n}\big|P(t,\,y,\,\alpha,\,\lambda)\big|\mathrm{d}y + M\int_{|s|\ge\delta}\big|P(t,\,y,\,\alpha,\,\lambda)\big|\mathrm{d}y \qquad (2.7)\end{aligned}$$

for some positive constant M and sufficiently small δ, where "$*$" is denoting a convolution. In terms of the analyticity of $K_\nu(z)$ on the whole plane except the origin and the asymptotic behavior in Lemma 2.2, $P(x,\,y,\,\alpha,\,\lambda)$ is absolutely integrable on $x\in\mathbf{R}^n$. Then, we obtain

$$\int_{\mathbf{R}^n}\big|P(t,\,s,\,\alpha,\,\lambda)\big|\mathrm{d}s < M$$

and for sufficiently small t,

$$\int_{|s|\ge\delta}\big|P(t,\,s,\,\alpha,\,\lambda)\big|\mathrm{d}s < \frac{\varepsilon}{M}.$$

Substituting above two inequalities into (2.7), then for sufficiently small t, we obtain

$$\Big|\phi(x) * P(t,\, x,\, \alpha,\, \lambda) - \phi(x)\Big| < 2\varepsilon. \tag{2.8}$$

This implies

$$\lim_{t\to 0} \phi(x) * P(t,\, x,\, \alpha,\, \lambda) = \phi(x) = \int_{\mathbf{R}^n} \phi(y)\delta(x-y)\mathrm{d}y. \tag{2.9}$$

which means that $P(t,\, x,\, \alpha,\, \lambda)$ converges to the δ-function when $t \to 0$ in a space of generalized functions $\mathfrak{D}'(\mathbf{R}^n)$

Hence, we complete the proof of Theorem 2.1.

3 Applications

The elliptic type Euler-Poisson-Darboux equation is obtained from axisymmetric Helmhotz equation for $\lambda = 0$ and the fundamental solution (1.3) of the corresponding Dirichlet boundary value problem is consistent with the formula derived by (1.2) when λ tends to zero.

The fundamental solution of Dirichlet boundary value problem of Helmhotz equation is obtained by passing here to the limit when α tends to zero

$$P(t,\, x,\, 0,\, \lambda) = \frac{(\mathrm{i}\lambda)^{\frac{1+n}{2}}}{2^{\frac{n-1}{2}}\pi^{\frac{n+1}{2}}} \cdot \frac{tK_{\frac{1+n}{2}}(\mathrm{i}\lambda\sqrt{t^2+|x|^2})}{(\sqrt{t^2+|x|^2})^{\frac{1+n}{2}}}.$$

Furthermore, passing here to the limit when α and λ tend to zero, we obtain the Poisson kernel

$$P(t,\, x,\, 0,\, 0) = \frac{\Gamma\Big(\dfrac{n+1}{2}\Big)}{\pi^{\frac{n+1}{2}}} \cdot \frac{t}{(t^2+|x|^2)^{\frac{n+1}{2}}},$$

which is the fundamental solution of Dirichlet boundary value problem of Laplace equation.

Last, considering Dirichlet boundary value problem

$$\begin{cases} \partial_t^2 u + \Delta_x u + \dfrac{\alpha}{t}\partial_t u + \lambda^2 u = 0 & \text{in } \mathbf{R}_+^{n+1}, \\ u(0,\, x) = \varphi(x) & \text{in } \mathbf{R}^n, \\ u(+\infty,\, x) \ \text{ is bounded} & \end{cases}$$

with $\alpha < 1$ and $\lambda > 0$, by use the fundamental solution given in Theorem 2.1, we have the exact solution

$$u(t,\, x) = \frac{(\mathrm{i}\lambda)^{\frac{1+n-\alpha}{2}}}{2^{\frac{n-\alpha-1}{2}}\pi^{\frac{n}{2}}\Gamma\Big(\dfrac{1-\alpha}{2}\Big)} t^{1-\alpha} \int_{\mathbf{R}^n} \frac{K_{\frac{1+n-\alpha}{2}}(\mathrm{i}\lambda\sqrt{t^2+|x-y|^2})}{(\sqrt{t^2+|x-y|^2})^{\frac{1+n-\alpha}{2}}}\varphi(y)\mathrm{d}y$$

if $\varphi(x)$ is a generalized function and the convolution exists.

References

[1] Chandler-Wilde S N, Ross C R. Scattering by rough surfaces: the Dirichlet problem for the Helmholtz equation in a non-locally perturbed half-plane. *Math. Methods Appl. Sci.*, 1996, **19**(12): 959–976.

[2] Colton D, Kress R. Integral Equation Methods in Scattering Theory. Society for Industrial and Applied Mathematics. New York: Wiley, 2013.

[3] O'Shaughnessy B, Procaccia I. Diffusion on fractals. *Phys. Rev. A*, 1985, **32**(5): 3073–3083.

[4] Koshlyakov N S, Gliner E B, Smirnov M M. Basic Differential Equations of Mathematical Physics (in Russian). Moscow: Fizmatgiz, 1962.

[5] Lin J, Chen W, Wang F. A new investigation into regularization techniques for the method of fundamental solutions. *Math. Comput. Simulation*, 2011, **81**(6): 1144–1152.

[6] Chen W, Lin J, Chen C S. The method of fundamental solutions for solving exterior axisymmetric Helmholtz problems with high wave-number. *Adv. Appl. Math. Mech.*, 2013, **5**(4): 477–493.

[7] Gilbert R P. Function Theoretic Methods in Partial Differential Equations. New York: Academic Press, 1969.

[8] Hasanov A. Fundamental solutions of generalized bi-axially symmetric Helmholtz equation. *Complex Var. Elliptic Equ.*, 2007, **52**(8): 673–683.

[9] Mavlyaviev R M, Garipov I B. Fundamental solution of multidimensional axisymmetric Helmholtz equation. *Complex Var. Elliptic Equ.*, 2017, **62**(3): 287–296.

[10] Tsai C C, Chen C S, Hsu T W. The method of particular solutions for solving axisymmetric polyharmonic and poly-Helmholtz equations. *Eng. Anal. Bound. Elem.*, 2009, **33**(12): 1396–1402.

[11] Algazin O D. Construction by similarity method of the fundamental solution of the Dirichlet problem for Keldysh type equation in the half-space. arXiv:1612.00205, *preprint*, 2016.

[12] McLachlan N W. Bessel Functions for Engineers. Oxford: Oxford Clarendon Press, 1955.

[13] Watson G N. A Treatise on the Theory of Bessel Functions. Cambridge: Cambridge University Press, 1995.