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On the Existence of Time-periodic Solution to the Compressible Heat-conducting Navier-Stokes Equations

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Abstract: We study in this article the compressible heat-conducting Navier-Stokes equations in periodic domain driven by a time-periodic external force. The existence of the strong time-periodic solution is established by a new approach. First, we reformulate the system and consider some decay estimates of the linearized system. Under some smallness and symmetry assumptions on the external force, the existence of the time-periodic solution of the linearized system is then identified as the fixed point of a Poincaré map which is obtained by the Tychonoff fixed point theorem. Although the Tychonoff fixed point theorem cannot directly ensure the uniqueness, but we could construct a set-valued function, the fixed point of which is the time-periodic solution of the original system. At last, the existence of the fixed point is obtained by the Kakutani fixed point theorem. In addition, the uniqueness of time-periodic solution is also studied.

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1 Introduction

In this paper, we prove the existence and uniqueness of the strong time-periodic solution to the Navier-Stokes equations for compressible heat-conducting fluids:

$$\rho_t + \operatorname{div}(\rho u) = 0, \tag{1.1}$$

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$$\rho(u_t + (u \cdot \nabla)u) + \nabla P(\rho, \theta) = \mu\Delta u + (\mu + \lambda)\nabla \mathrm{div} u + \rho f(x, t), \tag{1.2}$$

$$\rho C_\nu(\theta_t + u \cdot \nabla\theta) + \theta P_\theta(\rho, \theta)\mathrm{div} u = \kappa\Delta\theta + \Phi(u), \tag{1.3}$$

when the external force is time-periodic with T-period

$$f(t + T, x) = f(t, x)$$

for all t, x. Here, $\rho(x, t)$, $u(x, t) = (u_1, \cdots, u_n)(x, t)$, $\theta(x, t)$ represent the fluid density, velocity and temperature, respectively. $t \in \mathbf{R}$ is the time and x is the spatial variable confined to $\Omega \subset \mathbf{R}^n$ with $n \geq 3$. $P(\rho, \theta)$ is the pressure which is a smooth function of ρ, θ. μ, λ are the viscosity coefficients which are assumed to satisfy the physical restrictions

$$\mu > 0, \qquad \frac{n}{2}\lambda + \mu \geq 0.$$

The constants C_ν and κ are the heat capacity at constant volume and the coefficient of heat conductivity. The classical dissipation function $\Phi(u)$ is given by

$$\Phi(u) = \frac{\mu}{2}\sum_{i,j=1}^{n}(\partial_i u_j + \partial_j u_i)^2 + \lambda\sum_{j=1}^{n}(\partial_j u_j)^2.$$

Throughout the paper, we consider $\Omega := [-L, L]^n$. Let the density and the temperature satisfy the obvious physical requirements

$$\int_\Omega \rho \mathrm{d}x = \bar{\rho} > 0, \qquad \theta|_{\partial\Omega} = \bar{\theta}, \tag{1.4}$$

where $\bar{\rho}$ and $\bar{\theta}$ are given constants. Assume that the external force

$$f(x, t) = (f_1, \cdots, f_n)(x, t)$$

is spacial periodic with period $2L$ and satisfies

$$f_i(Y_i(x), t) = -f_i(x, t), \quad f_i(Y_j(x), t) = f_i(x, t), \qquad \forall\, i \neq j, \tag{1.5}$$

for all $i = 1, \cdots, n$ and $t \in \mathbf{R}$, where

$$Y_i[x_1, \cdots, x_i, \cdots, x_n] = [x_1, \cdots, -x_i, \cdots, x_n].$$

These conditions are to ensure that the Poincaré inequality holds. In fact, we can consider the following no-stick boundary conditions for the velocity:

$$u(t, x) \cdot n(x) = 0, \quad [\mathrm{D}u(t, x) \cdot n(x)]_\tau = 0 \qquad \text{on } \partial\Omega', \tag{1.6}$$

where $n(x)$ denotes the outer normal vector and $[w(x, t)]_\tau$ is the projection of a vector $w(t, x)$ on the tangent plane to $\partial\Omega'$ at the point x. In the n-dimensional case and the boundary is flat, (1.6) means that the vorticity is perpendicular to the boundary. For the physical background as well as further properties of flows on domains with frictionless boundary, we refer to [1]. To simplify the presentation, we restrict our attention to a particular class of spatial domains, specifically, we assume that Ω' is an n-dimensional cube:

$$\Omega' = [0, L]^n.$$

Then, the boundary conditions (1.6) read as

$$u_i = 0$$

on the opposite faces

$$\{x_i = 0, x_j = [0, L], i \neq j\} \bigcup \{x_i = L, x_j = [0, L], i \neq j\},$$

$$\frac{\partial u_j}{\partial x_i} = 0$$

for $i \neq j$ on the opposite faces

$$\{x_i = 0,\, x_j = [0,\, L],\, i \neq j\} \bigcup \{x_i = L,\, x_j = [0,\, L],\, i \neq j\}$$

for all $i = 1, \cdots, n$. Thus, a suitable function space framework is provided by the spatially periodic functions, i.e., the unknown functions are prolonged on the periodic domain Ω with the following geometrical conditions:

$$\rho(Y_i(x),\, t) = \rho(x,\, t), \qquad \theta(Y_i(x),\, t) = \theta(x,\, t), \tag{1.7}$$

$$u_i(Y_i(x),\, t) = -u_i(x,\, t), \quad u_i(Y_j(x),\, t) = u_i(x,\, t), \qquad \forall\, i \neq j,\ i = 1, \cdots, n, \tag{1.8}$$

where

$$Y_i[x_1, \cdots, x_i, \cdots, x_n] = [x_1, \cdots, -x_i, \cdots, x_n].$$

Therefore, it is worth to consider the external force with some structural condition (1.5).

Now, we are able to state the main result of this paper.

Theorem 1.1 *Let* $m \geq \left[\frac{n}{2}\right] + 1$, $n \geq 3$. *Assume that* $P(\rho,\, \theta)$ *is a smooth function near* $(\bar\rho,\, \bar\theta)$ *satisfying*

$$P_\rho(\bar\rho,\, \bar\theta) > 0, \qquad P_\theta(\bar\rho,\, \bar\theta) > 0.$$

If the T*-periodic external force* $f \in L^2(0,\, T;\, H^m(\Omega^n))$ *satisfies the geometric condition* (1.5) *with*

$$\int_0^T \|f\|_{H^m}^2 \mathrm{d}t \leq \eta,$$

η *appropriately small, then the problem* (1.1)–(1.4) *admits a unique* T*-periodic solution* $(\rho,\, u,\, \theta)$ *satisfying* (1.7)–(1.8) *and* $(\rho - \bar\rho,\, u,\, \theta - \bar\theta) \in X_\delta$, *where* X_δ *is defined in Section* 2, $\delta = \eta^{\frac{1}{4}}$.

As is known, time-periodic flow is one of the interesting phenomena in fluid mechanics. In the recent years, there has been lots of interest in the study of the time-periodic problems for fluid dynamical equations. A number of existence results on time-periodic solutions have been established in different ways. For the case of the incompressible Navier-Stokes equations, we only refer to [2]–[7] and reference cited therein.

For the case of the compressible Navier-Stokes equations, Feireisl *et al.* considered the Navier-Stokes equations for isentropic flows, and studied the no-stick boundary condition and the spatially flat boundary case (1.6) in three dimension (see [8]–[9]). By using the Faedo-Galerkin method and the vanishing viscosity method, they obtained the existence of weak time-periodic solutions. In [10], Cai *et al.* improved the result of [9] by extending the class of pressure functions. With similar ideas and techniques in [9], Yan[11] constructed a weak time-periodic solution of ferrofluids driven by time-periodic external forces.

On the other hand, we mention the work on the strong time-periodic solution of the compressible Navier-Stokes equations in a bounded domain. In [12], Valli established the existence of a strong time-periodic solution in a bounded domain with non-slip boundary condition by using Serrin's method. He also proved the stability of the time-periodic solution. For a spatially periodic domain, Jin *et al.* obtained the existence of a strong time-periodic solution to the three dimensional compressible Navier-Stokes system by employing the topological degree theory and energy method in [13]. Also, Cai *et al.* considered the compressible magnetohydrodynamic equations in [14]. For three dimensional compressible damped Euler equations in a periodic domain, Tan *et al.*[15] have proved the existence and uniqueness of a time periodic solution by adapting a regularized approximation scheme and applying the topological degree theory.

While for the whole space, Ma *et al.* showed that a strong time-periodic solution exists when the space dimension $n \geq 5$ under some smallness assumption in [16]. They proved the existence and uniqueness of the time-periodic solution by applying the energy method and the spectral analysis of the optimal decay estimates together with the contraction mapping theorem. In the past few years, Jin *et al.*[17] have investigated the compressible Navier-Stokes equations in $\mathbf{R}^3$ when external force satisfies the oddness condition. They have established the existence of a strong time-periodic solution by using the topological degree theory. Without symmetry of the external force, Jin[18] also showed the existence and uniqueness of the time-periodic solution of the non-isentropic Navier-Stokes equations in $\mathbf{R}^4$. Meanwhile, by the spectral properties, Tsuda *et al.*[19]–[21] obtained a time-periodic solution for the small time-periodic external force when the spatial dimension is greater than or equal to 3. Moreover, they obtained the asymptotically stability of the time-periodic solution.

To our best knowledge, there are only a few works on time-periodic solutions of the non-isentropic Navier-Stokes equations. In this paper, our goal is to seek a strong time-periodic solution around the constant state $(\bar{\rho}, 0, \bar{\theta})$ to the problem (1.1)–(1.4) by adopting a different approach. First, we reformulate the problem and give some energy estimates of the linearized system. Then, we construct a Poincaré map from an initial value $\varPhi$ to the state $U(T, \varPhi)$, where $U(t, \varPhi)$ is the solution of the linearized system corresponding to the initial data $\varPhi$. Under some smallness and symmetry assumptions on the external force, the existence of the time-periodic solution of the linearized system is identified as a fixed point of this Poincaré map by using the Tychonoff fixed point theorem. From the details of the proof, we can see the solution with some special initial data in a convex hull is exactly the time-periodic solution of the linearized system. Although the Tychonoff fixed point theorem cannot directly ensure the uniqueness of the time-periodic solution of the linearized system, we could construct a set-valued function, the fixed point of which is the time-periodic solution of the original system. The existence of the fixed point of the set-valued function is obtained by the Kakutani fixed point theorem. At last, we prove the uniqueness of the time-periodic solution.

Comparing with the previous result in [13], the regularity of the external force needed in our paper is lower. This difference is caused by the method. The authors in [10]–[13] first reformulated the problem and add a regularized term. They obtain the existence of the time-periodic solution by using a mild form. To obtain the existence of the time-periodic solution of the linearized system, we study a bounded solution with trivial initial data. This idea is from the Massera type criteria for linear periodic evolution equations [22]–[23].

The rest of this paper is organized as follows. We reformulate the problem and give some preliminaries in Section 2. In Section 3, we prove some energy estimates for later use. And the proof of Theorem 1.1 is given in Sections 4 and 5.

Notations Throughout this paper, we will omit the variables t, x of a function if it does not cause any confusion. We denote the spatial integral on Ω by $\int \mathrm{d}x$ for simplicity. Moreover, C denotes a generic positive constant which may vary in different estimates. $C_{a,b,\cdots} > 0$ is also a generic constant which depends on a, b, $\cdots$. For a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$, it is standard that

$$\partial_x^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\cdots\partial_{x_n}^{\alpha_n}, \qquad |\alpha| = \sum_{i=1}^n \alpha_i.$$

The norm of Hilbert space $H^s(\Omega)$ is denoted by $\|\cdot\|_{H^s}$. And we use $L^p(\Omega)$, $1 \le p \le \infty$ to denote the Sobolev space with norm $\|\cdot\|_p$.

2 Preliminaries

We reformulate the equations (1.1)–(1.3) as follows. Let $\sigma = \rho - \bar\rho$, $v = \theta - \bar\theta$. The system (1.1)–(1.3) can be rewritten as

$$\sigma_t + \bar\rho \mathrm{div} u + u\nabla\sigma = G_1(\sigma, u), \tag{2.1}$$

$$u_t - (\bar\mu\Delta u + (\bar\mu + \bar\lambda)\nabla\mathrm{div} u) + \gamma_1\bar\rho\nabla\sigma + \gamma_2\bar\rho\nabla v = G_2(\sigma, u, v) + f, \tag{2.2}$$

$$v_t - \bar\kappa\Delta v + \gamma_3\gamma_2\bar\rho\mathrm{div} u = G_3(\sigma, u, v), \tag{2.3}$$

where

$$\begin{aligned} G_1(\sigma, u) &= -\sigma\mathrm{div} u, \\ G_2(\sigma, u, v) &= -u\cdot\nabla u - g_1(\sigma, v)\nabla\sigma - g_2(\sigma, v)\nabla v - h(\sigma)(\bar\mu\Delta u + (\bar\mu + \bar\lambda)\nabla\mathrm{div} u), \\ G_3(\sigma, u, v) &= -u\cdot\nabla v - \bar\kappa h(\sigma)\Delta v - g_3(\sigma, v)\nabla u + \frac{\Phi(u)}{\bar\rho C_\nu} - \frac{h(\sigma)\Phi(u)}{\bar\rho C_\nu} \end{aligned}$$

and

$$h(\sigma) = \frac{\sigma}{\sigma + \bar\rho},$$

$$g_1(\sigma, v) = \frac{P_\rho(\sigma + \bar\rho, v + \bar\theta)}{\sigma + \bar\rho} - \frac{P_\rho(\bar\rho, \bar\theta)}{\bar\rho},$$

$$g_2(\sigma, v) = \frac{P_\theta(\sigma + \bar\rho, v + \bar\theta)}{\sigma + \bar\rho} - \frac{P_\theta(\bar\rho, \bar\theta)}{\bar\rho},$$

$$g_3(\sigma, v) = \frac{(v + \bar\theta)P_\theta(\sigma + \bar\rho, v + \bar\theta)}{C_\nu(\sigma + \bar\rho)} - \frac{\bar\theta P_\theta(\bar\rho, \bar\theta)}{C_\nu\bar\rho},$$

$$\bar\mu = \frac{\mu}{\bar\rho}, \quad \bar\lambda = \frac{\lambda}{\bar\rho}, \quad \gamma_1 = \frac{P_\rho(\bar\rho, \bar\theta)}{\bar\rho^2}, \quad \gamma_2 = \frac{P_\theta(\bar\rho, \bar\theta)}{\bar\rho^2}, \quad \gamma_3 = \frac{\bar\theta}{C_\nu}, \quad \bar\kappa = \frac{\kappa}{\bar\rho C_\nu}.$$

Obviously, G_1, G_2, G_3 have the following properties:

$$G_1(\sigma, u) = -\sigma\mathrm{div} u, \tag{2.4}$$

$$G_2(\sigma, u, v) \sim (u\cdot\nabla)u + \sigma\nabla\sigma + v\nabla\sigma + \sigma\nabla v + v\nabla v + \sigma\Delta u + \sigma\nabla\mathrm{div} u, \tag{2.5}$$

$$G_3(\sigma, u, v) \sim u\cdot\nabla v + \sigma\Delta v + \sigma\nabla\cdot u + v\nabla\cdot u + \Psi(u) + \sigma\Psi(u). \tag{2.6}$$

Here $\sim$ means that two sides are of the same order.

For $m \geq \left[\frac{n}{2}\right]+1$, we define the following suitable function space to deal with the problem:

$$\begin{aligned}X = \big\{&(\sigma,\, u,\, v) \,\big|\, (\sigma,\, u,\, v) \text{ is spatially periodic with } 2L,\ \sigma \in L^\infty(0,\, T;\, H^{m+1}(\Omega)),\\ &(u,\, v) \in L^\infty(0,\, T;\, H^{m+1}(\Omega)) \cap L^2(0,\, T;\, H^{m+2}(\Omega)),\\ &\sigma(Y_i(x),\, t) = \sigma(x,\, t),\ v(Y_i(x),\, t) = v(x,t),\ u_i(Y_i(x),\, t) = -u_i(x,\, t),\\ &u_i(Y_j(x),\, t) = u_i(x,\, t),\ \ \forall i \neq j,\ i = 1, \cdots, n,\\ &\text{where}\quad Y_i[x_1, \cdots, x_i, \cdots x_n] = [x_1, \cdots, -x_i, \cdots, x_n]\big\},\end{aligned}$$

$$X_\delta = \left\{(\sigma,\, u,\, v) \in X \middle| \|(\sigma,\, u,\, v)\|_X^2 := \sup_{t\in[0,\, T]} \|(\sigma,\, u,\, v)\|_{H^{m+1}}^2 + \int_0^T \|(u,\, v)\|_{H^{m+2}}^2 \mathrm{d}t \leq \delta^2\right\}.$$

And the space X_{-1} is defined by

$$\begin{aligned}X_{-1} = \big\{&(\sigma,\, u,\, v) \,\big|\, (\sigma,\, u,\, v) \text{ is spatially periodic with } 2L,\ \sigma \in L^\infty(0,\, T;\, H^m(\Omega)),\\ &(u,\, v) \in L^\infty(0,\, T;\, H^m(\Omega)) \cap L^2(0,\, T;\, H^{m+1}(\Omega)),\\ &\sigma(Y_i(x),\, t) = \sigma(x,t),\ v(Y_i(x),\, t) = v(x,\, t),\ u_i(Y_i(x),\, t) = -u_i(x,\, t),\\ &u_i(Y_j(x),\, t) = u_i(x,\, t), \qquad \forall\, i \neq j,\ i = 1, \cdots, n,\\ &\text{where } Y_i[x_1, \cdots, x_i, \cdots, x_n] = [x_1, \cdots, -x_i, \cdots, x_n]\big\}\end{aligned}$$

equipped with the norm

$$\|(\sigma,\, u,\, v)\|_{X_{-1}}^2 := \sup_{t\in[0,T]} \|(\sigma, u, v)\|_{H^m}^2 + \int_0^T \|(u, v)\|_{H^{m+1}}^2 \mathrm{d}t.$$

For convenience, we state some lemmas and fixed point theorem for later use.

Lemma 2.1 *Assume that* $m \geq \left[\frac{n}{2}\right] + 1$. *Then* $\|u\|_{L^\infty} \leq C\|u\|_{H^m}$.

Lemma 2.2[24] *If* $|\beta| + |\gamma| = k$, *then*

$$\|(\partial_x^\beta f)(\partial_x^\gamma g)\|_2 \leq C(\|f|_{-\infty}\|g\|_{H^k} + \|f\|_{H^k}\|g\|_\infty).$$

Remark 2.1 From Lemmas 2.1 and 2.2, we have the algebra property:

$$\|fg\|_{H^m} \leq C\|f\|_{H^m}\|g\|_{H^m}$$

for $m \geq \left[\frac{n}{2}\right] + 1$.

Lemma 2.3[25] *Assume that X and Y are Banach spaces with $X \hookrightarrow\hookrightarrow Y$. If $f_n \in L^\infty(0,\, T;\, X)$, $f_n \to f$ in $C^0([0,\, T],\, X$-weak), then $f_n \to f$ in $C^0([0,\, T],\, Y)$.*

Lemma 2.4[26] (Tychonoff fixed point theorem) *Let V be a locally convex topological vector space. For any nonempty compact convex set X in V, any continuous function $f: X \to X$ has a fixed point.*

Lemma 2.5[27] (Kakutani fixed point theorem) *Every correspondence that maps a compact convex subset of a locally convex space into itself with a closed graph and convex nonempty images has a fixed point.*

3 Energy Estimates

Firstly, we consider the following linearized system:

$$\sigma_t + \bar\rho \mathrm{div} u + u'\nabla\sigma = G_1(\sigma', u'), \tag{3.1}$$

$$u_t - (\bar\mu\Delta u + (\bar\mu+\bar\lambda)\nabla\mathrm{div}u) + \gamma_1\bar\rho\nabla\sigma + \gamma_2\bar\rho\nabla v = G_2(\sigma', u', v') + f, \tag{3.2}$$

$$v_t - \bar\kappa\Delta v + \gamma_3\gamma_2\bar\rho\mathrm{div}u = G_3(\sigma', u', v'), \qquad x\in\Omega \tag{3.3}$$

for any given T-periodic function $(\sigma', u', v') \in X_\delta$.

In what follows, we assume that

$$|\sigma'(x, t)| \le \frac{\bar\rho}{2}$$

for all $(x, t)\in\Omega\times[0, T]$ since that

$$\sup_{t\in[0, T]} \|\sigma'\|_\infty \le \sup_{t\in[0, T]} \|\sigma'\|_{H^{m+1}} \le \eta^{\frac14}$$

for η small enough. The energy estimates of lower order derivatives and high order derivatives is obtained respectively.

Lemma 3.1 *Let* $m \ge \left[\frac{n}{2}\right] + 1$. *Then there exists a constant* $C > 0$, *a suitably small constant* $\epsilon > 0$ *and a constant* C_ϵ *depending on* ϵ *such that*

$$\begin{aligned}
&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int(\gamma_1\gamma_3\sigma^2+\gamma_3u^2+v^2)\mathrm{d}x + \int\bar\mu\gamma_3|\nabla u|^2 + (\bar\mu+\bar\lambda)\gamma_3|\mathrm{div}u|^2 + \bar\kappa|\nabla v|^2\mathrm{d}x\\
\le& C(\|\mathrm{div}u'\|_2+\epsilon)\|\nabla\sigma\|_{H^{m-1}}^2 + \epsilon\|\nabla u\|_{H^{m-1}}^2 + \epsilon\|\nabla v\|_{H^{m-1}}^2 + C_\epsilon\|f\|_1^2 + C_\epsilon\|\nabla\sigma'\|_2^2\|\mathrm{div}u'\|_2^2\\
&+ C_\epsilon(\|u'\|_2\|\nabla u'\|_2 + \|\sigma'\|_2\|\nabla\sigma'\|_2 + \|v'\|_2\|\nabla\sigma'\|_2 + \|\sigma'\|_2\|\nabla v'\|_2 + \|v'\|_2\|\nabla v\|_2\\
&+ \|\sigma'\|_2\|\nabla u'\|_{H^1})^2 + C_\epsilon(\|u'\|_2\|\nabla v'\|_2 + \|\sigma'\|_2\|\nabla v'\|_{H^1} + \|\sigma'\|_2\|\nabla u'\|_2\\
&+ \|v'\|_2\|\nabla u'\|_2 + \|\nabla u'\|_2^2 + \|\nabla\sigma'\|_{H^{m-1}}\|\nabla u'\|_2^2)^2
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int u\nabla\sigma\mathrm{d}x + \frac{\gamma_1\bar\rho}{2}\int|\nabla\sigma|^2\mathrm{d}x\\
\le& \bar\rho\|\mathrm{div}u\|_2^2 + C_\epsilon(1+\|\nabla u'\|_{H^{m-1}}^2)\|\nabla u\|_{H^1}^2 + \epsilon\|\nabla\sigma\|_{H^{m-1}}^2\\
&+ C_\epsilon(\|f\|_1^2 + \|\nabla v\|_2^2 + \|\nabla\sigma'\|_{H^{m-1}}^2\|\mathrm{div}u'\|_2^2) + C_\epsilon(\|u'\|_2\|\nabla u'\|_2 + \|\sigma'\|_2\|\nabla\sigma'\|_2\\
&+ \|v'\|_2\|\nabla\sigma'\|_2 + \|\sigma'\|_2\|\nabla v'\|_2 + \|v'\|_2\|\nabla v\|_2 + \|\sigma'\|_2\|\nabla u'\|_{H^1})^2.
\end{aligned}$$

Proof. Multiplying the equations (3.1)–(3.3) by $\gamma_1\gamma_3\sigma$, γ_3u, v respectively, integrating them over Ω, and summing them up, we have

$$\begin{aligned}
&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int(\gamma_1\gamma_3\sigma^2+\gamma_3u^2+v^2)\mathrm{d}x + \int\bar\mu\gamma_3|\nabla u|^2 + (\bar\mu+\bar\lambda)\gamma_3|\mathrm{div}u|^2 + \bar\kappa|\nabla v|^2\mathrm{d}x\\
=&\frac{\gamma_1\gamma_3}{2}\int\sigma^2\mathrm{div}u'\mathrm{d}x + \gamma_1\gamma_3\int\sigma G_1(\sigma', u')\mathrm{d}x + \gamma_3\int G_2(\sigma', u', v')u\mathrm{d}x\\
&+\int G_3(\sigma', u', v')v\mathrm{d}x + \gamma_3\int fu\mathrm{d}x.
\end{aligned}$$

Due to (2.4)–(2.6), there holds

$$\|G_1(\sigma', u')\|_1 \le \|\sigma'\|_2\|\mathrm{div}u'\|_2.$$

$$\|G_2(\sigma', u', v')\|_1 \le C(\|u'\|_2\|\nabla u'\|_2 + \|\sigma'\|_2\|\nabla\sigma'\|_2 + \|v'\|_2\|\nabla\sigma'\|_2 + \|\sigma'\|_2\|\nabla v'\|_2$$

$$+ \|v'\|_2\|\nabla v\|_2 + \|\sigma'\|_2\|\nabla u'\|_{H^1}). \tag{3.4}$$

$$\|G_3(\sigma',\, u',\, v')\|_1 \leq C(\|u'\|_2\|\nabla v'\|_2 + \|\sigma'\|_2\|\nabla v'\|_{H^1} + \|\sigma'\|_2\|\nabla u'\|_2 + \|v'\|_2\|\nabla u'\|_2 + \|\nabla u'\|_2^2 + \|\sigma'\|_{H^m}\|\nabla u'\|_2^2).$$

Then, by Poincaré inequality and Lemma 2.1, we get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int(\gamma_1\gamma_3\sigma^2 + \gamma_3 u^2 + v^2)\mathrm{d}x + \int\bar{\mu}\gamma_3|\nabla u|^2 + (\bar{\mu}+\bar{\lambda})\gamma_3|\mathrm{div}u|^2 + \bar{\kappa}|\nabla v|^2\mathrm{d}x\\
\leq{}& \frac{\gamma_1\gamma_3}{2}\|\sigma\|_\infty\|\sigma\|_2\|\mathrm{div}u'\|_2 + \gamma_1\gamma_3\|\sigma\|_\infty\|\sigma'\|_2\|\mathrm{div}u'\|_2 + \gamma_3\|f\|_1\|u\|_\infty\\
&+ \gamma_3\|G_2(\sigma',u',v')\|_1\|u\|_\infty + \|G_3(\sigma',u',v')\|_1\|v\|_\infty\\
\leq{}& C\Big(\frac{\gamma_1\gamma_3}{2}\|\nabla\sigma\|_{H^{m-1}}^2\|\mathrm{div}u'\|_2 + \gamma_1\gamma_3\|\nabla\sigma\|_{H^{m-1}}\|\nabla\sigma'\|_2\|\mathrm{div}u'\|_2 + \gamma_3\|f\|_1\|\nabla u\|_{H^{m-1}}\\
&\quad + \gamma_3\|G_2(\sigma',u',v')\|_1\|\nabla u\|_{H^{m-1}} + \|G_3(\sigma',u',v')\|_1\|\nabla v\|_{H^{m-1}}\Big)\\
\leq{}& C(\|\mathrm{div}u'\|_2 + \epsilon)\|\nabla\sigma\|_{H^{m-1}}^2 + \epsilon\|\nabla u\|_{H^{m-1}}^2 + \epsilon\|\nabla v\|_{H^{m-1}}^2 + C_\epsilon\|f\|_1^2 + C_\epsilon\|\nabla\sigma'\|_2^2\|\mathrm{div}u'\|_2^2\\
&+ C_\epsilon(\|u'\|_2\|\nabla u'\|_2 + \|\sigma'\|_2\|\nabla\sigma'\|_2 + \|v'\|_2\|\nabla\sigma'\|_2 + \|\sigma'\|_2\|\nabla v'\|_2 + \|v'\|_2\|\nabla v\|_2\\
&\qquad + \|\sigma'\|_2\|\nabla u'\|_{H^1})^2\\
&+ C_\epsilon(\|u'\|_2\|\nabla v'\|_2 + \|\sigma'\|_2\|\nabla v'\|_{H^1} + \|\sigma'\|_2\|\nabla u'\|_2 + \|v'\|_2\|\nabla u'\|_2 + \|\nabla u'\|_2^2\\
&\qquad + \|\nabla\sigma'\|_{H^{m-1}}\|\nabla u'\|_2^2)^2.
\end{aligned}$$

Multiplying the equations (3.2) by $\nabla\sigma$ and then integrating them over Ω, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int u\nabla\sigma\mathrm{d}x + \gamma_1\bar{\rho}\int|\nabla\sigma|^2\mathrm{d}x\\
={}& \int\Big(\bar{\mu}\Delta u + (\bar{\mu}+\bar{\lambda})\nabla\mathrm{div}u - \gamma_2\bar{\rho}\nabla v + G_2(\sigma',\, u',\, v') + f\Big)\nabla\sigma\mathrm{d}x\\
&+ \int\mathrm{div}u(\bar{\rho}\mathrm{div}u + u'\nabla\sigma - G_1(\sigma',\, u'))\mathrm{d}x\\
\leq{}& \bar{\rho}\|\mathrm{div}u\|_2^2 + \epsilon\|\nabla\sigma\|_{H^{m-1}}^2 + \|\mathrm{div}u'\|_2^2\|\nabla\sigma'\|_{H^{m-1}}^2\\
&+ C_\epsilon\big(\|\nabla u\|_{H^1}^2 + \|f\|_1^2 + \|\nabla v\|_2^2 + \|\mathrm{div}u\|_2^2\|\nabla u'\|_{H^{m-1}}^2 + \|G_2(\sigma',u',v')\|_1^2\big).
\end{aligned}$$

From (3.4), there holds

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int u\nabla\sigma\mathrm{d}x + \frac{\gamma_1\bar{\rho}}{2}\int|\nabla\sigma|^2\mathrm{d}x\\
\leq{}& \bar{\rho}\|\mathrm{div}u\|_2^2 + C_\epsilon(1 + \|\nabla u'\|_{H^{m-1}}^2)\|\nabla u\|_{H^1}^2 + \epsilon\|\nabla\sigma\|_{H^{m-1}}^2\\
&+ C_\epsilon(\|f\|_1^2 + \|\nabla v\|_2^2 + \|\nabla\sigma'\|_{H^{m-1}}^2\|\mathrm{div}u'\|_2^2)\\
&+ C_\epsilon(\|u'\|_2\|\nabla u'\|_2 + \|\sigma'\|_2\|\nabla\sigma'\|_2 + \|v'\|_2\|\nabla\sigma'\|_2\\
&\qquad + \|\sigma'\|_2\|\nabla v'\|_2 + \|v'\|_2\|\nabla v\|_2 + \|\sigma'\|_2\|\nabla u'\|_{H^1})^2.
\end{aligned}$$

This completes the proof.

Next, we give the energy estimates on the high order derivatives of (σ, u, v).

Lemma 3.2 *Let* $m \geq \left[\frac{n}{2}\right] + 1$ *and multi-index* α *with* $|\alpha| = 1, \cdots, m+1$. *Then, there exists a constant* $C > 0$, *a suitably small constant* $\epsilon > 0$ *and a constant* C_ϵ *depending on* ϵ *such that*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int(\gamma_1\gamma_3|\partial_x^\alpha\sigma|^2 + \gamma_3|\partial_x^\alpha u|^2 + |\partial_x^\alpha v|^2)\mathrm{d}x + \gamma_3\int\big(\bar{\mu}|\partial_x^\alpha\nabla u|^2 + (\bar{\mu}+\bar{\lambda})|\partial_x^\alpha\mathrm{div}u|^2\big)\mathrm{d}x$$

$$
\begin{aligned}
&+\bar{\kappa}\int|\partial_x^\alpha\nabla v|^2\mathrm{d}x\\
\le\, & C\|\nabla\sigma\|_{H^m}^2\|\nabla u'\|_{H^m}+\epsilon(\|\partial_x^\alpha\sigma\|_2^2+\|\nabla u\|_{H^{m+1}}^2)\\
&+C_\epsilon\big(\|\nabla\sigma'\|_{H^m}^2\|\nabla u'\|_{H^{m+1}}^2+\|f\|_{H^m}^2+(\|\nabla u'\|_{H^m}^2+\|\nabla\sigma'\|_{H^m}^2+\|\nabla v'\|_{H^m}^2\\
&\qquad+\|\nabla\sigma'\|_{H^m}^2\|\Delta u'\|_{H^m}^2)^2+(\|\nabla\sigma'\|_{H^m}^2+\|\nabla u'\|_{H^m}^2+\|\nabla v'\|_{H^{m+1}}^2\\
&\qquad+\|\sigma'\|_{H^m}\|\nabla u'\|_{H^m}^2+\|\nabla\sigma'\|_{H^m}\|\nabla v'\|_{H^{m+1}})^2\big).
\end{aligned}
$$

Proof. For each multi-index α with $|\alpha|=1,\cdots,m+1$, by applying ∂_x^α to equations (3.1)–(3.3), multiplying them by $\gamma_1\gamma_3\partial_x^\alpha\sigma$, $\gamma_3\partial_x^\alpha u$, $\partial_x^\alpha v$ respectively and then integrating them, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int(\gamma_1\gamma_3|\partial_x^\alpha\sigma|^2+\gamma_3|\partial_x^\alpha u|^2+|\partial_x^\alpha v|^2)\mathrm{d}x+\gamma_3\int\big(\bar{\mu}|\partial_x^\alpha\nabla u|^2+(\bar{\mu}+\bar{\lambda})|\partial_x^\alpha\mathrm{div}u|^2\big)\mathrm{d}x\\
&+\bar{\kappa}\int|\partial_x^\alpha\nabla v|^2\mathrm{d}x\\
=&-\gamma_1\gamma_3\int\partial_x^\alpha(u'\cdot\nabla\sigma)\partial_x^\alpha\sigma\mathrm{d}x-\gamma_1\gamma_3\int\partial_x^\alpha(\sigma'\mathrm{div}u')\partial_x^\alpha\sigma\mathrm{d}x\\
&+\gamma_3\int\partial_x^\alpha f\partial_x^\alpha u\mathrm{d}x+\gamma_3\int\partial_x^\alpha G_2(\sigma',u',v')\partial_x^\alpha u\mathrm{d}x+\int\partial_x^\alpha G_3(\sigma',u',v')\partial_x^\alpha v\mathrm{d}x. \qquad (3.5)
\end{aligned}
$$

Now, we estimate the right hand terms in the above equation one by one. From Lemmas 2.1 and 2.2, we can get

$$
\begin{aligned}
&-\gamma_1\gamma_3\int\partial_x^\alpha(u'\cdot\nabla\sigma)\partial_x^\alpha\sigma\mathrm{d}x\\
=&\,\frac{\gamma_1\gamma_3}{2}\int|\partial_x^\alpha\sigma|^2\mathrm{div}u'\mathrm{d}x-\gamma_1\gamma_3\sum_{|l|=1}^{\alpha}\binom{\alpha}{l}\int\partial_x^l u'\partial_x^{\alpha-l}(\nabla\sigma)\partial_x^\alpha\sigma\mathrm{d}x\\
=&\,\frac{\gamma_1\gamma_3}{2}\int|\partial_x^\alpha\sigma|^2\mathrm{div}u'\mathrm{d}x-\gamma_1\gamma_3\sum_{|l-1|=0}^{\alpha-1}\binom{\alpha}{l}\int\partial_x^{l-1}\partial_x u'\partial_x^{\alpha-1-(l-1)}(\nabla\sigma)\partial_x^\alpha\sigma\mathrm{d}x\\
\le&\, C\big(\|\mathrm{div}u'\|_\infty\|\partial_x^\alpha\sigma\|_2^2+\|\partial_x^\alpha\sigma\|_2(\|\nabla\sigma\|_\infty\|\nabla u'\|_{H^{\alpha-1}}+\|\nabla\sigma\|_{H^{\alpha-1}}\|\nabla u'\|_\infty)\big)\\
\le&\, C\|\nabla\sigma\|_{H^m}^2\|\nabla u'\|_{H^m}.
\end{aligned}
$$

Using Young's inequality, there exists a suitably small constant $\epsilon>0$ such that

$$
\begin{aligned}
&-\gamma_1\gamma_3\int\partial_x^\alpha(\sigma'\mathrm{div}u')\partial_x^\alpha\sigma\mathrm{d}x\\
\le&\, C\|\partial_x^\alpha(\sigma'\mathrm{div}u')\|_2\|\partial_x^\alpha\sigma\|_2\\
\le&\, \epsilon\|\partial_x^\alpha\sigma\|_2^2+C_\epsilon\|\partial_x^\alpha(\sigma'\mathrm{div}u')\|_2^2\\
\le&\, \epsilon\|\partial_x^\alpha\sigma\|_2^2+C_\epsilon\|(\sigma'\mathrm{div}u')\|_{H^{m+1}}^2\\
\le&\, \epsilon\|\partial_x^\alpha\sigma\|_2^2+C_\epsilon\|\sigma'\|_{H^{m+1}}^2\|\mathrm{div}u'\|_{H^{m+1}}^2.
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
&\gamma_3\int\partial_x^\alpha f\partial_x^\alpha u\mathrm{d}x\le C\|f\|_{H^m}\|\nabla u\|_{H^{m+1}},\\
&\|\partial_x^{\alpha-1}(\sigma'\mathrm{div}u')\|_2^2\le C\|\sigma'\mathrm{div}u'\|_{H^m}^2\le C\|\sigma'\|_{H^m}^2\|\mathrm{div}u'\|_{H^m}^2.
\end{aligned}
$$

Using Remark 2.1 and Poincaré inequality, it concludes that

$$
\begin{aligned}
&-\gamma_3\int\partial_x^\alpha G_2(\sigma',u',v')\partial_x^\alpha u\mathrm{d}x\\
&\le C\|\partial_x^{\alpha-1}G_2(\sigma',u',v')\|_2\|\partial_x^\alpha\nabla u\|_2\\
&\le C\|\partial_x^\alpha\nabla u\|_2(\|u'\|_{H^m}\|\nabla u'\|_{H^m}+\|\sigma'\|_{H^m}\|\nabla\sigma'\|_{H^m}+\|\nabla\sigma'\|_{H^m}\|v'\|_{H^m}\\
&\qquad+\|\sigma'\|_{H^m}\|\nabla v'\|_{H^m}+\|v'\|_{H^m}\|\nabla v'\|_{H^m}^2+\|\sigma'\|_{H^m}\|\Delta u'\|_{H^m}\\
&\qquad+\|\sigma'\|_{H^m}\|\nabla\mathrm{div}u'\|_{H^m})\\
&\le C\|\partial_x^\alpha\nabla u\|_2(\|\nabla u'\|_{H^m}^2+\|\nabla\sigma'\|_{H^m}^2+\|\nabla v'\|_{H^m}^2+\|\nabla\sigma'\|_{H^m}\|\Delta u'\|_{H^m})
\end{aligned}
$$

and

$$
\begin{aligned}
&\int\partial_x^\alpha G_3(\sigma',u',v')\partial_x^\alpha v\mathrm{d}x\\
&\le C\|\partial_x^{\alpha-1}G_3(\sigma',u',v')\|_2\|\partial_x^\alpha\nabla u\|_2\\
&\le C\|\partial_x^\alpha\nabla u\|_2(\|u'\|_{H^m}\|\nabla v'\|_{H^m}+\|\sigma'\|_{H^m}\|\Delta v'\|_{H^m}+\|\sigma'\|_{H^m}\|\mathrm{div}u'\|_{H^m}\\
&\qquad+\|v'\|_{H^m}\|\mathrm{div}u'\|_{H^m}+\|\nabla u'\|_{H^m}^2+\|\sigma'\|_{H^m}\|\mathrm{div}u'\|_{H^m}^2)\\
&\le C\|\partial_x^\alpha\nabla u\|_2(\|\nabla\sigma'\|_{H^m}^2+\|\nabla u'\|_{H^m}^2+\|\nabla v'\|_{H^m}^2+\|\sigma'\|_{H^m}\|\nabla u'\|_{H^m}^2\\
&\qquad+\|\nabla\sigma'\|_{H^m}\|\nabla v'\|_{H^{m+1}}).
\end{aligned}
$$

Hence, substituting the above estimates into (3.5), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int(\gamma_1\gamma_3|\partial_x^\alpha\sigma|^2+\gamma_3|\partial_x^\alpha u|^2+|\partial_x^\alpha v|^2)\mathrm{d}x+\gamma_3\int\big(\bar\mu|\partial_x^\alpha\nabla u|^2+(\bar\mu+\bar\lambda)|\partial_x^\alpha\mathrm{div}u|^2\big)\mathrm{d}x\\
&+\bar\kappa\int|\partial_x^\alpha\nabla v|^2\mathrm{d}x\\
\le{}& C\|\nabla\sigma\|_{H^m}^2\|\nabla u'\|_{H^m}+\epsilon\|\partial_x^\alpha\sigma\|_2^2+C_\epsilon\|\nabla\sigma'\|_{H^m}^2\|\nabla u'\|_{H^{m+1}}^2+\|f\|_{H^m}\|\nabla u\|_{H^{m+1}}\\
&+\|\partial_x^\alpha\nabla u\|_2(\|\nabla u'\|_{H^m}^2+\|\nabla\sigma'\|_{H^m}^2+\|\nabla v'\|_{H^m}^2+\|\nabla\sigma'\|_{H^m}^2\|\Delta u'\|_{H^m}^2)\\
&+\|\partial_x^\alpha\nabla u\|_2(\|\nabla\sigma'\|_{H^m}^2+\|\nabla u'\|_{H^m}^2+\|\nabla v'\|_{H^{m+1}}^2+\|\sigma'\|_{H^m}\|\nabla u'\|_{H^m}^2\\
&\qquad+\|\nabla\sigma'\|_{H^m}\|\nabla v'\|_{H^{m+1}})\\
\le{}& C\|\nabla\sigma\|_{H^m}^2\|\nabla u'\|_{H^m}+\epsilon(\|\partial_x^\alpha\sigma\|_2^2+\|\nabla u\|_{H^{m+1}}^2)\\
&+C_\epsilon\big(\|\nabla\sigma'\|_{H^m}^2\|\nabla u'\|_{H^{m+1}}^2+\|f\|_{H^m}^2+(\|\nabla u'\|_{H^m}^2+\|\nabla\sigma'\|_{H^m}^2+\|\nabla v'\|_{H^m}^2\\
&\qquad+\|\nabla\sigma'\|_{H^m}\|\Delta u'\|_{H^m})^2+(\|\nabla\sigma'\|_{H^m}^2+\|\nabla u'\|_{H^m}^2+\|\nabla v'\|_{H^{m+1}}^2\\
&\qquad+\|\sigma'\|_{H^m}\|\nabla u'\|_{H^m}^2+\|\nabla\sigma'\|_{H^m}\|\nabla v'\|_{H^{m+1}})^2\big).
\end{aligned}
$$

It completes the proof.

Lemma 3.3 *Let $m\ge\left[\frac{n}{2}\right]+1$ and multi-index β with $|\beta|=1,\cdots,m$. Then there exists a constant $C>0$ and a constant $C_{\gamma_1,\bar\rho}$ depending on γ_1, $\bar\rho$ such that*

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int\partial_x^\beta u\cdot\partial_x^\beta\nabla\sigma\mathrm{d}x+\frac{\gamma_1\bar\rho}{2}\int|\partial_x^\beta\nabla\sigma|^2\mathrm{d}x\\
&\le(\bar\rho+C_{\gamma_1,\bar\rho}+C)\|u\|_{H^{m+2}}^2\\
&\quad+C_{\gamma_1,\bar\rho}\big(\|v\|_{H^{m+1}}^2+\|f\|_{H^m}^2+\|(\sigma',u',v')\|_{H^{m+1}}^4+\|\sigma'\|_{H^m}^2\|u'\|_{H^{m+2}}^2\\
&\qquad+\|\sigma\|_{H^{m+1}}^2\|u'\|_{H^{m+1}}^2+\|\sigma'\|_{H^{m+1}}^2\|u'\|_{H^{m+1}}^2\big).
\end{aligned}
$$

Proof. For each multi-index β with $|\beta| = 1, \cdots, m$, by applying ∂_x^β to equation (3.2), multiplying them by $\partial_x^\beta \nabla\sigma$ and then integrating them, it obtains that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int \partial_x^\beta u\cdot\partial_x^\beta\nabla\sigma \mathrm{d}x + \gamma_1\bar\rho\int|\partial_x^\beta\nabla\sigma|^2\mathrm{d}x\\
=&\int\partial_x^\beta(u'\cdot\nabla\sigma)\partial_x^\beta\mathrm{div}u\mathrm{d}x+\int\partial_x^\beta(\sigma'\mathrm{div}u')\partial_x^\beta\mathrm{div}u\mathrm{d}x+\bar\rho\|\partial_x^\beta\mathrm{div}u\|_2^2+\bar\mu\int\partial_x^\beta\Delta u\partial_x^\beta\nabla\sigma\mathrm{d}x\\
&+(\bar\mu+\bar\lambda)\int\partial_x^\beta\nabla\mathrm{div}u\partial_x^\beta\nabla\sigma\mathrm{d}x-\gamma\bar\rho\int\partial_x^\beta\nabla v\partial_x^\beta\nabla\sigma\mathrm{d}x+\int\partial_x^\beta G_2(\sigma',u',v')\partial_x^\beta\nabla\sigma\mathrm{d}x\\
&+\int\partial_x^\beta\nabla\sigma\partial_x^\beta f\mathrm{d}x\\
\le&\frac{\gamma_1\bar\rho}{2}\|\partial_x^\beta\nabla\sigma\|_2^2+\bar\rho\|\partial_x^\beta\mathrm{div}u\|_2^2+C_{\gamma_1,\bar\rho}\|\partial_x^\beta\Delta u\|_2^2+C_{\gamma_1,\bar\rho}\|\partial_x^\beta\nabla\mathrm{div}u\|_2^2\\
&+\|\partial_x^\beta\mathrm{div}u\|_2\|\partial_x^\beta(\sigma'\mathrm{div}u')\|_2+C_{\gamma_1,\bar\rho}\|\partial_x^\beta\nabla v\|_2^2+C_{\gamma_1,\bar\rho}\|\partial_x^\beta G_2(\sigma',u',v')\|_2^2\\
&+C_{\gamma_1,\bar\rho}\|\partial_x^\beta f\|_2^2+\|\partial_x^\beta\mathrm{div}u\|_2\|\partial_x^\beta(u'\cdot\nabla\sigma)\|_2.
\end{aligned}$$

Note that

$$\|\partial_x^\beta\mathrm{div}u\|_2\|\partial_x^\beta(u'\cdot\nabla\sigma)\|_2\le C(\|\partial_x^\beta\mathrm{div}u\|_2^2+\|u'\|_{H^m}^2\|\nabla\sigma\|_{H^m}^2)$$

and

$$\|\partial_x^\beta\mathrm{div}u\|_2\|\partial_x^\beta(\sigma'\mathrm{div}u')\|_2\le C(\|\partial_x^\beta\mathrm{div}u\|_2^2+\|\nabla u'\|_{H^m}^2\|\sigma'\|_{H^m}^2).$$

Therefore, with a similar argument in Lemma 3.2, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int\partial_x^\beta u\cdot\partial_x^\beta\nabla\sigma\mathrm{d}x+\frac{\gamma_1\bar\rho}{2}\int|\partial_x^\beta\nabla\sigma|^2\mathrm{d}x\\
\le&(\bar\rho+C_{\gamma_1,\bar\rho}+C)\|u\|_{H^{m+2}}^2\\
&+C_{\gamma_1,\bar\rho}\big(\|v\|_{H^{m+1}}^2+\|f\|_{H^m}^2+\|(\sigma',u',v')\|_{H^{m+1}}^4+\|\sigma'\|_{H^m}^2\|u'\|_{H^{m+2}}^2\\
&\qquad+\|\sigma\|_{H^{m+1}}^2\|u'\|_{H^{m+1}}^2+\|\sigma'\|_{H^{m+1}}^2\|u'\|_{H^{m+1}}^2\big).
\end{aligned}$$

This completes the proof.

Proposition 3.1 *Assume that* $m\ge\left[\frac{n}{2}\right]+1$ *and* (σ, u, v) *is the solution of linearized system* (3.1)–(3.3) *with initial data* (σ_0, u_0, v_0). *Then there exist two positive constants* C *and* C_1 *such that*

$$\begin{aligned}
&\|\sigma\|_{H^{m+1}}^2+\|u\|_{H^{m+1}}^2+\|v\|_{H^{m+1}}^2\\
\le& C\big(\|\sigma_0\|_{H^{m+1}}^2+\|u_0\|_{H^{m+1}}^2+\|v_0\|_{H^{m+1}}^2\big)\\
&+\int_0^t \mathrm{e}^{C_1(\tau-t)}C\big(\|\sigma'\|_{H^{m+1}}^4+\|u'\|_{H^{m+1}}^4+\|v'\|_{H^{m+1}}^4+\|f\|_{H^m}^2+\|\sigma'\|_{H^m}^2\|u'\|_{H^{m+1}}^4\\
&\qquad+\|\sigma'\|_{H^{m+1}}^2\|u'\|_{H^{m+2}}^2+\|\sigma'\|_{H^{m+1}}^2\|v'\|_{H^{m+2}}^2\big)\mathrm{d}\tau,\qquad t\in[0,+\infty).
\end{aligned}$$

Proof. By Lemmas 3.1, 3.2 and 3.3, summing up about $|\beta|$ and $|\alpha|$, taking $M>1$ large enough, we have

$$\begin{aligned}
&\frac{M}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\gamma_1\gamma_3\|\sigma\|_{H^{m+1}}^2+\gamma_3\|u\|_{H^{m+1}}^2+\|v\|_{H^{m+1}}^2\big)\\
&+M\gamma_3\Big(\frac{\mu}{2}\|\nabla u\|_{H^{m+1}}^2+(\bar\mu+\bar\lambda)\|\mathrm{div}u\|_{H^{m+1}}^2\Big)+\frac{M\bar\kappa}{2}\|\nabla v\|_{H^{m+1}}^2+\gamma_1\bar\rho\|\nabla\sigma\|_{H^m}^2
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{|\beta|=0}^{m}\frac{\mathrm{d}}{\mathrm{d}t}\int \partial_x^\beta u\partial_x^\beta \nabla\sigma \mathrm{d}x\\
&\le M(\epsilon+\|u'\|_{H^{m+1}})\|\sigma\|_{H^{m+1}}^2+MC\big[(\|\sigma'\|_{H^{m+1}}^4+\|u'\|_{H^{m+1}}^4+\|v'\|_{H^{m+1}}^4)\\
&\quad+\|f\|_{H^m}^2+\|\sigma'\|_{H^m}^2\|u'\|_{H^{m+1}}^4+\|\sigma'\|_{H^{m+1}}^2\|u'\|_{H^{m+2}}^2+\|\sigma'\|_{H^{m+1}}^2\|v'\|_{H^{m+2}}^2\big]\\
&\quad+C\big(\|\mathrm{div}u\|_2^2+(1+\|\nabla u'\|_{H^{m-1}})\|\nabla u\|_{H^1}^2+\|\nabla v\|_2^2+\|u\|_{H^{m+2}}^2+\|v\|_{H^{m+1}}^2\big).
\end{aligned}
$$

Then, choosing $\|u'\|_{H^{m+1}}, \epsilon$ appropriately small, we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|(\sigma,u,v)\|_{H^{m+1}}^2+\|u\|_{H^{m+2}}^2+\|v\|_{H^{m+2}}^2+\frac{1}{M}\|\sigma\|_{H^{m+1}}^2+\frac{1}{M}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{|\beta|=0}^{m}\int \partial_x^\beta u\partial_x^\beta\nabla\sigma \mathrm{d}x\\
&\le C\big(\|\sigma'\|_{H^{m+1}}^4+\|u'\|_{H^{m+1}}^4+\|v'\|_{H^{m+1}}^4+\|f\|_{H^m}^2+\|\sigma'\|_{H^m}^2\|u'\|_{H^{m+1}}^4\\
&\quad+\|\sigma'\|_{H^{m+1}}^2\|u'\|_{H^{m+2}}^2+\|\sigma'\|_{H^{m+1}}^2\|v'\|_{H^{m+2}}^2\big). \qquad (3.6)
\end{aligned}
$$

There exists a constant $0<C_1\le\dfrac{2}{2M+1}$ such that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\bigg(\|(\sigma,u,v)\|_{H^{m+1}}^2+\frac{1}{M}\sum_{|\beta|=0}^{m}\int \partial_x^\beta u\partial_x^\beta\nabla\sigma \mathrm{d}x\bigg)\\
&\quad+C_1\bigg(\|u\|_{H^{m+2}}^2+\|v\|_{H^{m+2}}^2+\|\sigma\|_{H^{m+1}}^2+\frac{1}{M}\sum_{|\beta|=0}^{m}\int \partial_x^\beta u\partial_x^\beta\nabla\sigma \mathrm{d}x\bigg)\\
&\le C\big(\|\sigma'\|_{H^{m+1}}^4+\|u'\|_{H^{m+1}}^4+\|v'\|_{H^{m+1}}^4+\|f\|_{H^m}^2+\|\sigma'\|_{H^m}^2\|u'\|_{H^{m+1}}^4\\
&\quad+\|\sigma'\|_{H^{m+1}}^2\|u'\|_{H^{m+2}}^2+\|\sigma'\|_{H^{m+1}}^2\|v'\|_{H^{m+2}}^2\big). \qquad (3.7)
\end{aligned}
$$

Multiplying the equation (3.7) by e^{C_1t}, we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\bigg(\mathrm{e}^{C_1t}(\|\sigma\|_{H^{m+1}}^2+\|u\|_{H^{m+1}}^2+\|v\|_{H^{m+1}}^2+\frac{1}{M}\sum_{|\beta|=0}^{m}\int \partial_x^\beta u\partial_x^\beta\nabla\sigma \mathrm{d}x)\bigg)\\
&\le \mathrm{e}^{C_1t}C\big(\|\sigma'\|_{H^{m+1}}^4+\|u'\|_{H^{m+1}}^4+\|v'\|_{H^{m+1}}^4+\|f\|_{H^m}^2+\|\sigma'\|_{H^m}^2\|u'\|_{H^{m+1}}^4\\
&\quad+\|\sigma'\|_{H^{m+1}}^2\|u'\|_{H^{m+2}}^2+\|\sigma'\|_{H^{m+1}}^2\|v'\|_{H^{m+2}}^2\big).
\end{aligned}
$$

Integrating from 0 to t, we have

$$
\begin{aligned}
&\mathrm{e}^{C_1\tau}\bigg(\|\sigma\|_{H^{m+1}}^2+\|u\|_{H^{m+1}}^2+\|v\|_{H^{m+1}}^2+\frac{1}{M}\sum_{|\beta|=0}^{m}\int \partial_x^\beta u\partial_x^\beta\nabla\sigma(x,\tau) \mathrm{d}x\bigg)\bigg|_0^t\\
&\le \int_0^t \mathrm{e}^{C_1\tau}C\big(\|\sigma'\|_{H^{m+1}}^4+\|u'\|_{H^{m+1}}^4+\|v'\|_{H^{m+1}}^4+\|f\|_{H^m}^2+\|\sigma'\|_{H^m}^2\|u'\|_{H^{m+1}}^4\\
&\quad+\|\sigma'\|_{H^{m+1}}^2\|u'\|_{H^{m+2}}^2+\|\sigma'\|_{H^{m+1}}^2\|v'\|_{H^{m+2}}^2\big)\mathrm{d}\tau.
\end{aligned}
$$

Thus, there exist two positive constants C_2 and C_3 such that

$$
\begin{aligned}
&\mathrm{e}^{C_1t}C_2(\|\sigma\|_{H^{m+1}}^2+\|u\|_{H^{m+1}}^2+\|v\|_{H^{m+1}}^2)\\
&\le C_3\big(\|\sigma_0\|_{H^{m+1}}^2+\|u_0\|_{H^{m+1}}^2+\|v_0\|_{H^{m+1}}^2\big)\\
&\quad+\int_0^t \mathrm{e}^{C_1\tau}C\big(\|\sigma'\|_{H^{m+1}}^4+\|u'\|_{H^{m+1}}^4+\|v'\|_{H^{m+1}}^4+\|f\|_{H^m}^2+\|\sigma'\|_{H^m}^2\|u'\|_{H^{m+1}}^4\\
&\quad+\|\sigma'\|_{H^{m+1}}^2\|u'\|_{H^{m+2}}^2+\|\sigma'\|_{H^{m+1}}^2\|v'\|_{H^{m+2}}^2\big)\mathrm{d}\tau,
\end{aligned}
$$

where we have used M appropriately large.

One can obtain the result by some simple calculations.

4 Existence

Here, we first state the framework of the proof to the existence of the periodic solution. Due to Proposition 3.1 and smallness assumptions, the solution of the linearized system (3.1)–(3.3) with trivial initial data will be estimated. Then, we will construct a convex hull. By Tychonoff fixed point theorem, one can see that the T-periodic solution of the linearized system (3.1)–(3.3) at $t = 0$ belongs to this set. Note that Tychonoff theorem cannot ensure the uniqueness of the fixed point. But we can define a set-valued function. The fixed point of this function is the T-periodic solution of the system (2.1)–(2.3) by using Kakutani fixed point theorem.

We rewrite the linearized system (3.1)–(3.3) in vector sense:

$$U_t = AU + G(W) + F,$$

where

$$A = \begin{pmatrix} -u'\nabla & -\bar{\rho}\mathrm{div} & 0 \\ -\gamma_1\bar{\rho}\nabla & \bar{\mu}\Delta + (\bar{\mu}+\bar{\lambda})\nabla\mathrm{div} & -\gamma_2\bar{\rho}\nabla \\ 0 & -\gamma_2\gamma_3\bar{\rho}\mathrm{div} & \bar{\kappa}\Delta \end{pmatrix},$$

$$G(W) = (G_1(\sigma', u'), G_2(\sigma', u', v'), G_3(\sigma', u', v'))^{\mathrm{T}}$$

and

$$U = (\sigma, u, v)^{\mathrm{T}}, \quad W = (\sigma', u', v')^{\mathrm{T}}, \quad F = (0, f, 0)^{\mathrm{T}}.$$

Let $\overline{U}(x, t)$ be the solution of (3.1)–(3.3) with trivial initial data (0, 0, 0). Then, for any given $t \geq 0$, there exists $n \in \mathbf{Z}^+$ such that $t \in [nT, (n+1)T)$. From Proposition 3.1 and T-periodic function $(\sigma', u', v') \in X_\delta$, we have

$$\begin{aligned}
&\|\overline{U}\|^2_{H^{m+1}} \\
\leq\ & \int_0^t \mathrm{e}^{-C_1(t-\tau)} C\big(\|\sigma'\|^4_{H^{m+1}} + \|u'\|^4_{H^{m+1}} + \|v'\|^4_{H^{m+1}} + \|f\|^2_{H^m} + \|\sigma'\|^2_{H^m}\|u'\|^4_{H^{m+1}} \\
&\qquad + \|\sigma'\|^2_{H^{m+1}}\|u'\|^2_{H^{m+2}} + \|\sigma'\|^2_{H^{m+1}}\|v'\|^2_{H^{m+2}}\big)\mathrm{d}\tau \\
\leq\ & \int_{nT}^t \mathrm{e}^{-C_1(t-\tau)} C\big(\|\sigma'\|^4_{H^{m+1}} + \|u'\|^4_{H^{m+1}} + \|v'\|^4_{H^{m+1}} + \|f\|^2_{H^m} + \|\sigma'\|^2_{H^m}\|u'\|^4_{H^{m+1}} \\
&\qquad + \|\sigma'\|^2_{H^{m+1}}\|u'\|^2_{H^{m+2}} + \|\sigma'\|^2_{H^{m+1}}\|v'\|^2_{H^{m+2}}\big)\mathrm{d}\tau \\
&+ \sum_{i=0}^{n-1} \int_{iT}^{(i+1)T} C\big(\|\sigma'\|^4_{H^{m+1}} + \|u'\|^4_{H^{m+1}} + \|v'\|^4_{H^{m+1}} + \|f\|^2_{H^m} + \|\sigma'\|^2_{H^m}\|u'\|^4_{H^{m+1}} \\
&\qquad + \|\sigma'\|^2_{H^{m+1}}\|u'\|^2_{H^{m+2}} + \|\sigma'\|^2_{H^{m+1}}\|v'\|^2_{H^{m+2}}\big)\mathrm{e}^{-C_1(t-\tau)}\mathrm{d}\tau \\
\leq\ & \int_{nT}^t \mathrm{e}^{-C_1(t-\tau)} C\big(\|\sigma'\|^4_{H^{m+1}} + \|u'\|^4_{H^{m+1}} + \|v'\|^4_{H^{m+1}} + \|f\|^2_{H^m} + \|\sigma'\|^2_{H^m}\|u'\|^4_{H^{m+1}} \\
&\qquad + \|\sigma'\|^2_{H^{m+1}}\|u'\|^2_{H^{m+2}} + \|\sigma'\|^2_{H^{m+1}}\|v'\|^2_{H^{m+2}}\big)\mathrm{d}\tau \\
&+ \sum_{i=0}^{n-1} \Big[\int_{iT}^{(i+1)T} C\big(\|\sigma'\|^4_{H^{m+1}} + \|u'\|^4_{H^{m+1}} + \|v'\|^4_{H^{m+1}} + \|f\|^2_{H^m} + \|\sigma'\|^2_{H^m}\|u'\|^4_{H^{m+1}}
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad + \|\sigma'\|_{H^{m+1}}^2\|u'\|_{H^{m+2}}^2 + \|\sigma'\|_{H^{m+1}}^2\|v'\|_{H^{m+2}}^2\big)\mathrm{d}\tau\cdot \mathrm{e}^{-C_1(t-(i+1)T)}\Big]\\
&\le \int_{nT}^t C\big(\|\sigma'\|_{H^{m+1}}^4 + \|u'\|_{H^{m+1}}^4 + \|v'\|_{H^{m+1}}^4 + \|f\|_{H^m}^2 + \|\sigma'\|_{H^m}^2\|u'\|_{H^{m+1}}^4\\
&\qquad + \|\sigma'\|_{H^{m+1}}^2\|u'\|_{H^{m+2}}^2 + \|\sigma'\|_{H^{m+1}}^2\|v'\|_{H^{m+2}}^2\big)\mathrm{d}\tau\\
&\quad + \sum_{i=0}^{n-1}\Big[\int_0^T C\big(\|\sigma'\|_{H^{m+1}}^4 + \|u'\|_{H^{m+1}}^4 + \|v'\|_{H^{m+1}}^4 + \|f\|_{H^m}^2 + \|\sigma'\|_{H^m}^2\|u'\|_{H^{m+1}}^4\\
&\qquad + \|\sigma'\|_{H^{m+1}}^2\|u'\|_{H^{m+2}}^2 + \|\sigma'\|_{H^{m+1}}^2\|v'\|_{H^{m+2}}^2\big)\mathrm{d}\tau\cdot \mathrm{e}^{-C_1(t-(i+1)T)}\Big]\\
&\le C\bigg(\sup_{t\in[0,T]}\big(\|\sigma'\|_{H^{m+1}}^4 + \|u'\|_{H^{m+1}}^4 + \|v'\|_{H^{m+1}}^4 + \|\sigma'\|_{H^m}^2\|u'\|_{H^{m+1}}^4\big)\\
&\qquad + \sup_{t\in[0,T]}\|\sigma'\|_{H^{m+1}}^2\int_0^T\big(\|u'\|_{H^{m+2}}^2 + \|v'\|_{H^{m+2}}^2\big)\mathrm{d}\tau + \int_0^T\|f\|_{H^m}^2\mathrm{d}\tau\bigg),
\end{aligned}
$$

where we have used

$$
\begin{aligned}
\sum_{i=0}^{n-1}\mathrm{e}^{-C_1(t-(i+1)T)} &\le \mathrm{e}^{-C_1t}\sum_{i=0}^{n-1}\mathrm{e}^{C_1(i+1)T}\\
&\le \mathrm{e}^{-C_1t}\mathrm{e}^{C_1nT}\frac{1}{1-\mathrm{e}^{-C_1T}}\\
&\le \frac{1}{1-\mathrm{e}^{-C_1T}}.
\end{aligned}
$$

The last inequality holds by $t\ge nT$.

Note that T-periodic function $(\sigma',\ u',\ v')\in X_\delta$ and $\delta^4=\eta<1$. It obtains that

$$
\|\overline{U}\|_{H^{m+1}}\le C\delta^2,\qquad \forall\, t\in[0,+\infty).
$$

Let $S_1=\{\overline{U}(kT),\ k=0,1,\cdots\}$. Obviously S_1 is nonempty and bounded in $H^{m+1}(\Omega)$. Thus, $\overline{Co}S_1$ is compact in $H^m(\Omega)$. Set

$$
P(\Phi)=U(T,\ \Phi),\qquad \Phi\in\overline{Co}S_1,
$$

where $U(T,\ \Phi)$ is the solution of linearized system (3.1)–(3.3) with the initial data Φ at time T. From the uniqueness, we have $P(x)\in S_1$ for $x\in S_1$. Then,

$$
P:\overline{Co}S_1\mapsto\overline{Co}S_1
$$

is continuous.

In fact, for any given $y\in Co\{\overline{U}(kT),\ k=0,1,\cdots\}$, there exists $\theta\in[0,\ 1]$ and $x_1,\ x_2\in\{\overline{U}(kT),\ k=0,1,\cdots\}$ such that $y=\theta x_1+(1-\theta)x_2$. Hence,

$$
\begin{aligned}
P(y) &= P(\theta x_1+(1-\theta)x_2)\\
&= \mathrm{e}^{-AT}(\theta x_1+(1-\theta)x_2)+\int_0^T \mathrm{e}^{A(\tau-T)}(G(W)+F)\mathrm{d}\tau\\
&= \theta(\mathrm{e}^{-AT}x_1+\int_0^T \mathrm{e}^{A(\tau-T)}(G(W)+F)\mathrm{d}\tau)\\
&\quad + (1-\theta)(\mathrm{e}^{-AT}x_2+\int_0^T \mathrm{e}^{A(\tau-T)}(G(W)+F)\mathrm{d}\tau)\\
&= \theta P(x_1)+(1-\theta)P(x_2).
\end{aligned}
$$

The continuity of P is from the continuous dependence of initial data.

From Tychonoff fixed point theorem, there exists a fixed point $U^* \in \overline{Co}S_1$ satisfying

$$\|U^*\|_{H^{m+1}} \le C\delta^2$$

such that $P(U^*) = U^*$, i.e., $U(t, U^*)$ is a periodic solution of the linearized system (3.1)–(3.3). Thus, with a similar argument above, there holds

$$\begin{aligned}
&\|U(t, U^*)\|^2_{H^{m+1}} \\
\le\,& e^{-C_1 t}\|U^*\|^2_{H^{m+1}} + \int_0^t e^{C_1(\tau-t)}C\big(\|\sigma'\|^4_{H^{m+1}} + \|u'\|^4_{H^{m+1}} + \|v'\|^4_{H^{m+1}} + \|f\|^2_{H^m} \\
&\qquad + \|\sigma'\|^2_{H^m}\|u'\|^4_{H^{m+1}} + \|\sigma'\|^2_{H^{m+1}}\|u'\|^2_{H^{m+2}} + \|\sigma'\|^2_{H^{m+1}}\|v'\|^2_{H^{m+2}}\big)\mathrm{d}\tau \\
\le\,& e^{-C_1 t}\|U^*\|^2_{H^{m+1}} + \sum_{i=0}^{n-1}\int_{iT}^{(i+1)T} e^{C_1(\tau-t)}C\big(\|\sigma'\|^4_{H^{m+1}} + \|u'\|^4_{H^{m+1}} + \|v'\|^4_{H^{m+1}} + \|f\|^2_{H^m} \\
&\qquad + \|\sigma'\|^2_{H^m}\|u'\|^4_{H^{m+1}} + \|\sigma'\|^2_{H^{m+1}}\|u'\|^2_{H^{m+2}} + \|\sigma'\|^2_{H^{m+1}}\|v'\|^2_{H^{m+2}}\big)\mathrm{d}\tau \\
&+ \int_{nT}^t e^{C_1(\tau-t)}C\big(\|\sigma'\|^4_{H^{m+1}} + \|u'\|^4_{H^{m+1}} + \|v'\|^4_{H^{m+1}} + \|f\|^2_{H^m} + \|\sigma'\|^2_{H^m}\|u'\|^4_{H^{m+1}} \\
&\qquad + \|\sigma'\|^2_{H^{m+1}}\|u'\|^2_{H^{m+2}} + \|\sigma'\|^2_{H^{m+1}}\|v'\|^2_{H^{m+2}}\big)\mathrm{d}\tau \\
\le\,& \|U^*\|^2_{H^{m+1}} + (C+1)\int_0^T\big(\|\sigma'\|^4_{H^{m+1}} + \|u'\|^4_{H^{m+1}} + \|v'\|^4_{H^{m+1}} + \|f\|^2_{H^m} + \|\sigma'\|^2_{H^m}\|u'\|^4_{H^{m+1}} \\
&\qquad + \|\sigma'\|^2_{H^{m+1}}\|u'\|^2_{H^{m+2}} + \|\sigma'\|^2_{H^{m+1}}\|v'\|^2_{H^{m+2}}\big)\mathrm{d}\tau \\
\le\,& C\delta^4.
\end{aligned}$$

Note that $U(t, U^*) := (\underline{\sigma}, \underline{u}, \underline{v})$ satisfies (3.6). Integrating it from 0 to T, we have

$$\begin{aligned}
&\int_0^T \Big(\|\underline{u}\|^2_{H^{m+2}} + \|\underline{v}\|^2_{H^{m+2}} + \frac{1}{M}\|\underline{\sigma}\|^2_{H^{m+1}}\Big)\mathrm{d}t \\
\le\,& C\int_0^T \big(\|\sigma'\|^4_{H^{m+1}} + \|u'\|^4_{H^{m+1}} + \|v'\|^4_{H^{m+1}} + \|f\|^2_{H^m} + \|\sigma'\|^2_{H^m}\|u'\|^4_{H^{m+1}} \\
&\qquad + \|\sigma'\|^2_{H^{m+1}}\|u'\|^2_{H^{m+2}} + \|\sigma'\|^2_{H^{m+1}}\|v'\|^2_{H^{m+2}}\big)\mathrm{d}t.
\end{aligned}$$

Therefore,

$$\int_0^T \big(\|\underline{u}\|^2_{H^{m+2}} + \|\underline{v}\|^2_{H^{m+2}} + \|\underline{\sigma}\|^2_{H^{m+1}}\big)\mathrm{d}\tau \le C\delta^4. \tag{4.1}$$

From above argument, we can see that the set

$$S_{2,W} = \big\{(\underline{\sigma}, \underline{u}, \underline{v}) \text{ is the } T\text{-periodic solution of (3.1)–(3.3) with } T\text{-periodic function } W \in X_\delta\big\}$$

is nonempty. And the T-periodic solution $(\underline{\sigma}, \underline{u}, \underline{v})$ of (3.1)–(3.3) satisfies

$$\|(\underline{\sigma}, \underline{u}, \underline{v})\|^2_X \le C\delta^4 \le \delta^2$$

for $\delta = \eta^{\frac{1}{4}}$ small enough. It obtains that $S_{2,W}$ is a subset of X_δ.

Lemma 4.1 *Assume that* $m \ge \left[\frac{n}{2}\right] + 1$ *and* $(\underline{\sigma}, \underline{u}, \underline{v})$ *is the* T*-periodic solution of the linearized system* (3.1)–(3.3) *with* T*-periodic function* $W \in X_\delta$*. Then, for each multi-index* β *with* $|\beta| = 0, \cdots, m$*, we have*

$$\int_0^T \|\partial_x^\beta \underline{\sigma}_t\|_2^2 + \|\partial_x^\beta \underline{u}_t\|_2^2 + \|\partial_x^\beta \underline{v}_t\|_2^2 \mathrm{d}t \le C\delta^4.$$

Proof. For each multi-index β with $|\beta| = 0, \cdots, m$, applying ∂_x^β to (3.1)–(3.3), multiplying $\partial_x^\beta \sigma_t$, $\partial_x^\beta u_t$, $\partial_x^\beta v_t$ respectively and integrating them over $\Omega \times [0, T]$, we have

$$\begin{aligned}
&\int_0^T\int |\partial_x^\beta \underline{\sigma}_t|^2 \mathrm{d}x\mathrm{d}t + \int_0^T\int \bar{\rho}\mathrm{div}\partial_x^\beta \underline{u}\partial_x^\beta \underline{\sigma}_t \mathrm{d}x\mathrm{d}t + \int_0^T\int \partial_x^\beta (u' \cdot \nabla \underline{\sigma})\partial_x^\beta \underline{\sigma}_t \mathrm{d}x\mathrm{d}t\\
&- \int_0^T\int (\bar{\mu}\Delta\partial_x^\beta \underline{u} + (\bar{\mu} + \bar{\lambda})\nabla\mathrm{div}\partial_x^\beta \underline{u})\partial_x^\beta \underline{u}_t \mathrm{d}x\mathrm{d}x\mathrm{d}t + \int_0^T\int (\gamma_1\bar{\rho}\nabla\partial_x^\beta \underline{\sigma} + \gamma_2\bar{\rho}\nabla\partial_x^\beta \underline{v})\partial_x^\beta \underline{u}_t \mathrm{d}x\mathrm{d}t\\
&+ \int_0^T\int |\partial_x^\beta \underline{v}_t|^2 \mathrm{d}x\mathrm{d}t - \int_0^T\int \bar{\kappa}\Delta\partial_x^\beta \underline{v}\partial_x^\beta \underline{v}_t \mathrm{d}x dt + \gamma_3\gamma_2\bar{\rho}\int \mathrm{div}\partial_x^\beta \underline{u}\partial_x^\beta \underline{v}_t \mathrm{d}x dt\\
&+ \int_0^T\int |\partial_x^\beta \underline{u}_t|^2 \mathrm{d}x\mathrm{d}t\\
=& \int_0^T\int \partial_x^\beta G_1(\sigma', u')\partial_x^\beta \underline{\sigma}_t \mathrm{d}x\mathrm{d}t + \int_0^T\int \partial_x^\beta G_2(\sigma', u', v')\partial_x^\beta \underline{u}_t \mathrm{d}x\mathrm{d}t\\
&+ \int_0^T\int \partial_x^\beta f \partial_x^\beta \underline{u}_t \mathrm{d}x + \int_0^T\int \partial_x^\beta G_3(\sigma', u', v')\partial_x^\beta \underline{v}_t \mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Then,

$$\begin{aligned}
&\int_0^T\int |\partial_x^\beta \underline{\sigma}_t|^2 \mathrm{d}x\mathrm{d}t + \int_0^T\int |\partial_x^\beta \underline{u}_t|^2 \mathrm{d}x\mathrm{d}t + \int_0^T\int |\partial_x^\beta \underline{v}_t|^2 \mathrm{d}x dt\\
\le& C\bigg(\int_0^T\int |\mathrm{div}\partial_x^\beta \underline{u}|^2 \mathrm{d}x\mathrm{d}t + \int_0^T\int |\partial_x^\beta (u' \cdot \nabla\underline{\sigma})|^2 \mathrm{d}x\mathrm{d}t + \int_0^T\int |\Delta\partial_x^\beta \underline{u}|^2 \mathrm{d}x\mathrm{d}t\\
&+ \int_0^T\int |\nabla\partial_x^\beta \underline{v}|^2 \mathrm{d}x\mathrm{d}t + \int_0^T\int |\Delta\partial_x^\beta \underline{v}|^2 \mathrm{d}x\mathrm{d}t + \int_0^T\int |\mathrm{div}\partial_x^\beta \underline{u}|^2 \mathrm{d}x\mathrm{d}t\\
&+ \int_0^T\int |\partial_x^\beta G_1(\sigma', u')|^2 \mathrm{d}x\mathrm{d}t + \int_0^T\int |\partial_x^\beta G_2(\sigma', u', v')|^2 \mathrm{d}x\mathrm{d}t + \int_0^T\int |\partial_x^\beta f|^2 \mathrm{d}x\\
&+ \int_0^T\int |\partial_x^\beta G_3(\sigma', u', v')|^2 \mathrm{d}x\mathrm{d}t + \int_0^T\int |\nabla\partial_x^\beta \underline{\sigma}|^2 \mathrm{d}x\mathrm{d}t\bigg).
\end{aligned}$$

From the assumptions, (4.1) and the arguments in Lemma 3.2, there holds

$$\int_0^T \|\partial_x^\beta \underline{\sigma}_t\|_2^2 + \|\partial_x^\beta \underline{u}_t\|_2^2 + \|\partial_x^\beta \underline{v}_t\|_2^2 \mathrm{d}t \le C\delta^4,$$

which completes the proof.

Therefore, by Lemma C.1 in [28], Lions-Aubin lemma[29] and Lemma 2.3, $S_{2,W}$ is precompact in X_{-1}. Moreover, $S_{2,W}$ is convex and closed because of the system (3.1)–(3.3) is linear.

Define

$$S_3 = \bigcup\{S_{2,W},\ \|W\|_X^2 \le \delta^2\},$$

and

$$S_4 = \overline{Co}S_3.$$

Proposition 4.1 *S_4 is nonempty, compact and convex in X_{-1}.*

Proof. The result is obtained by Mazur's theorem. Here, we omit the details of the proof.

Define a set-valued function:

$$Per(W) = S_{2,W}.$$

We can see for any given $\Phi \in S_4 \subset X_\delta$, $Per(\Phi) \subset S_4$. Moreover, the fixed point of $Per : S_4 \mapsto 2^{S_4}$ is the T-periodic solution of system (2.1)–(2.3).

Next, we prove $Per : S_4 \mapsto 2^{S_4}$ is a set-valued function which has a closed graph.

Proposition 4.2 *$Per : S_4 \mapsto 2^{S_4}$ has a closed graph. That is to say $\{(W, U) \mid W \in S_4,\ U \in Per(W)\}$ is closed subset in $X_{-1} \times X_{-1}$.*

Proof. Assume that

$$\begin{aligned} W_k &\to W \text{ in } X_{-1}, \qquad \text{as } k \to \infty,\\ U_k &\to U \text{ in } X_{-1}, \qquad \text{as } k \to \infty, \end{aligned}$$

where $U_k \in Per(W_k)$.

Since S_4 is compact, one has $W \in S_4$. We only need to prove $U \in Per(W)$.

Let

$$U_k = (\sigma_k,\, u_k,\, v_k), \qquad W_k = (\widetilde{\sigma}_k,\, \widetilde{u}_k,\, \widetilde{v}_k).$$

Since $\sigma_k \in L^\infty(0, T; H^{m+1}(\Omega))$, we have $\sigma_k \in C^\alpha(\Omega)$ with $\alpha \in (0, 1)$ for any fixed t.

Assume that $0 \le t_2 \le t_1$. Taking a ball B_r of radius $r = |t_1 - t_2|^\zeta$ centered at x with $\zeta > 0$, $\zeta = \dfrac{1}{n + 2\alpha}$, we have

$$\begin{aligned} \int_{B_r} |\sigma_k(y, t_1) - \sigma_k(y, t_2)|\mathrm{d}y &= \int_{B_r} \Big| \int_{t_2}^{t_1} \frac{\partial \sigma_k(y, t)}{\partial t} \mathrm{d}t \Big| \mathrm{d}y\\ &\le C\Big(\int_{B_r} \int_{t_2}^{t_1} \Big| \frac{\partial \sigma_k(y, t)}{\partial t} \Big|^2 \mathrm{d}t\mathrm{d}y \Big)^{\frac{1}{2}} |t_1 - t_2|^{\frac{1}{2}} r^{\frac{n}{2}}\\ &\le C|t_1 - t_2|^{\frac{1}{2}} r^{\frac{n}{2}}. \end{aligned}$$

By the mean value theorem, there exists x^* such that

$$|\sigma_k(x^*, t_1) - \sigma_k(x^*, t_2)| \le C|t_1 - t_2|^{\frac{1}{2}} r^{-\frac{n}{2}} \le C|t_1 - t_2|^{\frac{1 - n\zeta}{2}}.$$

Then,

$$\begin{aligned} &|\sigma_k(x, t_1) - \sigma_k(x, t_2)|\\ \le\,& |\sigma_k(x, t_1) - \sigma_k(x^*, t_1)| + |\sigma_k(x^*, t_1) - \sigma_k(x^*, t_2)| + |\sigma_k(x^*, t_2) - \sigma_k(x, t_2)|\\ \le\,& C(|x - x^*|^\alpha + |t_1 - t_2|^{\frac{1 - n\zeta}{2}})\\ \le\,& C(|t_1 - t_2|^{\alpha\zeta} + |t_1 - t_2|^{\frac{1 - n\zeta}{2}})\\ \le\,& C|t_1 - t_2|^{\frac{\alpha}{n + 2\alpha}}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} |u_k(x_1, t_1) - u_k(x_2, t_2)| &\le C(|x_1 - x_2|^\alpha + |t_1 - t_2|^\beta),\\ |v_k(x_1, t_1) - v_k(x_2, t_2)| &\le C(|x_1 - x_2|^\alpha + |t_1 - t_2|^\beta), \end{aligned}$$

where $\beta \in (0, 1)$.

Then, by Arzelà-Ascoli theorem, we get

$$(\sigma_k,\, u_k,\, v_k) \to (\sigma,\, u,\, v) \text{ uniformly}, \qquad \text{as } k \to \infty.$$

By Lemma 4.1, there holds

$$\begin{aligned}
&(\sigma_{kt},\, u_{kt},\, v_{kt}) \rightharpoonup (\sigma_t,\, u_t,\, v_t) \text{ in } L^2(0,\, T;\, H^m), \qquad \text{as } k \to \infty,\\
&(\sigma_k,\, u_k,\, v_k) \rightharpoonup (\sigma,\, u,\, v) \text{ weakly star in } L^\infty(0,\, T;\, H^{m+1}), \qquad \text{as } k \to \infty,\\
&(\sigma_k,\, u_k,\, v_k) \rightharpoonup (\sigma,\, u,\, v) \text{ in } L^2(0,\, T;\, H^{m+2}), \qquad \text{as } k \to \infty.
\end{aligned}$$

It implies that $U = (\sigma,\, u,\, v) \in X_\delta$.

Note that $U_k \in Per(W_k)$, i.e., U_k is a T-periodic solution of system (3.1)–(3.3) with T-periodic function W_k. Let $k \to \infty$. It obtains that $U = (\sigma,\, u,\, v)$ is a T-periodic solution of system (3.1)–(3.3) with W, i.e., $U \in Per(W)$.

From Propositions 4.1 and 4.2, $Per : S_4 \mapsto 2^{S_4}$ satisfies the hypotheses of the Kakutani fixed point theorem. By Lemma 2.5, there exists a fixed point U_{Per} of Per. It means that U_{Per} is a T-periodic solution of the system (2.1)–(2.3).

5 Uniqueness

In this section, we prove the uniqueness of the T-periodic solution.

Let $U_1 = (\sigma_1,\, u_1,\, v_1)$ and $U_2 = (\sigma_2,\, u_2,\, v_2)$ are T-periodic solutions of the system (2.1)–(2.3). Denote $\sigma = \sigma_1 - \sigma_2$, $u = u_1 - u_2$, $v = v_1 - v_2$. Then, $(\sigma,\, u,\, v)$ satisfies

$$\sigma_t + \bar\rho \operatorname{div} u + u_1 \nabla\sigma + u\nabla\sigma_2 = G_1(\sigma_1,\, u_1) - G_1(\sigma_2,\, u_2), \tag{5.1}$$

$$u_t - (\bar\mu\Delta u + (\bar\mu + \bar\lambda)\nabla \operatorname{div} u) + \gamma_1\bar\rho\nabla\sigma + \gamma_2\bar\rho\nabla v = G_2(\sigma_1,\, u_1,\, v_1) - G_2(\sigma_2,\, u_2,\, v_2), \tag{5.2}$$

$$v_t - \bar\kappa\Delta v + \gamma_3\gamma_2\bar\rho \operatorname{div} u = G_3(\sigma_1,\, u_1,\, v_1) - G_3(\sigma_2,\, u_2,\, v_2). \tag{5.3}$$

For each multi-index α with $|\alpha| = 0, \cdots, m+1$, applying ∂_x^α to (5.1)–(5.3), multiplying $\gamma_1\gamma_3\partial_x^\alpha\sigma$, $\gamma_3\partial_x^\alpha u$, $\partial_x^\alpha v$ respectively and integrating them over Ω, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int (\gamma_1\gamma_3|\partial_x^\alpha\sigma|^2 + \gamma_3|\partial_x^\alpha u|^2 + |\partial_x^\alpha v|^2)\mathrm{d}x + \gamma_3\int \big(\bar\mu|\partial_x^\alpha\nabla u|^2 + (\bar\mu+\bar\lambda)|\partial_x^\alpha \operatorname{div} u|^2\big)\mathrm{d}x\\
&\quad + \bar\kappa\int |\partial_x^\alpha\nabla v|^2\mathrm{d}x\\
=& -\gamma_1\gamma_3\int \partial_x^\alpha(u_1\cdot\nabla\sigma + u\cdot\nabla\sigma_2)\partial_x^\alpha\sigma\mathrm{d}x - \gamma_1\gamma_3\int \partial_x^\alpha(G_1(\sigma_1,\, u_1) - G_1(\sigma_2,\, u_2))\partial_x^\alpha\sigma\mathrm{d}x\\
&\quad - \gamma_3\int \partial_x^\alpha(G_2(\sigma_1, u_1, v_1) - G_2(\sigma_2, u_2, v_2))\partial_x^\alpha u\mathrm{d}x\\
&\quad + \int \partial_x^\alpha(G_3(\sigma_1,\, u_1,\, v_1) - G_3(\sigma_2,\, u_2,\, v_2))\partial_x^\alpha v\mathrm{d}x\\
\le& C\bigg[\int \big|\partial_x^\alpha(u_1\cdot\nabla\sigma + u\cdot\nabla\sigma_2)\big|^2\mathrm{d}x + \int \big|\partial_x^\alpha(G_1(\sigma_1,\, u_1) - G_1(\sigma_2,\, u_2))\big|^2\mathrm{d}x\\
&\qquad + \int \big|\partial_x^{\alpha-1}(G_2(\sigma_1,\, u_1,\, v_1) - G_2(\sigma_2,\, u_2,\, v_2))\big|^2\mathrm{d}x\\
&\qquad + \int \big|\partial_x^{\alpha-1}(G_3(\sigma_1,\, u_1,\, v_1) - G_3(\sigma_2,\, u_2,\, v_2))\big|^2\mathrm{d}x\bigg]
\end{aligned}$$

$$+\epsilon\int\left|\partial_x^\alpha\sigma\right|^2\mathrm{d}x+\frac{\gamma_3\bar{\mu}}{2}\int\left|\partial_x^\alpha\nabla u\right|^2\mathrm{d}x+\frac{\bar{\kappa}}{2}\int\left|\partial_x^\alpha\nabla v\right|^2\mathrm{d}x. \tag{5.4}$$

From Lemma 2.1 and 2.2, we have

$$\begin{aligned}&\int\left|\partial_x^\alpha(u_1\cdot\nabla\sigma+u\cdot\nabla\sigma_2)\right|^2\mathrm{d}x\\ \le\,& C\big(\|u_1\|_\infty\|\nabla\sigma\|_{H^m}+\|u_1\|_{H^m}\|\nabla\sigma\|_\infty+\|u\|_\infty\|\nabla\sigma_2\|_{H^m}+\|u\|_{H^m}\|\nabla\sigma_2\|_\infty\big)^2\\ \le\,& C\big(\|u_1\|_{H^m}\|\nabla\sigma\|_{H^m}+\|u\|_{H^m}\|\nabla\sigma_2\|_{H^m}\big)^2\end{aligned}$$

and

$$\int|\partial_x^\alpha(G_1(\sigma_1,\,u_1)-G_1(\sigma_2,\,u_2)|^2\mathrm{d}x\le C\big(\|\sigma_1\|_{H^m}\|\nabla u\|_{H^m}+\|\sigma\|_{H^m}\|\nabla u_2\|_{H^m}\big)^2.$$

With a similar argument in Lemma 3.2, we have

$$\begin{aligned}&\int\left|\partial_x^{\alpha-1}(G_2(\sigma_1,\,u_1,\,v_1)-G_2(\sigma_2,\,u_2,\,v_2))\right|^2\mathrm{d}x\\ \le\,& C\big[\|\nabla u\|_{H^m}(\|\nabla u_1\|_{H^m}+\|\nabla u_2\|_{H^m})+\|\nabla\sigma\|_{H^m}(\|\nabla\sigma_1\|_{H^m}+\|\nabla\sigma_2\|_{H^m})\\ &\quad+\|\nabla v\|_{H^m}(\|\nabla v_1\|_{H^m}+\|\nabla v_2\|_{H^m})+\|\nabla v\|_{H^m}\|\nabla\sigma_1\|_{H^m}+\|\nabla v_2\|_{H^m}\|\nabla\sigma\|_{H^m}\\ &\quad+\|\nabla\sigma\|_{H^m}\|\Delta u_1\|_{H^m}+\|\nabla\sigma_2\|_{H^m}\|\Delta u\|_{H^m}\big]^2\\ &+\|\nabla\sigma\|_{H^m}\|\Delta u_1\|_{H^m}+\|\nabla\sigma_2\|_{H^m}\|\Delta u\|_{H^m}\end{aligned}$$

and

$$\begin{aligned}&\int|\partial_x^{\alpha-1}(G_3(\sigma_1,\,u_1,\,v_1)-G_3(\sigma_2,\,u_2,\,v_2))|^2\mathrm{d}x\\ \le\,& C\big[\|u_1\|_{H^m}\|\nabla v\|_{H^m}+\|u\|_{H^m}\|\nabla v_2\|_{H^m}+\|\sigma_1\|_{H^m}\|\Delta v\|_{H^m}+\|\sigma\|_{H^m}\|\Delta v_2\|_{H^m}\\ &\quad+\|\sigma_1\|_{H^m}\|\nabla u\|_{H^m}+\|\sigma\|_{H^m}\|\nabla u_2\|_{H^m}+\|v_1\|_{H^m}\|\nabla u\|_{H^m}+\|v\|_{H^m}\|\nabla u_2\|_{H^m}\\ &\quad+\|\nabla u\|_{H^m}(\|\nabla u_1\|_{H^m}+\|\nabla u_2\|_{H^m})+\|\sigma_1\|_{H^m}\|\nabla u\|_{H^m}(\|\nabla u_1\|_{H^m}+\|\nabla u_2\|_{H^m})\\ &\quad+\|\nabla u_2\|_{H^m}^2\|\sigma\|_{H^m}\big]^2.\end{aligned}$$

Integrating (5.4) from 0 to T, it obtains that

$$\begin{aligned}&\gamma_3\int_0^T\Big(\frac{\bar{\mu}}{2}\|\partial_x^\alpha\nabla u\|_2^2+(\bar{\mu}+\bar{\lambda})|\partial_x^\alpha\mathrm{div}u\|_2^2\Big)\mathrm{d}t+\frac{\bar{\kappa}}{2}\int_0^T\|\partial_x^\alpha\nabla v\|_2^2\mathrm{d}t\\ \le\,& C\|(\sigma,\,u,\,v)\|_X^2\big(\|(\sigma_1,\,u_1,\,v_1)\|_X^2+\|(\sigma_2,\,u_2,\,v_2)\|_X^2\\ &+\int_0^T\|\sigma_1\|_{H^m}^2\mathrm{d}t\sup_{t\in[0,\,T]}(\|\nabla u_1\|_{H^m}^2+\|\nabla u_2\|_{H^m}^2)\\ &+\int_0^T\|\nabla u_2\|_{H^m}^2\mathrm{d}t\sup_{t\in[0,\,T]}\|\nabla u_2\|_{H^m}^2\big)+\epsilon\int\left|\partial_x^\alpha\sigma\right|^2\mathrm{d}x\\ \le\,& C\delta^2\|(\sigma,\,u,\,v)\|_X^2+\epsilon\int\left|\partial_x^\alpha\sigma\right|^2\mathrm{d}x.\end{aligned}\tag{5.5}$$

For each multi-index β with $|\beta|=1,\cdots,m$, by applying ∂_x^β to equations (5.2), multiplying them by $\partial_x^\beta\nabla\sigma$ and then integrating them, it obtains that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int\partial_x^\beta u\cdot\partial_x^\beta\nabla\sigma\mathrm{d}x+\gamma_1\bar{\rho}\int|\partial_x^\beta\nabla\sigma|^2\mathrm{d}x\\ =&-\int\partial_x^\beta u\big(\partial_x^\beta\nabla\mathrm{div}(\sigma_1u_1-\sigma_2u_2)\big)\mathrm{d}x\\ &+\bar{\rho}\|\partial_x^\beta\mathrm{div}u\|_2^2+\bar{\mu}\int\partial_x^\beta\Delta u\partial_x^\beta\nabla\sigma\mathrm{d}x+(\bar{\mu}+\bar{\lambda})\int\partial_x^\beta\nabla\mathrm{div}u\partial_x^\beta\nabla\sigma\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&- \gamma_2\bar\rho \int \partial_x^\beta \nabla v \partial_x^\beta \nabla\sigma \mathrm{d}x + \int \partial_x^\beta (G_2(\sigma_1,\, u_1,\, v_1) - G_2(\sigma_2,\, u_2,\, v_2))\partial_x^\beta \nabla\sigma \mathrm{d}x \\
\le& (\|\sigma_1\|_{H^{m+1}}\|u\|_{H^{m+1}} + \|\sigma\|_{H^{m+1}}\|u_2\|_{H^{m+1}}) + (1+\bar\rho)\|\partial_x^\beta \mathrm{div} u\|_2^2 + \frac{\gamma_2\bar\rho}{2}\int |\partial_x^\beta \nabla\sigma|^2 \mathrm{d}x \\
&+ C_{\bar\mu,\,\bar\lambda,\,\bar\rho,\,\gamma_2}\big(\|\partial_x^\beta \Delta u\|_2^2 + \|\partial_x^\beta \nabla \mathrm{div} u\|_2^2 + \|\partial_x^\beta \nabla v\|_2^2\big) \\
&+ C_{\gamma_2,\,\bar\rho}\big[\|\nabla u\|_{H^m}(\|\nabla u_1\|_{H^m} + \|\nabla u_2\|_{H^m}) + \|\nabla\sigma\|_{H^m}(\|\nabla\sigma_1\|_{H^m} + \|\nabla\sigma_2\|_{H^m}) \\
&\qquad + \|\nabla v\|_{H^m}\|\nabla\sigma_1\|_{H^m} + \|\nabla v_2\|_{H^m}\|\nabla\sigma\|_{H^m} + \|\nabla\sigma\|_{H^m}\|\Delta u_1\|_{H^m} \\
&\qquad + \|\nabla\sigma_2\|_{H^m}\|\Delta u\|_{H^m}\big]^2. \qquad (5.6)
\end{aligned}
$$

Integrating (5.6) from 0 to T, we have

$$
\begin{aligned}
\frac{\gamma_1\bar\rho}{2}\int_0^T \|\partial_x^\beta \nabla\sigma\|_2^2 \mathrm{d}t \le& C_{\bar\mu,\,\bar\lambda,\bar\rho,\,\gamma_2}\bigg(\int_0^T \|u\|_{H^{m+2}}^2 \mathrm{d}t + \int_0^T \|v\|_{H^{m+2}}^2 \mathrm{d}t\bigg) \\
&+ C\|(\sigma,\, u,\, v)\|_X(\|(\sigma_1,\, u_1,\, v_1)\|_X + \|(\sigma_2,\, u_2,\, v_2)\|_X). \qquad (5.7)
\end{aligned}
$$

From (5.5), (5.7), summing up about $|\alpha|$, $|\beta|$, taking M' appropriately large and letting ϵ appropriately small, there holds

$$
\begin{aligned}
&M'\bigg(\int_0^T \|u\|_{H^{m+2}}^2 \mathrm{d}t + \int_0^T \|v\|_{H^{m+2}}^2 \mathrm{d}t\bigg) + \frac{\gamma_1\bar\rho}{4}\int_0^T \|\sigma\|_{H^{m+1}}^2 \mathrm{d}t \\
\le& C_{M',\,\bar\mu,\,\bar\lambda,\,\bar\rho,\,\gamma_2,\,\gamma_3}\delta^2\|(\sigma,\, u,\, v)\|_X^2.
\end{aligned}
$$

Then, there exists a $t^* \in [0,\, T]$ such that

$$
\|u(t^*)\|_{H^{m+2}}^2 + \|v(t^*)\|_{H^{m+2}}^2 + \|\sigma(t^*)\|_{H^{m+1}}^2 \le C\delta^2\|(\sigma,\, u,\, v)\|_X^2.
$$

Combining (5.4) and (5.6), integrating over $[t^*,\, t]$ with $t \in [t^*,\, t^*+T]$, we have

$$
\begin{aligned}
&\|u(t)\|_{H^{m+1}}^2 + \|v(t)\|_{H^{m+1}}^2 + \|\sigma(t)\|_{H^{m+1}}^2 \\
\le& C\delta^2\|(\sigma,\, u,\, v)\|_X^2 + \|u(t^*)\|_{H^{m+1}}^2 + \|v(t^*)\|_{H^{m+1}}^2 + \|\sigma(t^*)\|_{H^{m+1}}^2 \\
\le& C\delta^2\|(\sigma,\, u,\, v)\|_X^2.
\end{aligned}
$$

Since $(\sigma,\, u,\, v)$ is T-periodic, it obtains that

$$
\begin{aligned}
\|(\sigma,\, u,\, v)\|_X^2 =& \sup_{t\in[0,T]}\big(\|u(t)\|_{H^{m+2}}^2 + \|v(t)\|_{H^{m+2}}^2 + \|\sigma(t)\|_{H^{m+1}}^2\big) \\
&+ \int_0^T \big(\|u\|_{H^{m+2}}^2 \mathrm{d}t + \|v\|_{H^{m+2}}^2 \mathrm{d}t + \|\sigma\|_{H^{m+1}}^2\big)\mathrm{d}t \\
\le& C\delta^2\|(\sigma,\, u,\, v)\|_X^2 \\
\le& C\eta^{\frac{1}{2}}\|(\sigma,\, u,\, v)\|_X^2.
\end{aligned}
$$

Since η small enough, we have

$$
\|(\sigma,\, u,\, v)\|_X^2 \equiv 0.
$$

It implies

$$
U_1 = U_2.
$$

Remark 5.1 From the proof of uniqueness, it is not difficult to see that there is no small non-trivial time periodic solution for the non-isentropic compressible Navier-Stokes equations without external force.

Remark 5.2 Assume that f is independent of t and $\|f\|_{H^m} \le \eta$ with η appropriately small. It means that f is periodic of any period $T > 0$. By Theorem 1.1, there exists a time periodic solution (ρ_1, u_1, θ_1) of period 1. On the other hand, there exists a time periodic solution (ρ_2, u_2, θ_2) of period $\frac{1}{2}$. By uniqueness, we must have $\rho_1 = \rho_2$, $u_1 = u_2$, $\theta_1 = \theta_2$. Going on this way, we can see (ρ_1, u_1, θ_1) is periodic of any rotational period. It concludes that (ρ_1, u_1, θ_1) is independent of $t \in \mathbf{Q}$. From a continuity argument, we get $(\rho_1(x), u_1(x), \theta_1(x))$ is independent of t. Therefore, $(\rho_1(x), u_1(x), \theta_1(x))$ is a unique stationary solution of the following system:

$$\begin{aligned}
&\mathrm{div}(\rho u) = 0,\\
&\rho(u\cdot\nabla)u + \nabla P(\rho, \theta) = \mu\Delta u + (\mu+\lambda)\nabla \mathrm{div} u + \rho f(x),\\
&\rho C_\nu(u\cdot\nabla\theta) + \theta P_\theta(\rho, \theta)\mathrm{div} u = \kappa\Delta\theta + \varPhi(u).
\end{aligned}$$

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