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On a Generalized Matrix Algebra over Frobenius Algebra

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Abstract: Let A be a Frobenius k-algebra. The matrix algebra $R = \begin{pmatrix} A & {}_AA_k \\ {}_kA_A & k \end{pmatrix}$ is called a generalized matrix algebra over a Frobenius algebra A. In this paper we show that R is also a Frobenius algebra.

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1 Introduction

Let A, B be two finite dimensional algebras over a field k, ${}_AM_B$, ${}_BN_A$ be two finitely generated bimodules. Assume that there are bimodule morphisms

$$\tau\colon M \otimes_B N \longrightarrow A\colon \tau(m \otimes n) = (m, n)$$
$$\mu\colon N \otimes_A M \longrightarrow B\colon \mu(n \otimes m) = [n, m]$$

satisfying

$$(m, n)m' = m[n, m'], \quad [n, m]n' = n(m, n'), \qquad m, m' \in M, \ n, n' \in N,$$

where addition and multiplication are defined as in customary for matrices, $R = \begin{pmatrix} A & M \\ N & B \end{pmatrix}$ is a k-algebra, which is called generalized matrix algebra. One point extension algebra and local extension algebra are also generalized matrix algebras. Generalized matrix algebra also comes up as a Morita Context. For more details see [1]–[3].

Frobenius bimodules are connected with Frobenius algebras and extensions. For instance, a ring extension $\phi\colon R \to S$ is a Frobenius extension if and only if ${}_RS_S$ is a Frobenius

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bimodule. Let A be a finite dimensional k-algebra. If ${}_kA_A$ is a Frobenius bimodule, there exists a bimodule isomorphism $\mathrm{Hom}_k({}_kA,\, k) \cong {}_AA_k$, then A is called a Frobenius algebra. Simple algebra over a field k, group algebra kG are also Frobenius algebra. By [1] (see p261), if A is a Frobenius algebra, then

$$\tau\colon {}_AA \otimes_k A_A \longrightarrow A, \qquad \mu\colon {}_kA \otimes_A k \longrightarrow A_k.$$

So $R = \begin{pmatrix} A & {}_AA_k \\ {}_kA_A & k \end{pmatrix}$ is a generalized matrix algebra over a Frobenius algebra A. In present paper, we show that R is also a Frobenius algebra. Throughout this paper, all rings have an identity element and all modules are unital, the following symbols can be referred in [4]–[6]. The latest related research on this subject can be found in [7]–[11].

2 The Functor Between mod-$A \times B$ and mod-R

For a ring A, the category of left A-modules is denoted by A-mod, mod-A denotes the category of the right A-modules. Let $\mathcal{A}(R)$ be the category whose objects are $(X,\, Y)_{\alpha,\,\beta}$, where $X \in$ mod-A, $Y \in$ mod-B, $\alpha \in \mathrm{Hom}_B(X \otimes_A M,\, Y)$, $\beta \in \mathrm{Hom}_A(Y \otimes_B N,\, X)$ such that

$$\alpha(\beta(y \otimes n) \otimes m) = y[n,\, m], \qquad \beta(\alpha(x \otimes m) \otimes n) = x(m,\, n)$$

for all $x \in X$, $y \in Y$, $m \in M$, $n \in N$.

Instead of α and β, it is more convenient to use the following homomorphisms $\overline{\alpha}$ and $\overline{\beta}$,

$$\overline{\alpha}\colon X \to \mathrm{Hom}_B(M,\, Y), \qquad \overline{\alpha}(x)m = \alpha(x \otimes m),$$
$$\overline{\beta}\colon Y \to \mathrm{Hom}_A(N,\, X), \qquad \overline{\beta}(y)n = \beta(y \otimes n).$$

The morphisms of $\mathcal{A}(R)$ are pairs of $(\sigma_1,\, \sigma_2)$, where $\sigma_1 \in \mathrm{Hom}_A(X,\, X')$, $\sigma_2 \in \mathrm{Hom}_B(Y,\, Y')$ such that the following diagrams are commutative.

$$\begin{array}{ccc} X \otimes M & \xrightarrow{\ \alpha\ } & Y \\ \Big\downarrow{\scriptstyle \sigma_1 \otimes 1_M} & & \Big\downarrow{\scriptstyle \sigma_2} \\ X' \otimes M & \xrightarrow{\ \alpha'\ } & Y' \end{array} \qquad\qquad \begin{array}{ccc} Y \otimes N & \xrightarrow{\ \beta\ } & X \\ \Big\downarrow{\scriptstyle \sigma_2 \times 1_N} & & \Big\downarrow{\scriptstyle \sigma_1} \\ Y' \otimes N & \xrightarrow{\ \beta'\ } & X' \end{array}$$

Green[4] proved that the category $\mathcal{A}(R)$ is equivalent to the category mod-R, i.e., there exists a categorical equivalent functor

$$F\colon \mathcal{A}(R) \Leftrightarrow \text{mod-}R$$

such that

$$F(X,\, Y)_{\alpha,\beta} = X \oplus Y,$$

where the right modular operation is

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} xa + \beta(y \otimes n) & \alpha(x \otimes n) + yb \end{pmatrix}.$$

Similarly, let $\mathcal{B}(R)$ be a left R-modules category, there is a categorical equivalent functor

$$G\colon \mathcal{B}(R) \Leftrightarrow R\text{-mod}$$

such that

$$G(X,\, Y)_{\alpha,\beta} = X \oplus Y.$$

Lemma 2.1 *The ring monomorphism*

$$\varPhi'\colon A \times B \to \begin{pmatrix} A & M \\ N & B \end{pmatrix}$$

induces a functor

$$\varPhi\colon \text{mod-}A \times B \to \text{mod-}R,$$

where

$$\begin{aligned}
\varPhi\colon \text{mod-}A \times B &\longrightarrow \text{mod-}R \\
X \times Y &\longmapsto \mathrm{Hom}_R(R,\, X \times Y), \\
\varPhi(X \times Y) = (X \oplus \mathrm{Hom}_B(M,\, Y),\ &\mathrm{Hom}_A(N,\, X) \oplus Y)_{\varepsilon,\delta}, \\
\varepsilon\colon (X \oplus \mathrm{Hom}_B(M,\, Y) \otimes M &\longrightarrow \mathrm{Hom}_A(N,\, X) \oplus Y) \\
(x,\, g) \otimes m &\longmapsto (x(m,\, \cdot),\, g(m)), \\
\delta\colon (\mathrm{Hom}_A(N,\, X) \oplus Y) \otimes N &\longrightarrow X \oplus \mathrm{Hom}_B(M,\, Y)) \\
(\eta,\, y) \otimes n &\longmapsto (\eta(n),\, y[n,\, \cdot\,]).
\end{aligned}$$

Proof. See [4] for details.

In fact, $\varPhi$ is a left exact functor. Since

$$\begin{aligned}
&\varepsilon((f,\, \mathrm{Hom}(M,\, g)\overline{\alpha}) \otimes 1)(x \otimes m) \\
&= \varepsilon(f(x),\, \mathrm{Hom}(M,\, g)\overline{\alpha}(x)) \otimes m \\
&= (f(x)(m,\, \cdot),\, \mathrm{Hom}(M,\, g)\overline{\alpha}(x)(m)),
\end{aligned}$$

we have

$$\begin{aligned}
&(\mathrm{Hom}(N,\, f)\overline{\beta},\, g)\alpha(x \otimes m) \\
&= (\mathrm{Hom}(N,\, f)\overline{\beta}\alpha(x \otimes m),\, g\alpha(x \otimes m)) \\
&= (f(x)(m,\, \cdot),\, \mathrm{Hom}(M,\, g)\overline{\alpha}(x)(m)).
\end{aligned}$$

Then

$$\varepsilon((f,\, \mathrm{Hom}(M,\, g)\overline{\alpha}) \otimes 1)(x \otimes m) = (\mathrm{Hom}(N,\, f)\overline{\beta},\, g)\alpha.$$

So we have the following commutative diagram:

$$\begin{array}{ccc}
X \otimes_A M & \xrightarrow{\quad\alpha\quad} & Y \\
\Big\downarrow{\scriptstyle \sigma_1 \otimes 1_M} & & \Big\downarrow{\scriptstyle \sigma_2} \\
(X' \oplus \mathrm{Hom}_B(M, Y')) \otimes M & \xrightarrow{\quad\varepsilon\quad} & \mathrm{Hom}_A(N, X') \oplus Y'
\end{array}$$

where

$$\sigma_1 = (f,\, \mathrm{Hom}(M,\, g)\overline{\alpha}), \qquad \sigma_2 = (\mathrm{Hom}(N,\, f)\overline{\beta},\, g).$$

Similarly,

$$\begin{array}{ccc} Y \otimes_B N & \xrightarrow{\quad\beta\quad} & X \\ \Big\downarrow{\scriptstyle \sigma_2 \otimes 1_N} & & \Big\downarrow{\scriptstyle \sigma_1} \\ (\mathrm{Hom}_A(N, X') \oplus Y') \otimes N & \xrightarrow{\quad\delta\quad} & \mathrm{Hom}_B(M, Y) \oplus X' \end{array}$$

Therefore, there is

$$\theta\colon \mathrm{Hom}_{A\times B}(X \times Y,\, X' \times Y') \to \mathrm{Hom}_R((X, Y)_{\alpha,\beta},\ \varPhi(X' \times Y'))$$

such that

$$\theta(f,\, g) = ((f,\, \mathrm{Hom}(M,\, g)\overline{\alpha}),\, \mathrm{Hom}(N,\, f)\overline{\beta},\, g))$$

is a morphism of additive group.

Now, let

$$\rho\colon \mathrm{Hom}_R((X,\, Y)_{\alpha,\beta},\ \varPhi(X' \times Y')) \to \mathrm{Hom}_{A\times B}(X \times Y,\, X' \times Y')$$

such that

$$\rho(f,\, g) = (\pi_{X'} f,\, \pi_{Y'} g),$$

where

$$\pi_{X'}\colon X' \oplus \mathrm{Hom}_B(M,\, Y') \to X', \qquad \pi_{Y'}\colon Y' \oplus \mathrm{Hom}_A(N,\, X') \to Y'$$

are projections. So we have

$$\rho\theta = 1, \qquad \theta\rho = 1.$$

Now we check $\theta\rho = 1$ directly.

For all $(f,\, g) \in \mathrm{Hom}_R((X,\, Y)_{\alpha,\beta},\ \varPhi(X' \times Y'))$,

$$\theta\rho(f,\, g) = \theta(\pi_{X'} f,\, \pi_{Y'} g),$$

by the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad 1\quad} & \mathrm{Hom}_B(M, Y) \\ \Big\downarrow{\scriptstyle \pi_{\mathrm{Hom}_B(M,Y')} f} & & \Big\downarrow{\scriptstyle \mathrm{Hom}_B(M, \pi_{Y'} g)} \\ \mathrm{Hom}_B(M, Y') & \xrightarrow{\quad 1\quad} & \mathrm{Hom}_B(M, Y') \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{\quad 1\quad} & \mathrm{Hom}_A(N, X) \\ \Big\downarrow{\scriptstyle \pi_{\mathrm{Hom}_A(N,X')} g} & & \Big\downarrow{\scriptstyle \mathrm{Hom}_A(N, \pi_{X'} f)} \\ \mathrm{Hom}_A(N, X') & \xrightarrow{\quad 1\quad} & \mathrm{Hom}_A(N, X') \end{array}$$

we have

$$\theta(\pi_{X'} f,\, \pi_{Y'} g) = (f,\, g).$$

Since the right adjoint functor admits a left exact functor, so $\varPhi$ is a left exact functor.

Proposition 2.1 *Let*

$$f=\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} : X\oplus \mathrm{Hom}_B(M,\,Y)\to X'\oplus \mathrm{Hom}_B(M,\,Y'),$$

$$g=\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} : Y\oplus \mathrm{Hom}_A(N,\,X)\to Y'\oplus \mathrm{Hom}_A(N,\,X').$$

Then,

$$(f,\,g)\in \mathrm{Hom}_R(\varPhi(X\times Y)_{\varepsilon,\delta},\ \varPhi(X\times Y)_{\varepsilon',\,\delta'})$$

if and only if

$$\begin{aligned} \overline{\varepsilon_1'}f_{11} &= \mathrm{Hom}_B(M,\,g_{22})\overline{\varepsilon_1}, & f_{22} &= \mathrm{Hom}_B(M,\,g_{11}),\\ \overline{\varepsilon_1'}f_{21} &= \mathrm{Hom}_B(M,\,g_{12}), & f_{12} &= \mathrm{Hom}_B(M,\,g_{21})\overline{\varepsilon_1'},\\ \overline{\delta_1'}g_{11} &= \mathrm{Hom}_A(N,\,f_{22})\overline{\delta_1}, & g_{22} &= \mathrm{Hom}_A(N,\,f_{11}),\\ \overline{\delta_1'}g_{21} &= \mathrm{Hom}_A(N,\,f_{12}), & g_{12} &= \mathrm{Hom}_A(N,\,f_{21})\overline{\delta_1'}, \end{aligned}$$

where

$$\begin{aligned} &f_{11}\colon X_A\to X', && f_{12}\colon X_A\to \mathrm{Hom}_B(M,\,Y'),\\ &f_{21}\colon \mathrm{Hom}_B(M,\,Y)\to X', && f_{22}\colon \mathrm{Hom}_B(M,\,Y)\to \mathrm{Hom}_B(M,\,Y'),\\ &g_{11}\colon Y_B\to Y', && g_{12}\colon Y_B\to \mathrm{Hom}_A(N,\,X'),\\ &g_{21}\colon \mathrm{Hom}_A(N,\,X')\to X', && g_{22}\colon \mathrm{Hom}_A(N,\,X)\to \mathrm{Hom}_A(N,\,X'), \end{aligned}$$

$$\begin{aligned} &\varepsilon_1 = X\otimes_A M\to \mathrm{Hom}_A(N,\,X)\colon \varepsilon_1(x\otimes m)(n)=x(m,\,n),\\ &\varepsilon_1' = X'\otimes_A M\to \mathrm{Hom}_A(N,\,X')\colon \varepsilon_1'(x\otimes m)(n)=x(m,\,n),\\ &\delta_1 = Y\otimes_B N\to \mathrm{Hom}_B(M,\,Y)\colon \delta_1(y\otimes n)(m)=y[n,\,m],\\ &\delta_1' = Y'\otimes_B N\to \mathrm{Hom}_B(M,\,Y')\colon \delta_1'(y\otimes n)(m)=y[n,\,m]. \end{aligned}$$

Proof. Put

$$\begin{aligned} &\varepsilon_2\colon \mathrm{Hom}_B(M,\,Y)\otimes_A M\to Y\colon \varepsilon_2(u\otimes m)=u(m),\\ &\varepsilon_2'\colon \mathrm{Hom}_B(M,\,Y')\otimes_A M\to Y'\colon \varepsilon_2'(u\otimes m)=u(m). \end{aligned}$$

$(f,\,g)\in \mathrm{Hom}_R(\varPhi(X\times Y)_{\varepsilon,\delta},\ \varPhi(X'\times Y')_{\varepsilon',\delta'})$ if and only if there exist the following commutative diagrams:

$$\begin{array}{ccc} X\otimes_A M & \xrightarrow{\ \varepsilon_1\ } & \mathrm{Hom}_A(N,X) \\ \big\downarrow{\scriptstyle f_{11}\otimes 1_M} & & \big\downarrow{\scriptstyle g_{22}} \\ X'\otimes_A M & \xrightarrow{\ \varepsilon_1'\ } & \mathrm{Hom}_A(N,X') \end{array} \iff \begin{array}{ccc} X & \xrightarrow{\ \overline{\varepsilon_1}\ } & \mathrm{Hom}_B(M,\mathrm{Hom}_A(N,X)) \\ {\scriptstyle f_{11}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Hom}(M,g_{22})} \\ X' & \xrightarrow{\ \overline{\varepsilon_1'}\ } & \mathrm{Hom}_B(M,\mathrm{Hom}_A(N,X')) \end{array}$$

$$
\begin{array}{ccc}
\mathrm{Hom}_B(M,Y)\otimes_A M & \xrightarrow{\varepsilon_2} & Y \\
{\scriptstyle f_{22}\otimes 1_M}\downarrow & & \downarrow{\scriptstyle g_{11}} \\
\mathrm{Hom}_B(M,Y')\otimes_A M & \xrightarrow{\varepsilon_2'} & Y'
\end{array}
\iff
\begin{array}{ccc}
\mathrm{Hom}_B(M,Y) & \xrightarrow{1} & \mathrm{Hom}_B(M,Y) \\
{\scriptstyle f_{22}}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}(M,g_{11})} \\
\mathrm{Hom}_B(M,Y') & \xrightarrow{1} & \mathrm{Hom}_B(M,Y')
\end{array}
$$

$$
\begin{array}{ccc}
X\otimes_A M & \xrightarrow{\varepsilon_1} & \mathrm{Hom}_A(N,X) \\
{\scriptstyle f_{12}\otimes 1_M}\downarrow & & \downarrow{\scriptstyle g_{21}} \\
\mathrm{Hom}_B(M,Y')\otimes_A M & \xrightarrow{\varepsilon_2'} & Y'
\end{array}
\iff
\begin{array}{ccc}
X & \xrightarrow{\overline{\varepsilon_1}} & \mathrm{Hom}_B(M,\mathrm{Hom}_A(N,X)) \\
{\scriptstyle f_{12}}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}(M,g_{21})} \\
\mathrm{Hom}_B(M,Y') & \xrightarrow{1} & \mathrm{Hom}_B(M,Y')
\end{array}
$$

$$
\begin{array}{ccc}
\mathrm{Hom}_B(M,Y)\otimes_A M & \xrightarrow{\varepsilon_2} & Y \\
{\scriptstyle f_{21}\otimes 1_M}\downarrow & & \downarrow{\scriptstyle g_{12}} \\
X'\otimes_A M & \xrightarrow{\varepsilon_1'} & \mathrm{Hom}_A(N,X')
\end{array}
\iff
\begin{array}{ccc}
\mathrm{Hom}_B(M,Y) & \xrightarrow{1} & \mathrm{Hom}_B(M,Y) \\
{\scriptstyle f_{21}}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}(M,g_{12})} \\
X' & \xrightarrow{\overline{\varepsilon_1'}} & \mathrm{Hom}_B(M,\mathrm{Hom}_A(N,X'))
\end{array}
$$

if and only if

$$
\begin{aligned}
&\overline{\varepsilon_1'}f_{11}=\mathrm{Hom}_B(M,\,g_{22})\overline{\varepsilon_1}, \qquad && f_{22}=\mathrm{Hom}_B(M,\,g_{11}),\\
&f_{12}=\mathrm{Hom}_B(M,\,g_{21})\overline{\varepsilon_1}, && \overline{\varepsilon_1'}f_{21}=\mathrm{Hom}_B(M,\,g_{12}).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\overline{\delta_1'}g_{11}=\mathrm{Hom}_A(N,\,f_{22})\overline{\delta_1'}, \qquad && g_{22}=\mathrm{Hom}_A(N,\,f_{11}),\\
&g_{12}=\mathrm{Hom}_A(N,\,f_{21})\overline{\delta_1}, && \overline{\delta_1'}g_{21}=\mathrm{Hom}_A(N,\,f_{12}).
\end{aligned}
$$

The following example shows that Φ is not epic.

Example 2.1 Let $A=k^{2\times 2}$ be an order 2 matrix algebra over field k, $B=k$, $M=k^2$. Put $X=\{(a,b)\mid a,\,b\in k\}$, $Y=k$, $R=\begin{pmatrix} A & M\\ 0 & B\end{pmatrix}$. Assume that $\boldsymbol{e}_1=(1,\,0)^{\mathrm{T}}$, $\boldsymbol{e}_2=(0,\,1)^{\mathrm{T}}\in M$, f_1, f_2 are the corresponding dual basis, i.e., $f_i(\boldsymbol{e}_i)=1$, $f_i(\boldsymbol{e}_j)=0$ ($i\neq j$; $i,j=1,2$) and $f_1,\,f_2\in\mathrm{Hom}_k(M,\,k)$. Let $f_{11}=1_X$, $f_{21}=\langle f_i\mapsto \boldsymbol{e}_i,\ i=1,2\rangle$, $f_{12}=0$, $f_{22}=1_{\mathrm{Hom}_k(M,k)}$, $g_{11}=1_Y$, $g_{12}=g_{21}=g_{22}=0$. So

$$f=\begin{pmatrix} f_{11} & f_{12}\\ f_{21} & f_{22}\end{pmatrix}: X\oplus \mathrm{Hom}_k(M,\,k)\to X\oplus \mathrm{Hom}_k(M,\,k),$$
$$g=\begin{pmatrix} g_{11} & g_{12}\\ g_{21} & g_{22}\end{pmatrix}\approx g_{11}\colon Y\to Y.$$

Then, $(f,\,g)\in \mathrm{End}_R(\varPhi(X\times Y))$.

However, since $f_{21}\neq 0$, so there is no $(f',\,g')\in \mathrm{Hom}_{A\times k}(\varPhi(X\times Y),\,\varPhi(X\times Y))$ such that $(f,\,g)=\varPhi(f',\,g')$.

Corollary 2.1 *The functor* $\varPhi\colon \text{mod-}A\times B\to \text{mod-}R$ *maps injective modules into injective modules.*

Proof. By the property of adjoint pair $(i,\,\varPhi)$, see [12], p406.

Muller[6] proved that all the injective right R-modules are precisely the R-modules $\varPhi'(I\times J)$, where I is the injective right A-module, J is the injective right B-module. Particularly, all the indecomposable injective right R-modules are $\varPhi(I\times 0)=(I,\,\mathrm{Hom}_A(N,\,I))_{\varepsilon,\delta}$ or $\varPhi(0\times J)=(\mathrm{Hom}_B(M,\,J),\,J)_{\gamma,\lambda}$, where I is the indecomposable injective A-module, J is the indecomposable injective B-module,

$$\varepsilon\colon I\otimes_A M\to \mathrm{Hom}_A(N,\,I))\colon \varepsilon(x\otimes m)=x\tau(m\otimes\,\cdot\,),$$
$$\delta\colon \mathrm{Hom}_A(N,\,I)\otimes N\to I\colon \delta(g\otimes n)=g(n),$$
$$\gamma\colon \mathrm{Hom}_{\mathrm{B}}(M,\,J)\otimes_A M\to J\colon \gamma(g\otimes m)=g(m),$$
$$\lambda\colon J\otimes_B N\to \mathrm{Hom}_B(M,\,J))\colon \lambda(x\otimes n)=x\mu(n\otimes\,\cdot\,).$$

3 Dual Functor D and the Main Results

Let $R=\begin{pmatrix} A & M\\ N & B\end{pmatrix}$, $D(\,\cdot\,)=\mathrm{Hom}_k(\,\cdot\,,\,k)$ be a usual k-dual functor, where k is a field.

Lemma 3.1 *If* $(X,\,Y)_{\alpha,\beta}\in \text{mod-}R$, *then there exist isomorphisms*

$$\varepsilon\colon M\otimes_B D(Y)\cong D\mathrm{Hom}_B(M,\,Y)$$

and

$$\delta\colon N\otimes_A D(X)\cong D\mathrm{Hom}_A(N,\,X)$$

such that

$$D(X,\,Y)_{\alpha,\beta}=(DX,\,DY)_{D(\overline{\beta})\delta,\,D(\overline{\alpha})\varepsilon}\in \text{mod-}R,$$

where

$$D(\overline{\alpha})\varepsilon\colon M\otimes DY\to DX \qquad s.t.\ \ D(\overline{\alpha})\varepsilon(m\otimes h)(x)=h(\alpha(x\otimes m)),$$
$$D(\overline{\beta})\delta\colon N\otimes DX\to DY \qquad s.t.\ \ D(\overline{\beta})\delta(n\otimes h)(y)=h(\beta(y\otimes n)).$$

Proof. Let $\varepsilon\colon M\otimes_B D(Y)\to D\mathrm{Hom}_B(M,\,Y)$ such that

$$\varepsilon(m\otimes h)(\lambda)=h(\lambda(m)),\qquad m\in M,\ h\in DY,\ \lambda\in \mathrm{Hom}_B(M,\,Y).$$

Then ε is a left A-module isomorphism. In fact, if $Y = D({}_BB)$, then $DY = B$ is a left B-module. Thus the isomorphisms are hold. Since $D({}_BB)$ is a injective co-generator over the right B-module category, there exist the natural number m, n such that

$$0 \to Y \to D(B)^m \to D(B)^n.$$

Applying the right exact functors $M \otimes_B D(\,\cdot\,)$ and $D\mathrm{Hom}_B(M,\,\cdot\,)$ respectively, we can get the following commutative diagram

$$\begin{array}{ccccccc}
M \otimes_B D(DB^n) & \longrightarrow & M \otimes_B D(DB^m) & \longrightarrow & M \otimes_B DY & \longrightarrow & 0 \\
\| & & \| & & \downarrow & & \\
D\mathrm{Hom}_B(M, DB^n) & \longrightarrow & D\mathrm{Hom}_B(M, DB^m) & \longrightarrow & D\mathrm{Hom}_B(M, Y) & \longrightarrow & 0
\end{array}$$

Therefore,

$$\varepsilon\colon M \otimes_B D(Y) \cong D\mathrm{Hom}_B(M,\, Y).$$

Similarly,

$$\delta\colon N \otimes_A D(X) \cong D\mathrm{Hom}_A(N,\, X)\colon \delta(n \otimes h)(\lambda) = h(\lambda(n)).$$

For $\alpha\colon X \otimes M \to Y$, $\beta\colon Y \otimes N \to X$, we have

$$\overline{\alpha}\colon X \to \mathrm{Hom}_B(M,\, Y), \qquad \overline{\beta}\colon Y \to \mathrm{Hom}_A(N,\, X).$$

So

$$D(\overline{\alpha})\colon D\mathrm{Hom}_B(M,\, Y) \to DX\colon D(\overline{\alpha})(\lambda)(x) = \lambda(\overline{\alpha}(x)),$$
$$D(\overline{\beta})\colon D\mathrm{Hom}_A(N,\, X) \to DY\colon D(\overline{\beta})(\lambda)(y) = \lambda(\overline{\beta}(y)).$$

Thus,

$$D(\overline{\beta})\delta\colon N \otimes DX \to DY\colon D(\overline{\beta})\delta(n \otimes h)(y) = h(\beta(y \otimes n)),$$
$$D(\overline{\alpha})\varepsilon\colon M \otimes DY \to DX\colon D(\overline{\alpha})\varepsilon(m \otimes h)(x) = h(\alpha(x \otimes m)).$$

So there exists a functor $\overline{D}\colon \mathcal{A}(R) \to \mathcal{B}(R)$ such that

$$\overline{D}\colon (X,\, Y)_{\alpha,\beta} \to (DX,\, DY)_{D(\overline{\beta})\delta,\, D(\overline{\alpha})\varepsilon}.$$

Lemma 3.2 (1) *Let $(X,\, Y)_{\alpha,\beta} \in \mathcal{A}(R)$, $f\colon X \to X'$ be a right A-module isomorphism, $g\colon Y \to Y'$ be a right B-module isomorphism. Then there exists*

$$(X,\, Y)_{\alpha,\beta} \cong (X',\, Y')_{g\alpha(f^{-1}\otimes 1_M),\, f\beta(g^{-1}\otimes 1_N)}$$

in $\mathcal{B}(R)$.

(2) *Let $(X,\, Y)_{\alpha,\beta} \in \mathcal{B}(R)$, $f\colon X \to X'$ be a left A-module isomorphism, $g\colon Y \to Y'$ be a left B-module isomorphism. Then there exists*

$$(X,\, Y)_{\alpha,\beta} \cong (X',\, Y')_{g\alpha(1_N\otimes f^{-1}),\, f\beta(1_N\otimes g^{-1})}$$

in $\mathcal{B}(R)$.

Proof. Check it directly.

Lemma 3.3 *If $X \in$ mod-A, $Y \in$ mod-B, then*

$$D\Phi(X \times Y) \cong (DX \oplus M \otimes_B DY,\, DY \oplus N \otimes_A DX)_{p,q},$$

where

$$p\colon N \otimes (DX \oplus M \otimes_B DY) \to DY \oplus N \otimes_A DX \colon n \otimes (u,\, m \otimes v) \mapsto ([n,\, m]v,\, n \otimes u),$$

$$q\colon M \otimes (DY \oplus N \otimes_A DX) \to DX \oplus M \otimes_B DY \colon m \otimes (v,\, n \otimes u) \mapsto ((m,\, n)u,\, m \otimes v).$$

Proof. By Lemma 3.1, assume

$$\mu\colon M \otimes_B DY \cong D\mathrm{Hom}_B(M,\, Y)\colon \mu(m \otimes h)(\lambda) = h(\lambda(m)),$$

$$\omega\colon N \otimes_A DX \cong D\mathrm{Hom}_A(N,\, X)\colon \omega(n \otimes h)(\lambda) = h(\lambda(n)).$$

So

$$\begin{aligned} D\Phi(X \times Y) &= D(X \oplus \mathrm{Hom}_B(M,\, Y),\, Y \oplus \mathrm{Hom}_A(N,\, X))_{\alpha,\beta} \\ &\cong (DX \oplus D\mathrm{Hom}_B(M,\, Y),\, DY \oplus D\mathrm{Hom}_A(N,\, X))_{D(\overline{\beta})\delta,D(\overline{\alpha})\varepsilon} \\ &\cong (DX \oplus M \otimes_B DY,\, DY \oplus N \otimes_A DX)_{p,q}, \end{aligned}$$

where

$$\delta\colon N \otimes_A D(X \oplus \mathrm{Hom}_B(M,\, Y)) \to D\mathrm{Hom}_A(N,\, X \oplus \mathrm{Hom}_B(M,\, Y)),$$

$$\varepsilon\colon M \otimes_B D(Y \oplus \mathrm{Hom}_A(N,\, X)) \to D\mathrm{Hom}_B(M,\, Y \oplus \mathrm{Hom}_A(N,\, X)),$$

$$p = (1_{DY},\, \omega^{-1})D(\overline{\beta})\delta(1_N \otimes (1_{DX},\, \mu)),$$

$$q = (1_{DX},\, \mu^{-1})D(\overline{\alpha})\varepsilon(1_M \otimes (1_{DY},\, \omega)).$$

Now, we only check

$$p\colon N \otimes_A (DX \oplus M \otimes DY)) \to DY \oplus N \otimes DX \colon n \otimes (u,\, m \otimes v) \mapsto ([n,\, m]v,\, n \otimes u).$$

Similarly, we can get

$$q\colon M \otimes_B (DY \oplus N \otimes DX)) \to DX \oplus M \otimes DY \colon m \otimes (v,\, n \otimes u) \mapsto ((m,\, n)u,\, m \otimes v).$$

Since $(1_{DY},\, \omega)\colon DY \oplus N \otimes_A DX \cong DY \oplus D\mathrm{Hom}_A(N,\, X)$, we have

$$(1_{DY},\, \omega)([n,\, m]v,\, n \otimes u) = ([n,\, m]v,\, \omega(n \otimes u)),$$

where $n \in N$, $m \in M$, $v \in DY$, $u \in \mathrm{Hom}_A(N, X)$, while

$$D(\overline{\beta})\delta(1_N \otimes (1_{DX},\, \mu))\colon N \otimes (DX \oplus M \otimes DY) \to DY \oplus D\mathrm{Hom}_A(N,\, X),$$

one has

$$D(\overline{\beta})\delta(1_N \otimes (1_{DX},\, \mu))(n \otimes (u,\, m \otimes v)) = \delta(n \otimes (u,\, \mu(m \otimes v)))\overline{\beta}.$$

So it is just to prove

$$\delta(n \otimes (u,\, \mu(m \otimes v)))\overline{\beta}(y,\, f) = (\omega(n \otimes u),\, [n,\, m]v)(y,\, f), \qquad (y,\, f) \in Y \oplus \mathrm{Hom}_A(N,\, X).$$

But

$$\beta((y,\, f) \otimes n) = (y[n,\, \cdot],\, f(n)),$$

so

$$\begin{aligned} \delta(n \otimes (u,\, \mu(m \otimes v)))\overline{\beta}(y,\, f) &= (u,\, \mu(m \otimes v))\overline{\beta}(y,\, f)(n) \\ &= (u,\, \mu(m \otimes v)(f(n),\, y[n,\, \cdot])) \\ &= (u(f(n)),\, v(y[n,\, m])) \\ &= (u(f(n)),\, [n,\, m]v(y)). \end{aligned}$$

On the other hand,

$$(\omega(n \otimes u),\, [n,\, m]v)(y,\, f) = (u(f(n)),\, [n,\, m]v(y)),$$

thus p is proved.

According to the symbols in Section 1, we have

$$\Phi(A \times k) = R$$

and

$$\begin{aligned} D\mathrm{Hom}_R((X,\, Y)_{\alpha,\beta}, R) &\cong D\mathrm{Hom}_R((X,\, Y)_{\alpha,\beta},\ \Phi(A \times k)) \\ &= D\mathrm{Hom}_A(X,\, A) \times D\mathrm{Hom}_k(Y,\, k) \\ &\cong D\mathrm{Hom}_A(X,\, A) \times Y. \end{aligned}$$

The following theorem is a main result of this paper.

Theorem 3.1 *A generalized matrix algebra R over a Frobenius algebra is also a Frobenius algebra.*

Proof. We only need to check $\mathrm{Hom}_k({}_kR,\, k) \cong {}_RR_k$. By Lemma 3.3,

$$D\Phi(A \times k) \cong (DA \oplus {}_AA, k \oplus {}_kA \otimes {}_ADA) \cong (A \oplus {}_AA_k, k \oplus {}_kA_A) = {}_RR_k.$$

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