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Singly Covered Minimal Elements of Linked Partitions and Cycles of Permutations

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Abstract: Linked partitions were introduced by Dykema (Dykema K J. Multilinear function series and transforms in free probability theory. *Adv. Math.*, 2005, 208(1): 351–407) in the study of the unsymmetrized T-transform in free probability theory. Permutation is one of the most classical combinatorial structures. According to the linear representation of linked partitions, Chen *et al.* (Chen W Y C, Wu S Y J, Yan C H. Linked partitions and linked cycles. *European J. Combin.*, 2008, 29(6): 1408–1426) defined the concept of singly covered minimal elements. Let $L(n, k)$ denote the set of linked partitions of $[n]$ with k singly covered minimal elements and let $P(n, k)$ denote the set of permutations of $[n]$ with k cycles. In this paper, we mainly establish two bijections between $L(n, k)$ and $P(n, k)$. The two bijections from a different perspective show the one-to-one correspondence between the singly covered minimal elements in $L(n, k)$ and the cycles in $P(n, k)$.

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1 Introduction

Linked partitions were introduced by Dykema[1] in the study of the unsymmetrized T-transform in free probability theory. Let $[n] = \{1, 2, \cdots, n\}$. A linked partition (see [2]) of $[n]$, is a collection of nonempty subsets $B_1, B_2, \cdots, B_k$ of $[n]$, called blocks, such that the union of $B_1, B_2, \cdots, B_k$ is $[n]$ and any two distinct blocks are nearly disjoint. Two blocks B_i and B_j are said to be nearly disjoint if for any $k \in B_i \cap B_j$, one of the following conditions holds:

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(1) $k = \min\{B_i\}$, $|B_i| > 1$ and $k \neq \min\{B_j\}$, or

(2) $k = \min\{B_j\}$, $|B_j| > 1$ and $k \neq \min\{B_i\}$.

The linear representation of a linked partition was introduced by Chen *et al.*[2]. For a linked partition τ of $[n]$, list n vertices in a horizonal line with labels $1, 2, \cdots, n$. For each block $B = \{i_1, i_2, \cdots, i_k\}$ with $k \geq 2$ and $i_1 < i_2 < \cdots < i_k$, we draw an arc from i_1 to i_j for any $j = 2, \cdots, k$. For $i < j$, we use a pair (i, j) to denote an arc from i_1 to i_j, and we call i and j the left-hand endpoint and the right-hand endpoint of (i, j), respectively. Chen *et al.*[3] introduced a classification of vertices in the linear representation of a linked partition as follows: For any vertex i, $1 \leqslant i \leqslant n$,

1. If i is only a left-hand endpoint, then we call it an origin;
2. If i is both a left-hand point and a right-hand point, then we call it a transient;
3. If i is isolated, then it is called a singleton;
4. If i is only a right-hand endpoint, then it is called a destination.

Fig. 1.1 illustrates a linked partition of [10], where the element 1 is an origin, 2, 3, 5, 6 and 8 are transients, 7 is a singleton and 4, 9 and 10 are destinations.

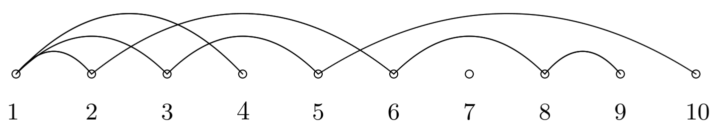

Fig. 1.1 The linear representation of $\tau = \{1, 2, 3, 4\}\{2, 6\}\{3, 5\}\{5, 10\}\{6, 8\}\{7\}\{8, 9\}$

The origins, singletons, and destinations are called singly covered elements, where origins and singletons are also called singly covered minimal elements. The transients are called doubly covered elements. The enumeration of permutations and linked partitions are both $n!$. Corteel[4] gave a graphical representation of permutations, in which there are interesting structures of crossings and nestings. Thanatipanonda[5] described the conceptions of inversions and major index of permutations. Stanley[6] introduced a variety of statistics of permutations, including cycle structures, descents, etc. He also introduced the relationship of cycle structures and Stirling number of the first kind. Define $c(n, k)$ to be the number of permutations with exactly k cycles, and $c(n, k)$ is called a signless Stirling number of the first kind.

In the following sections, we discuss the structural properties of linked partitions and permutations. Specifically, we present two different bijections between linked partitions and permutations such that the one-to-one correspondence between the singly covered minimal elements of linked partitions and the cycles of permutations is evident.

2 Bijection I

In this section, we exhibit our first bijection between linked partitions of n with k singly covered minimal elements and permutations of n with k cycles, which implies some interesting properties between connectivity of linked partitions and the cycles of permutations. Denote the set of linked partitions of n by $L(n)$ and denote the set of linked partitions of

$[n]$ with k singly covered minimal elements by $L(n, k)$, for $1 \leqslant k \leqslant n$. Let $P(n)$ denote the set of permutations of $[n]$. Let

$$\pi = \pi_1\pi_2 \cdots \pi_n \in P(n).$$

If we treat permutation π as a bijection $\pi\colon [n] \to [n]$, then for each $x \in [n]$, we consider the sequence x, $\pi(x)$, $\pi(x)^2$, $\cdots$. Then we can be sure it will definitely go back to x. We call $(x, \pi(x), \pi(x)^2, \cdots, \pi(x)^{l-1})$ a cycle with the length l of π. Let $P(n, k)$ denote the set of permutations of $[n]$ with k cycles. For example, if $\pi\colon [8] \to [8]$ is defined by

$$\pi(1) = 2,\ \pi(2) = 8,\ \pi(3) = 3,\ \pi(4) = 1,$$
$$\pi(5) = 7,\ \pi(6) = 5,\ \pi(7) = 6,\ \pi(8) = 4,$$

then

$$\pi = (2841)(3)(765) \in P(8, 3).$$

Here we adopt the standard representation of the cycle structure such that the element of π in each cycle are arranged in an ascending order, and all of the cycles in π are ordered by their minimal element from left to right. The above permutation of [8] should be written as $\pi = (1284)(3)(576)$.

Theorem 2.1 *For $n \geqslant 1$ and $1 \leqslant k \leqslant n$, there is a bijection ϕ between $L(n)$ and $P(n)$.*

Proof. We conduct induction on n. If n=1, that is,

$$\tau = \{\{1\}\} \in L(1),$$

then we set

$$\pi = \phi(\tau) = 1 = (1).$$

We now assume that for $n \geq 2$, the theorem holds for $m \leq n-1$. While $m = n$, let $\tau \in L(n)$ and let $\tau' \in L(n-1)$ be the linked partition obtained by removing n from τ (and the possible arcs connected with n). By induction hypothesis, $\phi(\tau') \in P(n-1)$ is constructed, and we just need to insert n into $\phi(\tau')$. If n is a singleton of τ, then we set

$$\phi(\tau) = \phi(\tau')n.$$

Otherwise, if there is an arc between n and some vertex i, $1 \leqslant i < n$, then we construct $\phi(\tau)$ by inserting n at the end of $\phi(\tau')$ and then interchanging n with i. Obviously, $\phi(\tau)$ in both cases are a permutation of $[n]$. On the other hand, let

$$\pi = \pi_1\pi_2 \cdots \pi_n \in P(n).$$

We shall construct $\phi(\pi)^{-1}$ by describing its liner representation. First, we draw n vertices labeled 1, 2, $\cdots$, n in a horizontal line. Then if $\pi_n = n$, then we set n as a singleton. Otherwise, if $\pi_n = i < n$, then we draw an arc connecting n with i. Note that there must be some $\pi_j = n$, for $1 \leqslant j \leqslant n-1$. We remove $\pi_j = n$ from π and let $\pi_j = i$, which results in a permutation of $[n-1]$. Repeat the above procedure on $n-1$, $n-2$, $\cdots$, 2, 1, respectively. Note that for any vertex j, there is only one vertex $i < j$ such that i connects with j. That is to say $\phi(\pi)^{-1}$ is exactly a linked partition of $[n]$. Above all the map ϕ is a bijection between $L(n)$ and $P(n)$. The proof is completed.

For example, the construction process of ϕ for the linked partition in Fig. 1.1 is as follows.

$$
\begin{aligned}
& 1 \\
\Longleftrightarrow\ & 2,\ 1 \\
\Longleftrightarrow\ & 2,\ 3,\ 1 \\
\Longleftrightarrow\ & 2,\ 3,\ 4,\ 1 \\
\Longleftrightarrow\ & 2,\ 5,\ 4,\ 1,\ 3 \\
\Longleftrightarrow\ & 6,\ 5,\ 4,\ 1,\ 3,\ 2 \\
\Longleftrightarrow\ & 6,\ 5,\ 4,\ 1,\ 3,\ 2,\ 7 \\
\Longleftrightarrow\ & 8,\ 5,\ 4,\ 1,\ 3,\ 2,\ 7,\ 6 \\
\Longleftrightarrow\ & 9,\ 5,\ 4,\ 1,\ 3,\ 2,\ 7,\ 6,\ 8 \\
\Longleftrightarrow\ & 9,\ 10,\ 4,\ 1,\ 3,\ 2,\ 7,\ 6,\ 8,\ 5
\end{aligned}
$$

For any linked partition τ, a connected component of τ is a maximal connected subgraph in the linear representation of τ. A connected linked partition has only one connected component that is itself, while a disconnected linked partition has at least two connected components. As an example, Fig. 1.1 is a disconnected linked partition containing 3 connected components.

Corollary 2.1 *For $n \geqslant 1$, let $\tau \in L(n)$. Then the elements in the same connected component of τ must be in the same cycle of $\phi(\tau)$.*

Proof. Suppose that τ is a connected linked partition of $[n]$. We only need to prove that $\phi(\tau)$ has only one cycle. We adopt induction on n. If $n = 1$, that is, $\tau = \{\{1\}\}$, then

$$\pi = \phi(\tau) = 1 = (1).$$

If $n = 2$, that is, $\tau = \{\{1, 2\}\}$, then

$$\pi = \phi(\tau) = 21 = (12).$$

So the theorem holds for $n = 1$ and $n = 2$.

For $n > 2$, we assume that the theorem holds for $m \leqslant n - 1$. Let $\tau' \in L(n-1)$ be the connected linked partition of $[n-1]$ formed by removing n from τ and the arc (i, n). By induction hypothesis, $\phi(\tau')$ is a permutation of $[n-1]$ which consists of one cycle. Let $\pi' = \phi(\tau') \in p(n-1)$ and $\pi = \phi(\tau) \in p(n)$. Denote $\pi'(t) = i$ and $\pi'(i) = t'$, where i connects to n in τ. It is to say that the unique cycle of $\pi' = \phi(\tau')$ is as $(\cdots, t, i, t', \cdots)$. According to the definition of bijection ϕ, we see that when inserting n into $\phi'(\tau')$, we only need to set

$$\pi(t) = n, \qquad \pi(j) = \pi'(j)$$

for any $j \in [n-1]$ but $j \neq t$, and set $\pi(n) = i$. Then $\pi = \phi(\tau)$ only consists of one cycle $(\cdots, t, n, i, t', \cdots)$ and both n and i are in the same cycle of $\pi = \phi(\tau)$. The proof is completed.

For example, Fig. 1.1 is a disconnected linked partition containing 3 connected components and its corresponding permutation is (10, 5, 3, 4, 1)(7)(9, 8, 6, 2) of three cycles.

Notice that for any $\tau \in L(n)$ the number of components of τ is equal to the total number of origins and singletons of τ. Combined with Corollary 2.1, we deduce our main results:

Theorem 2.2 *The bijection ϕ also shows a one-to-one correspondence between $L(n, k)$ and $P(n, k)$.*

Consequently, we have

Theorem 2.3 *For $n \geqslant 1$ and $1 \leqslant k \leqslant n$, $\#L(n,k) = \#P(n,k)$. That is, the number of linked partitions of $[n]$ with k singly covered minimal elements is equal to the number of permutations of $[n]$ with k cycles.*

For example see Fig. 2.2.

$L(3,k)$	$\{1,2,3\}$ 1 2 3	$\{1,2\}\{2,3\}$ 1 2 3	$\{1,3\}\{2\}$ 1 2 3	$\{1,2\}\{3\}$ 1 2 3	$\{1\}\{2,3\}$ 1 2 3	$\{1\}\{2\}\{3\}$ 1 2 3
$P(3,k)$	$231 = (123)$	$312 = (132)$	$321 = (13)(2)$	$213 = (12)(3)$	$132 = (1)(23)$	$123 = (1)(2)(3)$

Fig. 2.2 $L(3, k)$ and the corresponding $P(3, k)$, $1 \leqslant k \leqslant 3$

Moreover, based on Handshaking's lemma, we have

Corollary 2.2 *For any $\tau \in L(n, k)$, the total number of elements interchanging in the construction procedure of ϕ equals twice times the number of arcs of τ.*

3 Bijection II

In this section, we present a bijection between $L(n, k)$ and $P(n, k)$ from a different angle.

Theorem 3.1 *For $n \geqslant 1$ and $1 \leqslant k \leqslant n$, there is a bijection φ between $L(n, k)$ and $P(n, k)$.*

Proof. We prove the theorem by induction on n. When $n = 1$, $\tau = \{\{1\}\} \in L(1, 1)$, we set $\pi = \varphi(\tau) = (1)$, i.e., $\pi \in P(1, 1)$. Then the theorem holds for $n = 1$.

We now assume that $n \geqslant 2$ and the theorem holds for $n - 1$.

Let $\tau \in L(n, k)$ and let τ' denote the linked partition of $[n-1]$ with s singly covered minimal elements obtained from τ' by removing the vertex n along with the arcs possibly associated to n. By induction hypothesis, $\varphi(\tau')$ is a permutation of $[n-1]$ with s cycles, for $1 \leqslant s \leqslant n-1$. We assume that the singly covered minimal elements in τ' are $i_1 < i_2 < \cdots < i_s$. There are three cases.

Case 1. If the vertex n is a singleton in τ, then we set $\varphi(\tau) = \varphi(\tau')(n)$.

Case 2. If there is an arc connecting some i_j to n for $1 \leqslant j \leqslant s$, then we set $\varphi(\tau)$ as the permutation of $[n]$ obtained by inserting n into the rightmost position of the cycle containing i_j in $\varphi(\tau')$.

Case 3. If there is an arc from j to n, where j is a transient or a destination, then we let $\varphi(\tau)$ be the permutation of $[n]$ obtained from inserting n immediately to the left of j in $\varphi(\tau')$.

Clearly, in Case 1, we have $k = s + 1$, and $\varphi(\tau)$ is a permutation of $[n]$ with k cycles. For cases 2 and 3, we have $k = s$ such that $\varphi(\tau)$ is a permutation of $[n]$ with k cycles. That is to say φ is a map from $L(n, k)$ to $P(n, k)$.

To prove φ is a bijection, we describe the inverse map of φ as follows. Let $\pi \in P(n, k)$ be a permutation of $[n]$ with k cycles. Let π' be the permutation of $[n-1]$ obtained from $\pi \in (n, k)$ by removing n. By induction hypothesis, $\varphi^{-1}(\pi)$ has been constructed. Then according to three cases mentioned above. If n generate a cycle solely, then let n be a singleton in $\psi^{-1}(\pi)$. If n is the rightmost element of some cycle C, then there is an arc connecting n to the minimal element of cycle C. Otherwise, if either n is not the rightmost element in cycle l or n does not consist a cycle by itself, then we draw an arc between n and the element immediately to the right of n in cycle C. This completes the proof and ψ is a bijection between $L(n, k)$ and $P(n, k)$.

As an example, we illustrate the construction process of ψ for the linked partition in Fig. 1.1 as follows.

$$
\begin{aligned}
&(1)\\
\Longleftrightarrow\ &(1,2)\\
\Longleftrightarrow\ &(1,2,3)\\
\Longleftrightarrow\ &(1,2,3,4)\\
\Longleftrightarrow\ &(1,2,5,3,4)\\
\Longleftrightarrow\ &(1,6,2,5,3,4)\\
\Longleftrightarrow\ &(1,6,2,5,3,4)(7)\\
\Longleftrightarrow\ &(1,8,6,2,5,3,4)(7)\\
\Longleftrightarrow\ &(1,9,8,6,2,5,3,4)(7)\\
\Longleftrightarrow\ &(1,9,8,6,2,10,5,3,4)(7)
\end{aligned}
$$

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