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Rota-Baxter Operators on 3-dimensional Lie Algebras and Solutions of the Classical Yang-Baxter Equation

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Abstract: In this paper, we compute Rota-Baxter operators on the 3-dimensional Lie algebra g whose derived algebra's dimension is 2. Furthermore, we give the corresponding solutions of the classical Yang-Baxter equation in the 6-dimensional Lie algebras $g \ltimes_{ad^*} g^*$ and some new structures of left-symmetric algebra induced from g and its Rota-Baxter operators.

Key words: Rota-Baxter operators, 3-dimensional Lie algebra, classical Yang-Baxter equation, left-symmetric algebra
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1 Introduction

According to the Winternitz classification (see [1]), there are six kinds of 3-dimensional Lie algebras up to isomorphism over the complex field $\mathbb{C}$. That is,

$$
\begin{aligned}
&g_1\colon [e_1, e_2] = 0,\ [e_1, e_3] = 0,\ [e_2, e_3] = 0,\\
&g_2\colon [e_1, e_2] = 0,\ [e_1, e_3] = e_3,\ [e_2, e_3] = 0,\\
&g_3\colon [e_1, e_2] = e_3,\ [e_1, e_3] = 0,\ [e_2, e_3] = 0,\\
&g_4\colon [e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_2, e_3] = e_1,\\
&g_5\colon [e_1, e_2] = e_1,\ [e_1, e_3] = 0,\ [e_2, e_3] = e_1 + e_3,\\
&g_6\colon [e_1, e_2] = e_1,\ [e_1, e_3] = 0,\ [e_2, e_3] = ke_3 \quad (0 < |k| \leq 1).
\end{aligned}
$$

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We know that g_4 is the famous 3-dimensional simple Lie algebra $sl(2, \mathbb{C})$. The others are nonsimple. In [2], the authors gave all Rota-Baxter operators (of weight zero) on g_4 and the corresponding solutions of the classical Yang-Baxter equation. In [3], Rota-Baxter operators on another 3-dimensional non-simple Lie algebra g_5 were determined, the corresponding solutions of the classical Yang-Baxter equation and some new structures of left symmetric algebra are given. In [4], the authors determine the Rota-Baxter operators on g_2 and g_3. For g_1, it is clear that its Rota-Baxter operators are belong to its endomorphisms. Thus, in order to determine the Rota-Baxter operators on 3-dimensional Lie algebras, we just determine the Rota-Baxter operators on g_6. The aim of this paper is to determine the Rota-Baxter operators (of weight zero) on g_6 and the corresponding solutions of the Yang-Baxter equation. After this, we completely determine all of the Rota-Baxter operators (of weight zero) on all 3-dimensional Lie algebras. From now on, we denote g_6 as g.

A Rota-Baxter operator of weight zero on an associative algebra A is defined to be a linear map $P\colon A \to A$ satisfying

$$P(x)P(y) = P(P(x)y + xP(y)), \qquad x, y \in A. \tag{1.1}$$

Rota-Baxter operators on associative algebras were introduced by G. Baxter to solve an analytic formula in probability (see [5]). It has been related to other areas in Mathematics and Mathematical Physics (see [6]–[9]). A Rota-Baxter operator of weight zero on a Lie algebra $(g, [\,\cdot\,,\,\cdot\,])$ is a linear operator $P\colon g \to g$ such that

$$[P(x),\, P(y)] = P([P(x),\, y] + [x,\, P(y)]), \qquad x, y \in g. \tag{1.2}$$

In fact, a Rota-Baxter operator is also called the operator form of the classical Yang-Baxter equation (see [10] and [11]). Let g be a Lie algebra and

$$r = \sum_i a_i \otimes b_i \in g \otimes g.$$

Then r is called a solution of the classical Yang-Baxter equation (CYBE) in g if

$$[r_{12},\, r_{13}] + [r_{12},\, r_{23}] + [r_{13},\, r_{23}] = 0 \tag{1.3}$$

in $U(g)$, where $U(g)$ is the universal enveloping algebra of g and

$$r_{12} = \sum_i a_i \otimes b_i \otimes 1, \qquad r_{13} = \sum_i a_i \otimes 1 \otimes b_i, \qquad r_{23} = \sum_i 1 \otimes a_i \otimes b_i.$$

Semenov-Tian-Shansky[12] proved that a Rota-Baxter operator of weight 0 on a Lie algebra is exactly the operator form of the classical Yang-Baxter equation (1.3). On the one hand, Rota-Baxter operators of weight 0 on a Lie algebra g give rise to solutions of CYBE on the double Lie algebra $g \ltimes_{ad^*} g^*$ over the direct sum $g \bigoplus g^*$ of the Lie algebra g and its dual space g^* (see [2], [13]). Moreover, some solutions of CYBE in $g \ltimes_{ad^*} g^*$ Lie algebras through Rota-Baxter operators of any weight on g can be obtained (see [3], [14]). On the other hand, some certain interesting algebraic structures, such as left-symmetric algebras, coming out of the Rota-Baxter operators. In this paper, we determine the Rota-Baxter operators on 3-dimensional Lie algebra g and give a family of solutions of CYBE in $g \ltimes_{ad^*} g^*$. Finally, the induced left-symmetric algebraic structures from the Rota-Baxter operator of weight 0 on a Lie algebra g are obtained.

This paper is organized as follows. In Section 2, we give the Rota-Baxter operators (of weight zero) on g. In Section 3, according to Theorem 2.1, we give the corresponding solutions of CYBE in $g \ltimes_{ad^*} g^*$. In Section 4, we give the induced left-symmetric structure from the Rota-Baxter operators of weight 0 on g.

2 The Rota-Baxter Operators on g (of Weight Zero)

The main result of this section is the complete classification of Rota-Baxter operators of weight zero on g. As we will see, the problem of classification turns out to be solving a system of quadratic equations.

2.1 Notations and the Classification Theorem

According to the Winternitz classification (see [1]), let g be the 3-dimensional Lie algebras with a basis e_1, e_2, e_3 over the field of complex numbers $\mathbb{C}$ and the following Lie brackets

$$[e_1, e_2] = e_1, \quad [e_1, e_3] = 0, \quad [e_2, e_3] = ke_3 \qquad (0 < |k| \leq 1).$$

Here we note that the condition $0 < |k| \leq 1$ is not accurate. In fact, it is clear that any two Lie algebras with different k are not isomorphic over the real field, but might be isomorphic over the complex field for some special values of k. For example, the first complex Lie algebra L_1 has a basis $\{e_1, e_2, e_3\}$ with the following brackets

$$[e_1, e_2] = e_1, \quad [e_1, e_3] = 0, \quad [e_2, e_3] = \mathrm{e}^{\mathrm{i}\theta} e_3, \qquad 0 \leq \theta \leq \pi,$$

the second complex Lie algebra L_2 has a basis $\{f_1, f_2, f_3\}$ with the following brackets

$$[f_1, f_2] = f_1, \quad [f_1, f_3] = 0, \quad [f_2, f_3] = \mathrm{e}^{-\mathrm{i}\theta} f_3, \qquad 0 \leq \theta \leq \pi.$$

Let $\varphi\colon L_1 \to L_2$ be a linear transformation determined by

$$e_1 \to f_3, \qquad e_2 \to -\mathrm{e}^{\mathrm{i}\theta} f_2, \qquad e_3 \to f_1.$$

It is easy to check that L_1 is isomorphic to L_2 as complex Lie algebras. The example shows that the Winternitz classification condition for 3-dimensional Lie algebras is not accurate. Thus, we modify the 3-dimensional Lie algebras g as follows:

$$[e_1, e_2] = e_1, \quad [e_1, e_3] = 0, \quad [e_2, e_3] = ke_3 \qquad (0 < |k| < 1 \text{ or } k = \mathrm{e}^{\mathrm{i}\theta},\ 0 \leq \theta \leq \pi). \quad (2.1)$$

Let $P\colon g \to g$ be a linear operator determined by

$$\begin{pmatrix} P(e_1) \\ P(e_2) \\ P(e_3) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix},$$

where $a_{ij} \in \mathbb{C}$, $1 \leq i, j \leq 3$. The following is our main theorem.

Theorem 2.1 *All Rota-Baxter operators of weight zero on g are listed in their matrices form with respect to the basis below when $\theta = \pi$, where a, b, c are non-zero complex numbers*:

$$P_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad P_2 = \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad P_3 = \begin{pmatrix} 0 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$P_4=\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix},\qquad P_5=\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix},\qquad P_6=\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix},$$

$$P_7=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix},\qquad P_8=\begin{pmatrix}0&0&0\\a&b&0\\1&0&0\end{pmatrix},\qquad P_9=\begin{pmatrix}0&0&0\\0&0&0\\0&1&0\end{pmatrix},$$

$$P_{10}=\begin{pmatrix}0&0&0\\0&0&0\\a&1&0\end{pmatrix},\qquad P_{11}=\begin{pmatrix}0&0&0\\a&b&1\\c&0&0\end{pmatrix},\qquad P_{12}=\begin{pmatrix}0&0&0\\0&a&1\\b&0&0\end{pmatrix},$$

$$P_{13}=\begin{pmatrix}0&0&0\\a&0&1\\b&0&0\end{pmatrix},\qquad P_{14}=\begin{pmatrix}0&0&0\\a&b&1\\0&0&0\end{pmatrix},\qquad P_{15}=\begin{pmatrix}0&0&0\\0&0&1\\a&0&0\end{pmatrix},$$

$$P_{16}=\begin{pmatrix}0&0&0\\0&a&1\\0&0&0\end{pmatrix},\qquad P_{17}=\begin{pmatrix}0&0&0\\a&0&1\\0&0&0\end{pmatrix},\qquad P_{18}=\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix},$$

$$P_{19}=\begin{pmatrix}0&0&0\\0&-a&1\\0&-a^2&a\end{pmatrix},\qquad P_{20}=\begin{pmatrix}0&0&0\\-\frac{b}{a}&-a&1\\b&-a^2&a\end{pmatrix},\qquad P_{21}=\begin{pmatrix}0&1&0\\0&0&a\\0&b&0\end{pmatrix},$$

$$P_{22}=\begin{pmatrix}a&1&0\\-a^2&-a&0\\ab&b&0\end{pmatrix},\qquad P_{23}=\begin{pmatrix}a&1&0\\-a^2&-a&b\\0&0&0\end{pmatrix},\qquad P_{24}=\begin{pmatrix}0&1&0\\0&0&0\\0&a&0\end{pmatrix},$$

$$P_{25}=\begin{pmatrix}0&1&0\\0&0&a\\0&0&0\end{pmatrix},\qquad P_{26}=\begin{pmatrix}a&1&0\\-a^2&-a&0\\0&0&0\end{pmatrix},\qquad P_{27}=\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix},$$

$$P_{28}=\begin{pmatrix}a&1&0\\-a^2&-a&b\\ac&c&0\end{pmatrix},\qquad P_{29}=\begin{pmatrix}0&0&1\\0&a&b\\0&0&0\end{pmatrix},\qquad P_{30}=\begin{pmatrix}0&0&1\\a&0&b\\0&0&0\end{pmatrix},$$

$$P_{31}=\begin{pmatrix}0&0&1\\a&b&0\\0&0&0\end{pmatrix},\qquad P_{32}=\begin{pmatrix}0&0&1\\a&0&0\\0&0&0\end{pmatrix},\qquad P_{33}=\begin{pmatrix}0&0&1\\0&a&0\\0&0&0\end{pmatrix},$$

$$P_{34}=\begin{pmatrix}0&0&1\\0&0&a\\0&0&0\end{pmatrix},\qquad P_{35}=\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix},\qquad P_{36}=\begin{pmatrix}0&0&1\\a&b&c\\0&0&0\end{pmatrix},$$

$$P_{37} = \begin{pmatrix} 0 & a & 1 \\ 0 & -b & -\frac{b}{a} \\ 0 & ab & b \end{pmatrix}, \qquad P_{38} = \begin{pmatrix} 0 & a & 1 \\ -\frac{b}{a} & 0 & 0 \\ b & 0 & 0 \end{pmatrix}, \qquad P_{39} = \begin{pmatrix} 0 & a & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$P_{40} = \begin{pmatrix} 0 & a & 1 \\ -\frac{b}{a} & -c & -\frac{c}{a} \\ b & ac & c \end{pmatrix}, \qquad P_{41} = \begin{pmatrix} a & 0 & 1 \\ 0 & b & c \\ -a^2 & 0 & -a \end{pmatrix}, \qquad P_{42} = \begin{pmatrix} a & 0 & 1 \\ b & 0 & c \\ -a^2 & 0 & -a \end{pmatrix},$$

$$P_{43} = \begin{pmatrix} a & 0 & 1 \\ b & c & 0 \\ -a^2 & 0 & -a \end{pmatrix}, \qquad P_{44} = \begin{pmatrix} a & 0 & 1 \\ 0 & 0 & b \\ -a^2 & 0 & -a \end{pmatrix}, \qquad P_{45} = \begin{pmatrix} a & 0 & 1 \\ b & 0 & 0 \\ -a^2 & 0 & -a \end{pmatrix},$$

$$P_{46} = \begin{pmatrix} a & 0 & 1 \\ 0 & b & 0 \\ -a^2 & 0 & -a \end{pmatrix}, \qquad P_{47} = \begin{pmatrix} a & 0 & 1 \\ 0 & 0 & 0 \\ -a^2 & 0 & -a \end{pmatrix}, \qquad P_{48} = \begin{pmatrix} a & 0 & 1 \\ b & c & d \\ -a^2 & 0 & -a \end{pmatrix},$$

$$P_{49} = \begin{pmatrix} a & b & 1 \\ ac & bc & c \\ -a^2 - abc & -ab - b^2c & -a - bc \end{pmatrix}, \qquad P_{50} = \begin{pmatrix} a & b & 1 \\ 0 & 0 & 0 \\ -a^2 & -ab & -a \end{pmatrix}$$

Theorem 2.2 *All Rota-Baxter operators of weight zero on g are listed in their matrices form with respect to the basis below when $0 < |k| < 1$ or $k = e^{i\theta}$, $\theta \neq \pi$, a, b, c are non-zero complex numbers*:

$$P_{51} = \begin{pmatrix} ka & 0 & ka^2 \\ b & 0 & 0 \\ 1 & 0 & a \end{pmatrix}, \qquad P_{52} = \begin{pmatrix} ka & 0 & ka^2 \\ 0 & 0 & b \\ 1 & 0 & a \end{pmatrix}, \qquad P_{53} = \begin{pmatrix} ka & 0 & ka^2 \\ 0 & 0 & 0 \\ 1 & 0 & a \end{pmatrix},$$

$$P_{54} = \begin{pmatrix} ka & 0 & ka^2 \\ b & 0 & c \\ 1 & 0 & a \end{pmatrix}, \qquad P_{55} = \begin{pmatrix} a & 1 & b \\ -a^2 & -a & -ab \\ 0 & 0 & 0 \end{pmatrix}.$$

2.2 Reduction to a System of Quadratic Equations

In fact, we can reduce the problem to be solving a system of quadratic equations. We first need to check on g

$$[P(e_1), P(e_3)] = P([P(e_1), e_3] + [e_1, P(e_3)]),$$
$$[P(e_1), P(e_2)] = P([P(e_1), e_2] + [e_1, P(e_2)]),$$
$$[P(e_2), P(e_3)] = P([P(e_2), e_3] + [e_2, P(e_3)]).$$

It follows from (2.1) that

$$\begin{aligned}[P(e_1), P(e_3)] &= [a_{11}e_1 + a_{12}e_2 + a_{13e_3}, a_{31}e_1 + a_{32}e_2 + a_{33}e_3] \\ &= (a_{11}a_{32} - a_{12}a_{31})e_1 + (ka_{12}a_{33} - ka_{32}a_{13})e_3, \end{aligned} \tag{2.2}$$

while

$$\begin{aligned} & P([P(e_1),\, e_3] + [e_1,\, P(e_3)]) \\ = \; & P(ka_{12}e_3 + a_{32}e_1) \\ = \; & (ka_{12}a_{31} + a_{32}a_{11})e_1 + (ka_{12}a_{32} + a_{32}a_{12})e_2 + (ka_{12}a_{33} + a_{32}a_{13})e_3. \end{aligned} \tag{2.3}$$

Comparing the coefficients in (2.2) and (2.3), we have

$$(k+1)a_{12}a_{31} = 0, \tag{2.4}$$

$$(k+1)a_{12}a_{32} = 0, \tag{2.5}$$

$$(k+1)a_{13}a_{32} = 0. \tag{2.6}$$

Similarly, from

$$[P(e_1),\, P(e_2)] = P([P(e_1),\, e_2] + [e_1,\, P(e_2)]),$$

$$[P(e_2),\, P(e_3)] = P([P(e_2),\, e_3] + [e_2,\, P(e_3)]),$$

we obtain the following six equations:

$$ka_{13}a_{31} - a_{12}a_{21} - a_{11}a_{11} = 0, \tag{2.7}$$

$$ka_{13}a_{32} - a_{12}a_{22} - a_{11}a_{12} = 0, \tag{2.8}$$

$$(k+1)a_{13}a_{22} - ka_{13}a_{33} - ka_{12}a_{23} + a_{11}a_{13} = 0, \tag{2.9}$$

$$(k+1)a_{22}a_{31} + ka_{33}a_{31} - a_{31}a_{11} - a_{21}a_{32} = 0, \tag{2.10}$$

$$ka_{22}a_{32} + ka_{33}a_{32} - a_{31}a_{12} = 0, \tag{2.11}$$

$$ka_{33}a_{33} + ka_{23}a_{32} - a_{31}a_{13} = 0. \tag{2.12}$$

2.3 Solving the Quadratic Equation

In order to solve the quadratic equations (2.4)–(2.12), we consider two cases depending on k.

Case 1. $k = -1$, that is, $\theta = \pi$.

In this case, there are two subcases: $a_{13} = 0$ and $a_{13} \neq 0$.

(A) $a_{13} = 0$.

In this case, (2.9) implies

$$a_{12}a_{23} = 0.$$

(A_1) Assume $a_{12} = 0$, $a_{23} = 0$. It follows from (2.7) and (2.12) that $a_{11} = a_{33} = 0$, (2.10) implies $a_{21}a_{32} = 0$, (2.11) implies $a_{22}a_{32} = 0$.

(A_{11}) If $a_{32} = 0$, then we obtain

$$P = \begin{pmatrix} 0 & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & 0 & 0 \end{pmatrix}.$$

Taking $a_{21} = 0$, $a_{22} = a$ and $a_{31} = 1$. We obtain P_1. Taking $a_{21} = a$, $a_{22} = 0$ and $a_{31} = 1$. We obtain P_2. Taking $a_{21} = a$, $a_{22} = 1$ and $a_{31} = 0$. We obtain P_3. Taking $a_{21} = 0$, $a_{22} = 0$ and $a_{31} = 1$. We obtain P_4. Taking $a_{21} = 0$, $a_{22} = 1$ and $a_{31} = 0$. We obtain P_5. Taking

$a_{21} = 1$, $a_{22} = 0$ and $a_{31} = 0$. We obtain P_6. Taking $a_{21} = 0$, $a_{22} = 0$ and $a_{31} = 0$. We obtain P_7. Taking $a_{21} = a$, $a_{22} = b$ and $a_{31} = 1$. We obtain P_8.

(A_{12}) If $a_{32} \neq 0$, taking $a_{32} = 1$, then (2.10) and (2.11) implies $a_{21} = a_{22} = 0$. We obtain

$$P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_{31} & 1 & 0 \end{pmatrix}.$$

Taking $a_{31} = 0$, we obtain P_9. Taking $a_{31} = a$, we obtain P_{10}.

(A_2) If $a_{12} = 0$, $a_{23} \neq 0$, taking $a_{23} = 1$, then (2.7) implies $a_{11} = 0$, (2.11) implies $(a_{22} + a_{33})a_{32} = 0$.

(A_{21}) If $a_{32} = 0$, then (2.12) implies $a_{33} = 0$. We obtain

$$P = \begin{pmatrix} 0 & 0 & 0 \\ a_{21} & a_{22} & 1 \\ a_{31} & 0 & 0 \end{pmatrix}.$$

Taking $a_{21} = a$, $a_{22} = b$, $a_{31} = c$. We obtain P_{11}. Taking $a_{21} = 0$, $a_{22} = a$, $a_{31} = b$. We obtain P_{12}. Taking $a_{21} = a$, $a_{22} = 0$, $a_{31} = b$. We obtain P_{13}. Taking $a_{21} = a$, $a_{22} = b$, $a_{31} = 0$. We obtain P_{14}. Taking $a_{21} = 0$, $a_{22} = 0$, $a_{31} = a$. We obtain P_{15}. Taking $a_{21} = 0$, $a_{22} = a$, $a_{31} = 0$. We obtain P_{16}. Taking $a_{21} = a$, $a_{22} = 0$, $a_{31} = 0$. We obtain P_{17}. Taking $a_{21} = 0$, $a_{22} = 0$, $a_{31} = 0$. We obtain P_{18}.

(A_{22}) If $a_{32} \neq 0$, then $a_{22} = -a_{33}$, and (2.12) implies $a_{32} = -a_{33}^2 \neq 0$. Taking $a_{33} = a$. Then (2.10) implies

$$a_{21} = \frac{-a_{33}a_{31}}{a_{32}} = \frac{a_{31}}{a}.$$

We obtain

$$P = \begin{pmatrix} 0 & 0 & 0 \\ \frac{a_{31}}{a} & -a & 1 \\ a_{31} & -a^2 & a \end{pmatrix}.$$

Taking $a_{31} = 0$, we obtain P_{19}. Taking $a_{31} = b$, we obtain P_{20}.

(A_3) If $a_{12} \neq 0$, $a_{23} = 0$, taking $a_{12} = 1$, then (2.12) implies $a_{33} = 0$, (2.7) implies $a_{21} = -a_{11}^2$, (2.8) implies $a_{22} = -a_{11}$, (2.11) implies $a_{31} = a_{11}a_{32}$. We obtian

$$P = \begin{pmatrix} a_{11} & 1 & 0 \\ -a_{11}^2 & -a_{11} & a_{23} \\ a_{11}a_{32} & a_{32} & 0 \end{pmatrix}.$$

Taking $a_{11} = 0$, $a_{23} = a$, $a_{32} = b$. We obtain P_{21}. Taking $a_{11} = a$, $a_{23} = 0$, $a_{32} = b$. We obtain P_{22}. Taking $a_{11} = a$, $a_{23} = b$, $a_{32} = 0$. We obtain P_{23}. Taking $a_{11} = 0$, $a_{23} = 0$, $a_{32} = a$. We obtain P_{24}. Taking $a_{11} = 0$, $a_{23} = a$, $a_{32} = 0$. We obtain P_{25}. Taking $a_{11} = a$, $a_{23} = 0$, $a_{32} = 0$. We obtain P_{26}. Taking $a_{11} = 0$, $a_{23} = 0$, $a_{32} = 0$. We obtain P_{27}. Taking $a_{11} = a$, $a_{23} = b$, $a_{32} = c$. We obtain P_{28}.

(B) Assume $a_{13} \neq 0$, taking $a_{13} = 1$.

(B_1) If $a_{11} = 0$, then (2.7) implies $a_{31} = -a_{12}a_{21}$, (2.8) implies $a_{32} = -a_{12}a_{22}$, (2.9) implies $a_{33} = -a_{12}a_{23}$.

(B_{11}) If $a_{12}=0$, then $a_{31}=0$, $a_{32}=0$, $a_{33}=0$. We obtain

$$P=\begin{pmatrix} 0 & 0 & 1\\ a_{21} & a_{22} & a_{23}\\ 0 & 0 & 0 \end{pmatrix}.$$

Taking $a_{21}=0$, $a_{22}=a$, $a_{23}=b$, we obtain P_{29}. Taking $a_{21}=a$, $a_{22}=0$, $a_{23}=b$, we obtain P_{30}. Taking $a_{21}=a$, $a_{22}=b$, $a_{23}=0$, we obtain P_{31}. Taking $a_{21}=a$, $a_{22}=0$, $a_{23}=0$, we obtain P_{32}. Taking $a_{21}=0$, $a_{22}=a$, $a_{23}=0$, we obtain P_{33}. Taking $a_{21}=0$, $a_{22}=0$, $a_{23}=a$, we obtain P_{34}. Taking $a_{21}=0$, $a_{22}=0$, $a_{23}=0$, we obtain P_{35}. Taking $a_{21}=a$, $a_{22}=b$, $a_{23}=c$, we obtain P_{36}.

(B_{12}) If $a_{12}\neq 0$, taking $a_{12}=a$, then (2.7) implies $a_{21}=-\dfrac{a_{31}}{a}$, (2.8) implies $a_{22}=-\dfrac{a_{32}}{a}$, (2.9) implies $a_{23}=-\dfrac{a_{33}}{a}$, (2.12) implies $a_{31}=\dfrac{a_{33}a_{32}}{a}-a_{33}^2$, then (2.10) implies $a_{33}(a_{32}-a_{12}a_{33})^2=0$, (2.11) implies $(a_{32}-aa_{33})^2=0$, then $a_{32}=aa_{33}$. We obtain

$$P=\begin{pmatrix} 0 & a & 1\\ -\dfrac{a_{31}}{a} & -a_{33} & -\dfrac{a_{33}}{a}\\ a_{31} & aa_{33} & a_{33} \end{pmatrix}.$$

Taking $a_{31}=0$, $a_{33}=b$. We obtain P_{37}. Taking $a_{31}=b$, $a_{33}=0$. We obtain P_{38}. Taking $a_{31}=0$, $a_{33}=0$. We obtain P_{39}. Taking $a_{31}=b$, $a_{33}=c$. We obtain P_{40}.

(B_2) If $a_{11}\neq 0$, taking $a_{11}=a$.

(B_{21}) If $a_{12}=0$, then (2.7) implies $a_{31}=-a^2$, (2.8) implies $a_{32}=0$, (2.9) implies $a_{33}=-a$. We obtain

$$P=\begin{pmatrix} a & 0 & 1\\ a_{21} & a_{22} & a_{23}\\ -a^2 & 0 & -a \end{pmatrix}.$$

Taking $a_{21}=0$, $a_{22}=b$, $a_{23}=c$. We obtain P_{41}. Taking $a_{21}=b$, $a_{22}=0$, $a_{23}=c$. We obtain P_{42}. Taking $a_{21}=b$, $a_{22}=c$, $a_{23}=0$. We obtain P_{43}. Taking $a_{21}=0$, $a_{22}=0$, $a_{23}=b$. We obtain P_{44}. Taking $a_{21}=b$, $a_{22}=0$, $a_{23}=0$. We obtain P_{45}. Taking $a_{21}=0$, $a_{22}=b$, $a_{23}=0$. We obtain P_{46}. Taking $a_{21}=0$, $a_{22}=0$, $a_{23}=0$. We obtain P_{47}. Taking $a_{21}=b$, $a_{22}=c$, $a_{23}=d$. We obtain P_{48}.

(B_{22}) Assume $a_{12}\neq 0$, taking $a_{12}=b$. Then (2.12) implies $a_{21}=a_{12}a_{23}^2+a_{11}a_{23}-a_{22}a_{23}$, (2.11) implies $(a_{22}-a_{12}a_{23})^2=0$, (2.10) implies $a_{23}(a_{22}-a_{12}a_{23})^2=0$, then $a_{22}=ba_{23}$, $a_{21}=aa_{23}$, and so (2.7) implies $a_{31}=-a^2-aba_{23}$, (2.8) implies $a_{32}=-ab-b^2a_{23}$, (2.9) implies $a_{33}=-a-ba_{23}$. We obtain

$$P=\begin{pmatrix} a & b & 1\\ aa_{23} & ba_{23} & a_{23}\\ -a^2-aba_{23} & -ab-b^2a_{23} & -a-ba_{23} \end{pmatrix}.$$

Taking $a_{23}=c$, we obtain P_{49}. Taking $a_{23}=0$, we obtain P_{50}.

Case 2. $k\neq -1$, that is, $\theta\neq\pi$.

According to (2.4), there are three subcases: (C) $a_{12} = 0$, $a_{31} \neq 0$; (D) $a_{12} = 0$, $a_{31} = 0$; (E) $a_{12} \neq 0$, $a_{31} = 0$.

(C) Assume $a_{12} = 0$, $a_{31} \neq 0$, taking $a_{31} = 1$. Then (2.6) implies $a_{13}a_{32} = 0$.

(C_1) If $a_{13} = 0$, $a_{32} = 0$, then (2.7) implies $a_{11} = 0$, (2.12) implies $a_{33} = 0$, and so (2.10) implies $a_{22} = 0$. We obtain

$$P = \begin{pmatrix} 0 & 0 & 0 \\ a_{21} & 0 & a_{23} \\ 1 & 0 & 0 \end{pmatrix}.$$

Taking $a_{21} = a$, $a_{23} = 0$. We obtain P_2. Taking $a_{21} = 0$, $a_{23} = a$. We obtain P_{15}. Taking $a_{21} = 0$, $a_{23} = 0$. We obtain P_4. Taking $a_{21} = a$, $a_{23} = b$. We obtain P_{13}.

(C_2) If $a_{13} = 0$, $a_{32} \neq 0$, taking $a_{32} = a$, then (2.7) implies $a_{11} = 0$, (2.11) implies $a_{22} = -a_{33}$, (2.12) implies $a_{21} = \dfrac{-a_{33}}{a}$ and $a_{23} = \dfrac{-a_{33}^2}{a}$. We obtain

$$P = \begin{pmatrix} 0 & 0 & 0 \\ -\dfrac{a_{33}}{a} & -a_{33} & -\dfrac{a_{33}^2}{a} \\ 1 & a & a_{33} \end{pmatrix}.$$

Taking $a_{33} = 0$. We obtain P_{10}. Taking $a_{33} = a$. We obtain P_{20}.

(C_3) If $a_{13} \neq 0$, $a_{32} = 0$, then (2.12) implies $a_{13} = ka_{33}^2$. So we have $a_{33} \neq 0$. Taking $a_{33} = a$. (2.10) implies $a_{22} = \dfrac{a_{11} - ka_{33}}{k+1}$, (2.9) implies $a_{33}^2(a_{11} - ka_{33}) = 0$. And then

$$a_{11} = ka_{33}, \qquad a_{22} = 0.$$

We obtain

$$P = \begin{pmatrix} ka & 0 & ka^2 \\ a_{21} & 0 & a_{23} \\ 1 & 0 & a \end{pmatrix}.$$

Taking $a_{21} = b$, $a_{23} = 0$. We obtain P_{51}. Taking $a_{21} = 0$, $a_{23} = b$. We obtain P_{52}. Taking $a_{21} = 0$, $a_{23} = 0$. We obtain P_{53}. Taking $a_{21} = b$, $a_{23} = c$. We obtain P_{54}.

(D) If $a_{12} = 0$, $a_{31} = 0$, then (2.7) implies $a_{11} = 0$, (2.6) implies $a_{13}a_{32} = 0$.

(D_1) If $a_{32} = 0$, $a_{13} \neq 0$, taking $a_{13} = 1$, then (2.12) implies $a_{33} = 0$, (2.9) implies $a_{22} = 0$. We obtain

$$P = \begin{pmatrix} 0 & 0 & 0 \\ a_{21} & 0 & a_{23} \\ 0 & 0 & 0 \end{pmatrix}.$$

Taking $a_{21} = a$, $a_{23} = 0$. We obtain P_{32}. Taking $a_{21} = 0$, $a_{23} = a$. We obtain P_{34}. Taking $a_{21} = 0$, $a_{23} = 0$. We obtain P_{35}. Taking $a_{21} = a$, $a_{23} = b$. We obtain P_{31}.

(D_2) If $a_{13} = 0$, $a_{32} = 0$, then (2.7) implies $a_{11} = 0$, (2.11) implies $a_{33} = 0$. We obtain

$$P = \begin{pmatrix} 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 0 \end{pmatrix}.$$

Taking $a_{21} = a$, $a_{22} = 1$, $a_{23} = 0$. We obtain P_3. Taking $a_{21} = 0$, $a_{22} = a$, $a_{23} = 1$. We obtain P_{16}. Taking $a_{21} = a$, $a_{22} = 0$, $a_{23} = 1$. We obtain P_{17}. Taking $a_{21} = 0$, $a_{22} = 0$, $a_{23} = 1$. We obtain P_{18}. Taking $a_{21} = 0$, $a_{22} = 1$, $a_{23} = 0$. We obtain P_5. Taking $a_{21} = 1$, $a_{22} = 0$, $a_{23} = 0$. We obtain P_6. Taking $a_{21} = a$, $a_{22} = b$, $a_{23} = 1$. We obtain P_{14}. Taking $a_{21} = 0$, $a_{22} = 0$, $a_{23} = 0$. We obtain P_7.

(D_3) If $a_{32} \neq 0$, taking $a_{32} = 1$, $a_{13} = 0$, then (2.7) implies $a_{11} = 0$, (2.10) implies $a_{21} = 0$, (2.11) implies $a_{22} = -a_{33}$, (2.12) implies $a_{23} = -a_{33}^2$. We obtain

$$P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -a_{33} & -a_{33}^2 \\ 0 & a & a_{33} \end{pmatrix}.$$

Taking $a_{33} = 0$. We obtain P_9. Taking $a_{33} = b$. We obtain P_{19}.

(E) If $a_{12} \neq 0$, $a_{13} = 0$, taking $a_{12} = 1$, then (2.5) implies $a_{32} = 0$, (2.7) implies $a_{21} = -a_{11}^2$, (2.8) implies $a_{22} = -a_{11}$, (2.9) implies $a_{23} = -a_{11}a_{13}$. We obtain

$$P = \begin{pmatrix} a_{11} & 1 & a_{13} \\ -a_{11}^2 & -a_{11} & -a_{11}a_{13} \\ 0 & 0 & 0 \end{pmatrix}.$$

Taking $a_{11} = a$, $a_{13} = 0$. We obtain P_{26}. Taking $a_{11} = 0$, $a_{13} = a$. We obtain P_{39}. Taking $a_{11} = 0$, $a_{13} = 0$. We obtain P_{27}. Taking $a_{11} = a$, $a_{13} = b$. We obtain P_{55}.

3 Solutions of CYBE in $g \ltimes_{ad^*} g^*$

In this section, we give some solutions of CYBE in $g \ltimes_{ad^*} g^*$ from the previous section. Let $(g, [\cdot])$ be a Lie algebra and $\beta\colon g \to gl(V)$ be a representation of g. On the vector space $g \oplus V$, there is, natural Lie algebra structure (denoted by $g \ltimes_\beta V$) given by

$$[x_1 + v_1,\, x_2 + v_2] = [x_1,\, x_2] + \beta(x_1)v_2 - \beta(x_2)v_1, \qquad x_1, x_2 \in g,\ v_1, v_2 \in V. \tag{3.1}$$

Let $\beta^*\colon g \to gl(V^*)$ be the dual representation of β. A linear map $P\colon V \to g$ can be identified as an element $\widetilde{P}$ in $g \otimes V^* \subset (g \ltimes_{\beta^*} V^*) \otimes (g \ltimes_{\beta^*} V^*)$ as follows. Let $\{v_1, v_2, \cdots, v_m\}$ be a basis of V, and $\{v_1^*, v_2^* \cdots, v_m^*\}$ be the dual basis in V^*, that is, $v_i^*(v_j) = \delta_{ij}$. Let $\{e_1, e_2, \cdots, e_n\}$ be a basis of g. Set

$$P(v_i) = \sum_{i=1}^{n} a_{ij} e_j, \qquad 1 \le i \le n.$$

Since as a vector space, $\mathrm{Hom}(V, g) \cong g \otimes V^*$, then

$$\widetilde{P} = \sum_{i=1}^{n} P(v_i) \otimes v_i^* = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} e_j \otimes v_i^* \subseteq (g \ltimes_{\beta^*} V^*) \otimes (g \ltimes_{\beta^*} V^*). \tag{3.2}$$

For any tensor element $r = \sum_i a_i \otimes b_i \in V \otimes V$, denote $r^{21} = \sum_i b_i \otimes a_i$.

Lemma 3.1 *Let L be a Lie algebra. A linear map $P\colon L \to L$ is a Rota-Baxter operator weight 0 if and only if $r = P - P^{21}$ is a skew-symmetric solution of the CYBE in $L \ltimes_{ad^*} L^*$.*

Let L be an algebra equipped with a bilinear product, its formal characteristic matrix is defined by

$$\begin{pmatrix} \sum_{k=1}^{n} a_{11}^{k} v_k & \cdots & \sum_{k=1}^{n} a_{1n}^{k} v_k \\ \vdots & & \vdots \\ \sum_{k=1}^{n} a_{n1}^{k} v_k & \cdots & \sum_{k=1}^{n} a_{nn}^{k} v_k \end{pmatrix},$$

where $\{v_1, v_2, \cdots, v_n\}$ is a basis of L and the multiplication

$$v_i v_j = \sum_{k=1}^{n} a_{ij}^{k} v_k.$$

For 3-dimensional Lie algebras g, let e_1, e_2, e_3 be the basis of g and e_1^*, e_2^*, e_3^* be the dual basis of e_1, e_2, e_3. Now we consider the adjoint representation $ad\colon g \to gl(g)$ of g and its dual adjoint representation $ad^*\colon g \to gl(g^*)$ defined by

$$ad^*(X) = -(adX)^{\mathrm{T}}, \qquad X \in g.$$

Then, by (3.1), the characteristic matrix of 6-dimensional Lie algebra $g \ltimes_{ad^*} g^*$ with respect to the basis $\{e,\ f,\ h,\ e^*,\ f^*,\ h^*\}$ is

$$\begin{pmatrix} 0 & e_1 & 0 & -e_2^* & 0 & 0 \\ -e_1 & 0 & ke_3 & e_1^* & 0 & -ke_3^* \\ 0 & -ke_3 & 0 & 0 & 0 & ke_2^* \\ e_2^* & -e_1^* & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & ke_3^* & -ke_2^* & 0 & 0 & 0 \end{pmatrix}. \tag{3.3}$$

Using Lemma 3.1 and relation (3.2), we can obtain a family of solutions of CYBE in $g \ltimes_{ad^*} g^*$ through the Rota-Baxter operators on g given in Theorem 2.1.

Theorem 3.1 *The following tensors are solutions of the classical Yang-Baxter equation in $g \ltimes_{ad^*} g^*$, where a, b, c are non-zero complex numbers.*

$r_1 = ae_2 \otimes e_2^* + be_1 \otimes e_3^* - e_2^* \otimes ae_2 - e_3^* \otimes be_1,$

$r_2 = ae_1 \otimes e_2^* + be_1 \otimes e_3^* - e_2^* \otimes ae_1 - e_3^* \otimes be_1,$

$r_3 = (ae_1 + be_2) \otimes e_2^* - e_2^* \otimes (ae_1 + be_2),$

$r_4 = ae_1 \otimes e_3^* - e_3^* \otimes ae_1,$

$r_5 = ae_2 \otimes e_2^* - e_2^* \otimes ae_2,$

$r_6 = ae_1 \otimes e_2^* - e_2^* \otimes ae_1,$

$r_7 = 0,$

$r_8 = (ae_1 + be_2) \otimes e_2^* + ce_1 \otimes e_3^* - e_2^* \otimes (ae_1 + be_2) - e_3^* \otimes ce_1,$

$r_9 = e_2 \otimes e_3^* - e_3^* \otimes e_2,$

$r_{10} = (ae_1 + e_2) \otimes e_3^* - e_3^* \otimes (ae_1 + e_2),$

$r_{11} = (ae_1 + be_2 + e_3) \otimes e_2^* + ce_1 \otimes e_3^* - e_2^* \otimes (ae_1 + be_2 + e_3) - e_3^* \otimes ce_1,$

$$r_{12} = (ae_2 + e_3) \otimes e_2^* + be_1 \otimes e_3^* - e_2^* \otimes (ae_2 + e_3) - e_3^* \otimes be_1,$$
$$r_{13} = (ae_1 + e_3) \otimes e_2^* + be_1 \otimes e_3^* - e_2^* \otimes (ae_1 + e_3) - e_3^* \otimes be_1,$$
$$r_{14} = (ae_1 + be_2 + e_3) \otimes e_2^* - e_2^* \otimes (ae_1 + be_2 + e_3),$$
$$r_{15} = e_3 \otimes e_2^* + ae_1 \otimes e_3^* - e_2^* \otimes e_3 - e_3^* \otimes ae_1,$$
$$r_{16} = (ae_2 + e_3) \otimes e_2^* - e_2^* \otimes (ae_2 + e_3),$$
$$r_{17} = (ae_1 + e_3) \otimes e_2^* - e_2^* \otimes (ae_1 + e_3),$$
$$r_{18} = e_3 \otimes e_2^* - e_2^* \otimes e_3,$$
$$r_{19} = (-ae_2 + e_3) \otimes e_2^* + (-a^2e_2 + ae_3) \otimes e_3^* - e_2^* \otimes (-ae_2 + e_3) - e_3^* \otimes (-a^2e_2 + ae_3),$$
$$r_{20} = \left(\frac{b}{a}e_1 - ae_2 + e_3\right) \otimes e_2^* + (be_1 - a^2e_2 + ae_3) \otimes e_3^* - e_2^* \otimes \left(\frac{b}{a}e_1 - ae_2 + e_3\right) - e_3^* \otimes (be_1 - a^2e_2 + ae_3),$$
$$r_{21} = e_2 \otimes e_1^* + ae_3 \otimes e_2^* + be_2 \otimes e_3^* - e_1^* \otimes e_2 - e_2^* \otimes ae_3 - e_3^* \otimes be_2,$$
$$r_{22} = (ae_1 + e_2) \otimes e_1^* - (a^2e_1 + ae_2) \otimes e_2^* + (abe_1 + be_2) \otimes e_3^* - e_1^* \otimes (ae_1 + e_2) + e_2^* \otimes (a^2e_1 + ae_2) - e_3^* \otimes (abe_1 + be_2),$$
$$r_{23} = (ae_1 + e_2) \otimes e_1^* - (a^2e_1 + ae_2 - be_3) \otimes e_2^* - e_1^* \otimes (ae_1 + e_2) + e_2^* \otimes (a^2e_1 + ae_2 - be_3),$$
$$r_{24} = e_2 \otimes e_1^* + ae_2 \otimes e_3^* - e_1^* \otimes e_2 - e_3^* \otimes ae_2,$$
$$r_{25} = e_2 \otimes e_1^* + ae_3 \otimes e_2^* - e_1^* \otimes e_2 - e_2^* \otimes ae_3,$$
$$r_{26} = (ae_1 + e_2) \otimes e_1^* - (a^2e_1 + ae_2) \otimes e_2^* - e_1^* \otimes (ae_1 + e_2) + e_2^* \otimes (a^2e_1 + ae_2),$$
$$r_{27} = e_2 \otimes e_1^* - e_1^* \otimes e_2,$$
$$r_{28} = (ae_1 + e_2) \otimes e_1^* - (a^2e_1 + ae_2 - be_3) \otimes e_2^* + (ace_1 + ce_2) \otimes e_3^* - e_1^* \otimes (ae_1 + e_2) + e_2^* \otimes (a^2e_1 + ae_2 - be_3) - e_3^* \otimes (ace_1 + ce_2),$$
$$r_{29} = e_3 \otimes e_1^* + (ae_2 + be_3) \otimes e_2^* - e_1^* \otimes e_3 - e_2^* \otimes (ae_2 + be_3),$$
$$r_{30} = e_3 \otimes e_1^* + (ae_1 + be_3) \otimes e_2^* - e_1^* \otimes e_3 - e_2^* \otimes (ae_1 + be_3),$$
$$r_{31} = e_3 \otimes e_1^* + (ae_1 + be_2) \otimes e_2^* - e_1^* \otimes e_3 - e_2^* \otimes (ae_1 + be_2),$$
$$r_{32} = e_3 \otimes e_1^* + ae_1 \otimes e_2^* - e_1^* \otimes e_3 - e_2^* \otimes ae_1,$$
$$r_{33} = e_3 \otimes e_1^* + ae_2 \otimes e_2^* - e_1^* \otimes e_3 - e_2^* \otimes ae_2,$$
$$r_{34} = e_3 \otimes e_1^* + ae_3 \otimes e_2^* - e_1^* \otimes e_3 - e_2^* \otimes ae_3,$$
$$r_{35} = e_3 \otimes e_1^* - e_1^* \otimes e_3,$$
$$r_{36} = e_3 \otimes e_1^* + (ae_1 + be_2 + ce_3) \otimes e_2^* - e_1^* \otimes e_3 - e_2^* \otimes (ae_1 + be_2 + ce_3),$$
$$r_{37} = (ae_2 + e_3) \otimes e_1^* - \left(be_2 + \frac{b}{a}e_3\right) \otimes e_2^* + (abe_2 + be_3) \otimes e_3^* - e_1^* \otimes (ae_2 + e_3) + e_2^* \otimes \left(be_2 + \frac{b}{a}e_3\right) - e_3^* \otimes (abe_2 + be_3),$$
$$r_{38} = (ae_2 + e_3) \otimes e_1^* - \frac{b}{a}e_1 \otimes e_2^* + be_1 \otimes e_3^* - e_1^* \otimes (ae_2 + e_3)$$

$$
\begin{aligned}
&+ e_2^* \otimes \frac{b}{a}e_1 - e_3^* \otimes be_1,\\
r_{39} =\ & (ae_2 + e_3) \otimes e_1^* - e_1^* \otimes (ae_2 + e_3),\\
r_{40} =\ & (ae_2 + e_3) \otimes e_1^* - \left(\frac{b}{a}e_1 + ce_2 + \frac{c}{a}e_3\right) \otimes e_2^* + (be_1 + ace_2 + ce_3) \otimes e_3^*\\
&- e_1^* \otimes (ae_2 + e_3) + e_2^* \otimes \left(\frac{b}{a}e_1 + ce_2 + \frac{c}{a}e_3\right) - e_3^* \otimes (be_1 + ace_2 + ce_3),\\
r_{41} =\ & (ae_1 + e_3) \otimes e_1^* + (be_2 + ce_3) \otimes e_2^* - (a^2e_1 + ae_3) \otimes e_3^*\\
&- e_1^* \otimes (ae_1 + e_3) - e_2^* \otimes (be_2 + ce_3) + e_3^* \otimes (a^2e_1 + ae_3),\\
r_{42} =\ & (ae_1 + e_3) \otimes e_1^* + (be_1 + ce_3) \otimes e_2^* - (a^2e_1 + ae_3) \otimes e_3^*\\
&- e_1^* \otimes (ae_1 + e_3) - e_2^* \otimes (be_1 + ce_3) + e_3^* \otimes (a^2e_1 + ae_3),\\
r_{43} =\ & (ae_1 + e_3) \otimes e_1^* + (be_1 + ce_2) \otimes e_2^* - (a^2e_1 + ae_3) \otimes e_3^*\\
&- e_1^* \otimes (ae_1 + e_3) - e_2^* \otimes (be_1 + ce_2) + e_3^* \otimes (a^2e_1 + ae_3),\\
r_{44} =\ & (ae_1 + e_3) \otimes e_1^* + be_3 \otimes e_2^* - (a^2e_1 + ae_3) \otimes e_3^*\\
&- e_1^* \otimes (ae_1 + e_3) - e_2^* \otimes be_3 + e_3^* \otimes (a^2e_1 + ae_3),\\
r_{45} =\ & (ae_1 + e_3) \otimes e_1^* + be_1 \otimes e_2^* - (a^2e_1 + ae_3) \otimes e_3^*\\
&- e_1^* \otimes (ae_1 + e_3) - e_2^* \otimes be_1 + e_3^* \otimes (a^2e_1 + ae_3),\\
r_{46} =\ & (ae_1 + e_3) \otimes e_1^* + be_2 \otimes e_2^* - (a^2e_1 + ae_3) \otimes e_3^* - e_1^* \otimes (ae_1 + e_3)\\
&- e_2^* \otimes be_2 + e_3^* \otimes (a^2e_1 + ae_3),\\
r_{47} =\ & (ae_1 + e_3) \otimes e_1^* - (a^2e_1 + ae_3) \otimes e_3^* - e_1^* \otimes (ae_1 + e_3) + e_3^* \otimes (a^2e_1 + ae_3),\\
r_{48} =\ & (ae_1 + e_3) \otimes e_1^* + (be_1 + ce_2 + de_3) \otimes e_2^* - (a^2e_1 + ae_3) \otimes e_3^*\\
&- e_1^* \otimes (ae_1 + e_3) - e_2^* \otimes (be_1 + ce_2 + de_3) + e_3^* \otimes (a^2e_1 + ae_3),\\
r_{49} =\ & (ae_1 + be_2 + e_3) \otimes e_1^* + (ace_1 + bce_2 + ce_3) \otimes e_2^*\\
&- ((a^2 + abc)e_1 + (ab + b^2c)e_2 + (a + bc)e_3) \otimes e_3^* - e_1^* \otimes (ae_1 + be_2 + e_3)\\
&- e_2^* \otimes (ace_1 + bce_2 + ce_3) + e_3^* \otimes ((a^2 + abc)e_1 + (ab + b^2c)e_2 + (a + bc)e_3),\\
r_{50} =\ & (ae_1 + be_2 + e_3) \otimes e_1^* - (a^2e_1 + abe_2 + ae_3) \otimes e_3^* - e_1^* \otimes (ae_1 + be_2 + e_3)\\
&+ e_3^* \otimes (a^2e_1 + abe_2 + ae_3),\\
r_{51} =\ & (kae_1 + ka^2e_3) \otimes e_1^* + be_1 \otimes e_2^* + (e_1 + ae_3) \otimes e_3^* - e_1^* \otimes (kae_1 + ka^2e_3) - e_2^* \otimes be_1\\
&- e_3^* \otimes (e_1 + ae_3),\\
r_{52} =\ & (kae_1 + ka^2e_3) \otimes e_1^* + be_3 \otimes e_2^* + (e_1 + ae_3) \otimes e_3^* - e_1^* \otimes (kae_1 + ka^2e_3) - e_2^* \otimes be_3\\
&- e_3^* \otimes (e_1 + ae_3),\\
r_{53} =\ & (kae_1 + ka^2e_3) \otimes e_1^* + (e_1 + ae_3) \otimes e_3^* - e_1^* \otimes (kae_1 + ka^2e_3)\\
&- e_3^* \otimes (e_1 + ae_3),\\
r_{54} =\ & (kae_1 + ka^2e_3) \otimes e_1^* + (be_1 + ce_3) \otimes e_2^* + (e_1 + ae_3) \otimes e_3^* - e_1^* \otimes (kae_1 + ka^2e_3)\\
&- e_2^* \otimes (be_1 + ce_3) - e_3^* \otimes (e_1 + ae_3),
\end{aligned}
$$

$$r_{55} = (ae_1 + e_2 + be_3) \otimes e_1^* - (a^2e_1 + ae_2 + abe_3) \otimes e_2^* - e_1^* \otimes (ae_1 + e_2 + be_3) + e_2^* \otimes (a^2e_1 + ae_2 + abe_3).$$

One can check that all of the tensor above are solutions of the classical Yang-Baxter equation in $g \ltimes_{ad^*} g^*$.

4 Induced Left-symmetric Algebras from the Rota-Baxter Operators of Weight 0 on g

A left-symmetric algebra structure on g is a bilinear product $\cdot\colon g \otimes g \to g$ satisfying the condition:

$$x \cdot (y \cdot z) - (x \cdot y) \cdot z = y \cdot (x \cdot z) - (y \cdot x) \cdot z, \qquad x, y, z \in g. \tag{4.1}$$

Given a Lie algebra g, it is a fundamental problem to decide whether g admits a left-symmetric product and to give a classification of such products (see [15]). As an application of the Rota-Baxter operators on g, we can obtain the induced left-symmetric algebras from the Rota-Baxter operators of weight 0 on g. The next lemma comes from [15].

Lemma 4.1 *Let g be a Lie algebra and $P\colon g \to g$ be a Rota-Baxter operator of weight 0. Define a new operation on g by*

$$x * y = [P(x), y], \qquad x, y \in g.$$

Then $(g, *)$ is a left-symmetric algebra.

The following two theorems can be proved easily by a direct computation.

Theorem 4.1 *In the sense of Lemma 4.1, the Rota-Baxter operators of weight 0 on g obtained in Theorem 2.1 give the following left-symmetric algebras.*

(1) $e_2 * e_1 = -ae_1$, $e_2 * e_3 = -ae_3$, $e_3 * e_2 = e_1$;
(2) $e_2 * e_2 = ae_1$, $e_3 * e_2 = e_1$;
(3) $e_2 * e_1 = -e_1$, $e_2 * e_2 = ae_1$, $e_2 * e_3 = -e_3$;
(4) $e_3 * e_2 = e_1$;
(5) $e_2 * e_1 = -e_1$, $e_2 * e_3 = -e_3$;
(6) $e_2 * e_2 = e_1$;
(8) $e_2 * e_1 = -be_1$, $e_2 * e_2 = ae_1$, $e_2 * e_3 = -be_3$;
(9) $e_3 * e_1 = -e_1$, $e_3 * e_3 = -e_3$;
(10) $e_3 * e_1 = -e_1$, $e_3 * e_2 = ae_1$, $e_3 * e_3 = -e_3$;
(11) $e_2 * e_1 = -be_1$, $e_2 * e_2 = ae_1 + e_3$, $e_2 * e_3 = -be_3$, $e_3 * e_2 = ce_1$;
(12) $e_2 * e_1 = -ae_1$, $e_2 * e_2 = e_3$, $e_2 * e_3 = -ae_3$, $e_3 * e_2 = be_1$;
(13) $e_2 * e_2 = ae_1 + e_3$, $e_3 * e_2 = be_1$;
(14) $e_2 * e_1 = -be_1$, $e_2 * e_2 = ae_1 + e_3$, $e_2 * e_3 = -be_3$;
(15) $e_2 * e_2 = e_3$, $e_3 * e_2 = ae_1$;
(16) $e_2 * e_1 = -ae_1$, $e_2 * e_2 = e_3$, $e_2 * e_3 = -ae_3$;

(17) $e_2 * e_2 = ae_1 + e_3$;

(18) $e_2 * e_2 = e_3$;

(19) $e_2 * e_1 = ae_1$, $e_2 * e_2 = e_3$, $e_2 * e_3 = ae_3$, $e_3 * e_1 = a^2e_1$, $e_3 * e_2 = ae_3$, $e_3 * e_3 = a^2e_3$;

(20) $e_2 * e_1 = ae_1$, $e_2 * e_2 = -\frac{b}{a}e_1 + e_3$, $e_2 * e_3 = ae_3$, $e_3 * e_1 = a^2e_1$, $e_3 * e_2 = be_1 + ae_3$, $e_3 * e_3 = a^2e_3$;

(21) $e_1 * e_1 = -e_1$, $e_1 * e_3 = -e_3$, $e_2 * e_2 = ae_3$, $e_3 * e_1 = -be_1$, $e_3 * e_3 = -be_3$;

(22) $e_1*e_1 = -e_1$, $e_1*e_2 = ae_1$, $e_1*e_3 = -e_3$, $e_2*e_1 = ae_1$, $e_2*e_2 = -a^2e_1$, $e_2*e_3 = ae_3$, $e_3 * e_1 = -be_1$, $e_3 * e_2 = abe_1$, $e_3 * e_3 = -be_3$;

(23) $e_1 * e_1 = -e_1$, $e_1 * e_2 = ae_1$, $e_1 * e_3 = -e_3$, $e_2 * e_1 = ae_1$, $e_2 * e_2 = -a^2e_1 + be_3$, $e_2 * e_3 = ae_3$;

(24) $e_1 * e_1 = -e_1$, $e_1 * e_3 = -e_3$, $e_3 * e_1 = -ae_1$, $e_3 * e_3 = -ae_3$;

(25) $e_1 * e_1 = -e_1$, $e_1 * e_3 = -e_3$, $e_2 * e_2 = ae_3$;

(26) $e_1*e_1 = -e_1$, $e_1*e_2 = ae_1$, $e_1*e_3 = -e_3$, $e_2*e_1 = ae_1$, $e_2*e_2 = -a^2e_1$, $e_2*e_3 = ae_3$;

(27) $e_1 * e_1 = -e_1$, $e_1 * e_3 = -e_3$;

(28) $e_1 * e_1 = -e_1$, $e_1 * e_2 = ae_1$, $e_1 * e_3 = -e_3$, $e_2 * e_1 = ae_1$, $e_2 * e_2 = -a^2e_1 + be_3$, $e_2 * e_3 = ae_3$, $e_3 * e_1 = -ce_1$, $e_3 * e_2 = ace_1$, $e_3 * e_3 = -ce_3$;

(29) $e_1 * e_2 = e_3$, $e_2 * e_1 = -ae_1$, $e_2 * e_2 = be_3$, $e_2 * e_3 = -ae_3$;

(30) $e_1 * e_2 = e_3$, $e_2 * e_2 = ae_1 + be_3$;

(31) $e_1 * e_2 = e_3$, $e_2 * e_1 = -be_1$, $e_2 * e_2 = ae_1$, $e_2 * e_3 = -be_3$;

(32) $e_1 * e_2 = e_3$, $e_2 * e_2 = ae_1$;

(33) $e_1 * e_2 = e_3$, $e_2 * e_1 = -ae_1$, $e_2 * e_3 = -ae_3$;

(34) $e_1 * e_2 = e_3$, $e_2 * e_2 = ae_3$;

(35) $e_1 * e_2 = e_3$;

(36) $e_1 * e_2 = e_3$, $e_2 * e_1 = -be_1$, $e_2 * e_2 = ae_1 + ce_3$, $e_2 * e_3 = -be_3$;

(37) $e_1*e_1 = -ae_1$, $e_1*e_2 = e_3$, $e_1*e_3 = -ae_3$, $e_2*e_1 = be_1$, $e_2*e_2 = -\frac{b}{a}e_3$, $e_2*e_3 = be_3$, $e_3 * e_1 = -abe_1$, $e_3 * e_2 = be_3$, $e_3 * e_3 = -abe_3$;

(38) $e_1 * e_1 = -ae_1$, $e_1 * e_2 = e_3$, $e_1 * e_3 = -ae_3$, $e_2 * e_2 = -\frac{b}{a}e_1$, $e_3 * e_2 = be_1$;

(39) $e_1 * e_1 = -ae_1$, $e_1 * e_2 = e_3$, $e_1 * e_3 = -ae_3$;

(40) $e_1 * e_1 = -ae_1$, $e_1 * e_2 = e_3$, $e_1 * e_3 = -ae_3$, $e_2 * e_1 = ce_1$, $e_2 * e_2 = -\frac{b}{a}e_1 - \frac{c}{a}e_3$, $e_2 * e_3 = ce_3$, $e_3 * e_1 = -ace_1$, $e_3 * e_2 = be_1 + ce_3$, $e_3 * e_3 = -ace_3$;

(41) $e_1 * e_2 = ae_1 + e_3$, $e_2 * e_1 = -be_1$, $e_2 * e_2 = ce_3$, $e_2 * e_3 = -be_3$, $e_3 * e_2 = -a^2e_1 - ae_3$;

(42) $e_1 * e_2 = ae_1 + e_3$, $e_2 * e_1 = -be_1$, $e_2 * e_3 = -be_3$, $e_3 * e_2 = -a^2e_1 - ae_3$;

(43) $e_1 * e_2 = ae_1 + e_3$, $e_2 * e_1 = -ce_1$, $e_2 * e_2 = be_1$, $e_2 * e_3 = ce_3$, $e_3 * e_2 = -a^2e_1 - ae_3$;

(44) $e_1 * e_2 = ae_1 + e_3$, $e_2 * e_2 = be_3$, $e_3 * e_2 = -a^2e_1 - ae_3$;

(45) $e_1 * e_2 = ae_1 + e_3$, $e_2 * e_2 = be_1$, $e_3 * e_2 = -a^2e_1 - ae_3$;

(46) $e_1 * e_2 = ae_1 + e_3$, $e_2 * e_1 = -be_1$, $e_2 * e_3 = -be_3$, $e_3 * e_2 = -a^2e_1 - ae_3$;

(47) $e_1 * e_2 = ae_1 + e_3$, $e_3 * e_2 = -a^2e_1 - ae_3$;

(48) $e_1 * e_2 = ae_1 + e_3$, $e_2 * e_1 = -ce_1$, $e_2 * e_2 = be_1 + de_3$, $e_2 * e_3 = -ce_3$, $e_3 * e_2 = -a^2e_1 - ae_3$;

(49) $e_1 * e_1 = -be_1$, $e_1 * e_2 = ae_1 + e_3$, $e_1 * e_3 = -be_3$, $e_2 * e_1 = -bce_1$, $e_2 * e_2 = ace_1 + ce_3$, $e_2 * e_3 = -bce_3$, $e_3 * e_1 = (ab + b^2c)e_1$, $e_3 * e_2 = -(abc + a^2)e_1 - (a + bc)e_3$, $e_3 * e_3 = (ab + b^2c)e_3$;

(50) $e_1 * e_1 = -be_1$, $e_1 * e_2 = ae_1 + e_3$, $e_1 * e_3 = -be_3$, $e_3 * e_1 = abe_1$, $e_3 * e_2 = -a^2e_1 - ae_3$, $e_3 * e_3 = abe_3$.

Theorem 4.2 *In the sense of Lemma* 4.1, *the Rota-Baxter operators of weight* 0 *on* g *obtained in Theorem* 2.2 *give the following left-symmetric algebras.*

(51) $e_1 * e_2 = kae_1 - k^2a^2e_3$, $e_2 * e_2 = be_1$, $e_3 * e_2 = e_1 - kae_3$;

(52) $e_1 * e_2 = kae_1 - k^2a^2e_3$, $e_2 * e_2 = -kbe_3$, $e_3 * e_2 = e_1 - kae_3$;

(53) $e_1 * e_2 = kae_1 - k^2a^2e_3$, $e_3 * e_2 = e_1 - kae_3$;

(54) $e_1 * e_2 = kae_1 - k^2a^2e_3$, $e_2 * e_2 = be_1 - kce_3$, $e_3 * e_2 = e_1 - kae_3$;

(55) $e_1 * e_1 = -e_1$, $e_1 * e_2 = ae_1 - kbe_3$, $e_1 * e_3 = ke_3$, $e_2 * e_1 = ae_1$, $e_2 * e_2 = -a^2e_1 + kabe_3$, $e_2 * e_3 = -kae_3$.

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