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Review Article

On Current Developments in Partial Differential Equations

Fanghua Lin*

Courant Institute, New York University, New York, NY 10012, USA.

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Abstract. The first part of this article is an overview on some recent major developments in the field of analysis and partial different equations. It is a brief presentation given by the author at a round table discussion. The second part is a supplement of various details provided by several outstanding researchers on subjects.

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*Corresponding author. *Email address:* `linf@cims.nyu.edu` (F. Lin)

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1 An overview

During the period of January 5–15, 2020, there was a special program on the analysis and partial differential equations at the Southern University of Science and Technology, China organized by Professor Tao Tang, Xiao-Ming Wang, Linlin Su and myself. In the first seven days, there were special lecture series on the quantitative periodic and stochastic homogenizations and its applications. It is followed by a conference "On the recent trend in the analysis of partial differential equations". The program also included a rather special Round Table discussion on some important current developments in the field of partial differential equations and the current state of corresponding researches in China.

I am particularly happy that we had such a round table discussions. Many friends and colleagues get together with wonderful conversations about old and new things as well as works in our field of research. Each of us can open its mind to freely express individual views on the past and current developments and visions for the future. Thinking back, it's truly worth to have such an event once a while. Last year was also special in some sense at my personal level. I arrived at 60, and it's a natural time to have a reflection. I first returned to China after my Ph.D in USA in 1989, so it is also the 30th year anniversary for me to participate and to organize various forms of academic programs in China. There are a lot great memories and there have been a lot wonderful friends and young collaborators. It has become an important aspect of why I and all of us love mathematics.

On this reflection something has been noticed, for whatever reasons, our generations may have probably not done very well; and history may not necessarily be so friendly. One can no longer have excuses as it had been, at a very different time and situation, for our teacher's generation 30 years ago. Currently in China, there are thousands of researchers in the field of Analysis & PDE, and in each year over the past decades there have been also in thousands of research papers got published. But one may have to admit that the impacts and visibilities of these works are rather limited. It is not about the activities, but it is not very good. In fact, it's quite worry some. One needs to do something about it. At least, one needs to know why? What could one do differently? And how can one do about it?

For mathematics in China, the field of computational and applied math seems doing much better in comparison. It may be so for geometry/geometric analysis too. Noticeable advances and sharp improvements have also been made in certain parts of pure and abstract mathematics. One may ask that: does the field of Analysis & PDE loss its importance and relevance? No one could be certain about the future. But one may have good ideas as what it would most likely be in the next 5, 10,$\cdots$, years. So we run this round table to discuss the current developments in the field of Analysis & PDEs. In particular we discuss the following questions for our field of research:

(1) What are most outstanding works with high impacts over the past 10-15 years?

(2) What are important directions of researches with exciting developments over the last decade or two?

(3) What are important and representative works done by researchers in China over the past 10-20 years?

As outcomes of our discussions, we have noticed that the Italian School of Analysis (since the time of De Giorgi, Stampachia and others) has re-emerged as a leading force which drives the developments in the Calculus of Variations, Geometric Measure Theory and Nonlinear PDEs with such top mathematicians as: L. Ambrosio, C. De Lellis, A. Figalli, G. De Philippis,$\cdots$. We have again observed the tradition and the strength of the French school of analysis, the rising of ETH-Zurich school and leading figures from other European schools with high impact contributions. We also impressed by top harmonic analysts in USA and their fundamental contributions in dispersive equations (nonlinear wave, Schrödinger,$\cdots$). We also see again, once a while, young genius coming out from the former Soviet Union. Then we realize, in several major directions with most exciting and important developments, there are unfortunately not many researchers in China being involved in, and there are not many visible work done in such directions. It is not because the field of analysis/PDEs has lost its relevance and its high impacts in both pure and applied mathematics. To the contrary, the field has grown strikingly fast with many remarkably deep and important new developments.

In the discussions, I have listed some (which are major developments from my own view) important recent works:

(1) Works around Onsager Conjecture (the idea goes back to Nash's isometric embedding, see Subsection 2.4 for a detailed explanation). It started with works by C. De Lellis and L. Székelyhidi on the 3D incompressible Euler equations and some very weak solutions (modeling turbulence) and have now been extended to the Navier Stokes and other incompressible fluids by T. Buckmaster, V. Vicol and P. Isset. One believes that the technique (from Nash) may go much beyond in many problems from physics and geometry (see Subsection 2.9 for more details).

(2) Recent resolution of the Nadirashvili and Yau conjectures by A. Logunov [91, 92] and difficult new results by A. Logunov and E. Malinnikova [93] on the propagation of smallness and Remez estimates for solutions of the second order elliptic equations are truly remarkable (see Section 2.5).

(3) Recent important breakthrough on the finite time blow up of $C^{1,\alpha}$ solutions of 3D incompressible Euler equations by T. Elgindi et al. and its extensions, in particular the works by J. Chen and T. Hou and others have opened a new door to this outstanding problem (see Subsection 2.3).

(4) D. Christodolou have written two big papers (which resulted in two books of total pages close to 2000). One is on 3D compressible relativistic fluids and how smooth solutions develop shocks, the other is about compressible Euler and Maximal Cauchy developments (singularities). One of these work have very recently been greatly simplified and improved by F. Merle, I. Rodinaski and others). And the other simplification with a much direct approach is done by T. Buckmaster and V. Vicol. These two recent work used two completely different approaches (see Subsection 2.6).

One observes that three of the above major developments are in some sense motivated by the Mellenion problem on the 3-D Navier-Stokes and Euler equations (see Subsection 2.8 for more details). While the idea in (1) uses Nash's iterative construction and higher and higher oscillations, the work in (3) as well as in (4) by Buckmaster-Vicol uses so called modulation method. The method in the work of F. Merle, I. Rodinanski and others goes back to some earlier work by German physicist, very classical and difficult ODE systems with rational nonlinearity autonomous.

We have to say something about the modulation method (again very classical method) which has first used not so effectively in the works by F. Merle (over last 25 years) on KDV, Nonlinear Wave, Schrödinger equations for blow up solutions near critical masses. The method has now used in works toward the so-called Soliton resolution Conjecture (see Subsection 2.1). It is used in the study of phase transition problem, De Giorgi conjecture and related geometric problem (in particular, works by J. C. Wei and his collaborators), harmonic map heat flows (by J. C. Wei et al.) 2D finite time blow ups, finite time blow ups in 2D-liquid crystal flows (J. C. Wei joint with me and others). It has now also been used in many geometric variational problems and its associated flows and other nonlinear evolution equations. It would be remarkable if one can extend these studies for constructions of finite time blow ups for the Navier-Stokes equations, Sverak and Jia may be quite relevant, we refer to the Subsection 2.7 for more explanations on this method.

I also discussed many other directions that obviously worth much more and further studies:

(a) Free boundary problems, linear and nonlinear, local and nonlocal (like SQG equation in Subsection 2.12), elliptic, parabolic and even hyperbolic (for example, the aforementioned work by Christodolou in some sense is to study shocks/free boundaries). There are also a large class of problems involving phase transitions in vectorial setting with more complex nonlinearly constrained cases.

(b) Control and Inverse problems often involve ill-posed nature and a lot deep harmonic, microlocal and PDE analysis(see for instance Subsection 2.10 for the Lions' problem on the control of Navier-Stokes equations). One believes that such problems may also be relevant in applications in geometry, data analysis and nonlinear partial differential equations,$\cdots$, particularly when solutions may possess singularities.

(c) Complex fluids and fluid + object (which can be a map such as for liquid crystal flows; it can be a vector like that for MHD; and it can be a deformation like that in nonlinear elasticity or viscoelastic fluids). In general, it can be a fluid + a geometric object of various dimensions, and its coupled flows can be of various type with applications in immersed boundary problems, active particle evolutions, neurons in electric-magnetic fields, phonetic dynamics$\cdots$. We refer to Subsection 2.11 for more details.

(d) Quantum physics and Nonlinear Schrödinger equations and other equations from classical and quantum field theory are remarkably important and interesting. These problems also include studies of various focusing and oscillating phenomena. For examples, knotted solitons, wave localizations or de-localizations, and their interactions and hydrodynamics are fascinating issues from both analysis and applications.

(e) Microlocal analysis and geometric nature of PDE constrained measures. Recent geometric study of Tartar's H-measure [116] (or Gerard's microlocal measures [52]) by G. De Philippis et al. [40], and some study on shock structure in general via GMT seem to be an interesting new direction that worth much further researches.

The above list is far from being exclusive. The only purpose to make this list of scientific issues and to have above discussions is to attract attentions of our colleagues and friends in China to a larger picture of some current developments. Based on these we also made following suggestions.

(i) Beside one's own research interests (and dig deep in some particular problems which are very important), one must stand on a high ground with a very far and globally view. One can't miss the large picture and not to understand the relevance of ones own research in this large picture.

(ii) The value of a research work at end must either solve a problem or to rise an interesting aspect of it including that to develop a new theory. For example, for old problems one needs to ask new questions, and look at truly new challenges. Works are always better to have their intrinsic values. For example, many have looked at compressible 3D flows, however, recent major developments as described in item (3) tell us that one has to ask important and more fundamental questions.

(iii) Training students and young researchers are most critical for changing the current status. It needs to start at the undergraduate and graduate trainings. To provide excellent trainings from the undergraduate levels all the way to the postdoctoral studies are most important for all of us as teachers.

(iv) It is truly useful to organize in groups of students and post-doctors to study most important and high impact works with strikingly new ideas/results. Such organized working seminars have proved again and again in the past (for examples, the Chen and Su seminars at the old Zhejiang University in 1930s and 1940s, and its tradition had been kept by the old Hangzhou University in 1980s and 1990s; and extended and improved by various top institutions in China as well as in other countries) to be one of the most effective ways to guide our students. It is truly worth to invest our time and energy into this key task as it could be the most critical in changing the current status of our research field in China.

2 Some challenging problems

In this section, we shall present more details on some particular problems. Some have been mentioned in the previous section. For more concrete and precise presentations, I have invited: Chenjie Fan from University of Chicago to write Subsection 2.1 on the soliton resolution conjecture; Jun Geng from Lanzhou University to write Subsection 2.2 on homogenization; Yanlin Liu from The Chinese Academy of Sciences to write Subsection 2.3 on the blow-up solutions of 3-D Euler equations; Xinan Ma from University of Sciences and Technology of China to write Subsections 2.4 and 2.5; Shuang Miao from Wuhan University to write Subsection 2.6 on general relativity; Kelei Wang from Wuhan University to write Subsection 2.7 concerning modulation method in heat flow; Ping Zhang from The Chinese Academy of Sciences to write Subsections 2.8, 2.9, 2.10 and 2.11; Zhifei Zhang from Beijing University to write Subsection 2.12 concerning the well-posedness of SQG equation. I thank all of them for their contributions.

2.1 Dispersive equation: Soliton resolution Conjecture

Soliton resolution Conjecture is one major conjecture in the field of dispersive PDEs and predicts the generic behaviors for the so-called Type-II solutions for focusing nonlinear dispersive equations. Here, Type-II means certain critical Sobolev norm of the solution stays bounded within the evolution. Focusing means the nonlinearity competes with the linear part, and blow up behaviors are expected in such models. The conjecture suggests that all such solutions will asymptotically decouple to several solitary waves living at different scales plus a regular radiation term. This conjecture is widely open and particularly hard when the model is not integrable. Typical models covered by this conjecture include energy critical nonlinear wave equations, generalized KdV, mass critical nonlinear Schrödinger equations, and energy critical nonlinear Schrödinger equations.

For example, let u solves (focusing) energy critical nonlinear Schrödinger equations in $\mathbb{R}^d$

$$iu_t - \Delta u = |u|^{\frac{4}{d-2}} u. \tag{2.1}$$

Assuming u blows up at finite time T, and $\limsup_{t\to T} \|u(t,\cdot)\|_{\dot{H}^1} < \infty$, then as t approaches T, Soliton resolution Conjecture predicts that

$$u(t,x) = \sum_{j=1}^{J} \frac{1}{\lambda_j(t)^{\frac{d}{2}-1}} P_j\Big(\frac{x - x_j(t)}{\lambda_j(t)}\Big) e^{i\gamma_j(t)} + v(x) + r(t,x), \tag{2.2}$$

for some J in $\mathbb{N}$, $x_j(t) \in \mathbb{R}^d, \gamma_i(t) \in \mathbb{R}$, $\lambda_j(t) > 0$, $v(x) \in \dot{H}^1$ and $\lim_{t\to T} \|r(t,\cdot)\|_{\dot{H}^1} = 0$. And each $P_i \in \dot{H}^1$, and solves the following stationary energy critical NLS

$$-\Delta f = |f|^{\frac{4}{d-2}} f, \tag{2.3}$$

and

$$\frac{|x_j - x_{j'}|}{\lambda_j} + \left|\ln\frac{\lambda_j}{\lambda_{j'}}\right| \xrightarrow{t\to T} \infty, \quad \forall\, j \neq j'. \tag{2.4}$$

This is wide open even in the radial case and dimension $d=3$. It should be noted that, the radial case are more approachable since there is essentially only one $\dot{H}^1$ solution to (2.3) and is explicit. Meanwhile, (2.3) admits infinite many non-radial solutions and they are yet to be classified. There is also a parallel conjecture for solutions to (2.1) which are global, where the $v(x)$ in (2.2) should be replaced by a solution $v(t,x)$ to linear Schrödinger equations.

The parallel conjecture for (nonradial) wave is of same spirits but slightly more complicated to state due to Lorentz symmetry. But we will state a precise version for radial wave below. Indeed, one of the most complete results in Soliton resolution Conjecture is on radial energy critical wave equations in odd dimensions, where this conjecture is confirmed.

Let u solves radial (focusing) energy critical nonlinear wave equation in $\mathbb{R}^d$

$$u_{tt} - \Delta u = |u|^{\frac{4}{d-2}} u, \tag{2.5}$$

and let W be the unique (up to scaling) positive radial $\dot{H}^1$ solutions to (2.3). Assuming u blows up at finite time T, and $\limsup_{t\to T}\left(\|u(t,\cdot)\|_{\dot{H}^1} + \|\partial_t u(t,\cdot)\|_{L^2_x}\right) < \infty$, then as t approaches T, the Soliton resolution Conjecture predicts that

$$\begin{aligned} u(t,x) &= \sum_{j=1}^{J} \frac{1}{\lambda_j^{\frac{d}{2}-1}(t)} \ell_j W\left(\frac{x}{\lambda_j(t)}\right) + v_0(x) + r_0(t,x), \\ \partial_t u(t,x) &= v_1(x) + r_1(t,x). \end{aligned} \tag{2.6}$$

For some J in $\mathbb{N}$, $\ell_j = \pm 1$, $\lambda_j(t) > 0$, $v(x) \in \dot{H}^1$, $v_1(x) \in L^2$ and $\lim_{t\to T}\left(\|r_0(t,\cdot)\|_{\dot{H}^1} + \|r_1(t,\cdot)\|_{L^2}\right) = 0$. and

$$\left|\ln\frac{\lambda_j}{\lambda_{j'}}\right| \xrightarrow{t\to T} \infty, \quad \forall\, j \neq j'. \tag{2.7}$$

This conjecture was proved first when $d=3$, by Duyckaerts, Kenig and Merle, in [43]. And very recently, in the end of December 2019, Duyckaerts, Kenig and Merle proved this conjecture for all $d \geq 3$ which is odd, in [44]. A parallel result for global solutions was also proved in [43, 44]. Note that only for $d \geq 3$ can one make sense of energy critical. One may also refer to the introduction of those two papers for a lot of related works.

2.2 Homogenization

Partial differential equations and systems with rapidly oscillating coefficients are used to model various physical phenomena in inhomogeneous or heterogeneous media, such as composite and perforated materials. For a family of second-order linear operators in divergence form with rapidly oscillating periodic coefficients

$$\mathcal{L}_{\uparrow} = -\frac{\partial}{\partial x_i}\left[a_{ij}^{\alpha\beta}\left(\frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_j}\right] = -\mathrm{div}\left[A\left(\frac{x}{\varepsilon}\right)\nabla\right], \quad \varepsilon > 0, \tag{2.8}$$

where the coefficient matrix $A(y) = (a_{ij}^{\alpha\beta}(y))$ is real, elliptic and periodic (i.e. $A(y+z) = A(y)$, $z \in \mathbb{Z}^d$, $y \in \mathbb{R}^d$). Under these assumptions, the operators $\mathcal{L}_\varepsilon$ converges as $\varepsilon \to 0$ to a "homogenized" constant coefficient elliptic operator $\mathcal{L}_0 = -\mathrm{div}(\hat{A}\nabla)$ and the constant coefficient $\hat{A}$ in fact can be explicitly computed [9]. During 1987-1991, in a series of papers [3–7], with extra assumption that A verifies the regularity hypothesis $\|A\|_{C^\alpha(\mathbb{R}^d)} \leq C$, Avellaneda and Lin obtained the uniform estimates such as boundary Lipschitz estimate and $W^{1,p}$ estimate etc. for the Dirichlet problem

$$\mathcal{L}_{\uparrow}(u_\varepsilon) = F \quad \text{in } \Omega, \qquad u_\varepsilon = g \quad \text{on } \partial\Omega \tag{DP}$$

in $C^{1,\alpha}$ domain Ω. In particular, if $F=0$, they also proved that the non-tangential maximal function estimates

$$\|(u_\varepsilon)^*\|_{L^p(\partial\Omega)} \leq C\|g\|_{L^p(\partial\Omega)} \tag{2.9}$$

holds for any $1 < p < \infty$ with C independent of ε. In 1994, Kenig [66] conjectured that (2.9) holds for strongly elliptic, periodic, C^α systems and $p=2$ for arbitrary bounded Lipschitz domains $\Omega \subset \mathbb{R}^d$. And this problem was solved by Kenig and Shen [72] by using the layer potential method. One may notice that the Neumann problem

$$\mathcal{L}_{\uparrow}(u_\varepsilon) = F \quad \text{in } \Omega, \qquad \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = g \quad \text{on } \partial\Omega \tag{NP}$$

is more difficult than the Dirichlet problem (DP) since the boundary conditions in (NP) are ε-dependent. Extending quantitative theory to homogenization of elliptic systems with Neumann boundary conditions has been a longstanding open problem for nearly thirty years. In 2013, this was solved by Kenig, Lin and Shen [67–70] and [72].

Once the qualitative and quantitative homogenization of (DP) and (NP) are established, it is natural for us to be interested in the qualitative and quantitative homogenization theory for oblique derivative problem

$$\mathcal{L}_{\uparrow}(u_\varepsilon)=F \quad \text{in } \Omega, \qquad \nabla u_\varepsilon\cdot\nu_\varepsilon=g \quad \text{on } \partial\Omega, \tag{ODP}$$

where $\nu_\varepsilon(x)$ is nowhere tangential to the boundary $\partial\Omega$ and $\mathcal{L}_{\uparrow}=-a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial^2}{\partial x_i\partial x_j}$ is a second-order linear operators in non-divergence form with rapidly oscillating periodic coefficients. However, to my best knowledge, even the energy estimates for the oblique derivative problem is not trivial even when $\mathcal{L}_{\uparrow}$ is the Laplacian Δ. Thus interesting challenges arise in dealing with (ODP). The natural question is: can one have the boundary Lipschitz estimates, boundary Hölder estimates, and $W^{1,p}$ estimates as well as the nontangential maximal function estimates (2.9)? One may see [17] and [71] for the nontangential maximal function estimates for oblique derivative problem of Laplace equation in Lipschitz domains.

2.3 $C^{1,\alpha}$ blow-up solutions of 3D incompressible Euler equations

Consider the incompressible Euler equations on $\mathbb{R}^3$:

$$\begin{cases} \partial_t u+u\cdot\nabla u+\nabla p=0, \\ \operatorname{div} u=0, \\ u\,|_{t=0}=u_0, \end{cases} \tag{2.10}$$

where $u:[0,\infty)\times\mathbb{R}^3\to\mathbb{R}^3$ stands for the velocity of the fluid flow, f for the external force, and p designates scalar pressure function, which guarantees the divergence free condition of the velocity field. The incompressibility ensures that any smooth solution u of (2.10) on $[0,T)\times\mathbb{R}^3$ satisfies

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|u(t,x)|^2dx=\int_{\mathbb{R}^3}u(t,x)\cdot f(t,x)dx, \quad \forall\, t\in[0,T). \tag{2.11}$$

The question of global regularity for the solutions to (2.10) has been studied by numerous mathematicians and is considered to be one of the biggest open problems in the field of mathematical fluid mechanics. It dates back to 1920's that Gunther [55] and Lichtenstein [80] showed that if $u_0\in C^{1,\alpha}(\mathbb{R}^3)$ for some $0<\alpha<1$ and the initial vorticity decays sufficiently rapidly at infinity, then there exists a time $T>0$ so that (2.10) has a unique solution $u\in C^{1,\alpha}([0,T)\times\mathbb{R}^3)$. We will call this solution "classical solution". Later, similar results were established by many authors in Sobolev spaces, Besov spaces, etc.

The well-known Beale-Kato-Majda criterion [8] tells us that a classical solution loses its regularity at some finite time T if and only if

$$\lim_{t\to T}\int_0^t \|\omega(s,\cdot)\|_{L^\infty}\,ds=\infty, \tag{2.12}$$

where $\omega \overset{\text{def}}{=} \mathrm{curl}\, u$ is the vorticity of the fluid, which satisfies

$$\partial_t\omega+u\cdot\nabla\omega=\omega\cdot\nabla u+\mathrm{curl} f.$$

It is easy to use this criterion to show that the classical solution of 2-D Euler equation cannot develop singularity at finite time. Indeed, for 2-D case, the vortex stretching term $\omega\cdot\nabla u\equiv 0$, thus we have the following *a priori* bound

$$\|\omega(t,\cdot)\|_{L^\infty}\le\int_0^t \|\mathrm{curl} f(s,\cdot)\|_{L^\infty}\,ds.$$

However, this bound is not available for 3-D case, see [47]. What's worse, recently Elgindi [46] showed that the condition (2.12) can be satisfied for some well-designed classical solutions. Precisely, Elgindi proved that:

Theorem 2.1. *There exists some sufficiently small $\alpha>0$, a divergence-free and odd*[†] *$u_0\in C^{1,\alpha}(\mathbb{R}^3)$ with $|\omega_0(x)|\le\frac{C}{1+|x|^\alpha}$ for some constant $C>0$, so that the unique local odd solution to* (2.10) *(with $f=0$) belongs to $C^{1,\alpha}([0,1)\times\mathbb{R}^3)$ and satisfies*

$$\lim_{t\to 1}\int_0^t \|\omega(s,\cdot)\|_{L^\infty}\,ds=\infty.$$

Elgindi's work [46] is a very important progress in studying the global regularity for solutions to Euler equations. However, there are still several pities. First, the solutions in Theorem 2.1 have infinite energy and do not satisfy the energy equality (2.11). But if one allows a uniformly $C^{1,\alpha}$ external force, then his construction can exactly give some finite energy solutions which blow up at finite time. Second, although the blow-up solutions here have $C^{1,\alpha}$ regularity, but this α has to be sufficiently small, so the following basic problem still remains open:

Open Problem: Given a solution $u\in C^\infty([0,T)\times\mathbb{R}^3)$ to (2.10) satisfying the energy equality (2.11) and the external force $f\in(C^\infty\cap L^2)([0,T]\times\mathbb{R}^3)$, is it possible that

$$\limsup_{t\to T}\|\nabla u(t,\cdot)\|_{L^\infty}=\infty?$$

[†]Here we say a vector field $u:\mathbb{R}^3\to\mathbb{R}^3$ is odd if u_i is odd in x_i and even in the other two variables for each $1\le i\le 3$.

2.4 Nash isometric imbedding

The problem of isometric embeddings of Riemannian manifolds consists of a global topological condition (being an embedding) and a system of partial differential equations (being an isometry). For concreteness let us consider a smooth n-dimensional Riemannian manifold (Σ^n,g). A continuous map $u:\Sigma\longrightarrow R^N$is isometric if it preserves the length of curves, namely if

$$l_{g(\gamma)}=l_e(u(\gamma))$$

for any C^1 curve $\gamma\subset\Sigma$, where $g(\gamma)$ denotes the length of γ with respect to the metric g:

$$g(\gamma)=\int\sqrt{g(\gamma(t))[\dot\gamma(t),\dot\gamma(t)]}dt.$$

As customary, in local coordinates we can express the metric tensor g as $g=g_{ij}dx_i\bigotimes dx_j$. Then, if u is C^1, is equivalent to a system of partial differential equations, which in local coordinates takes the following form:

$$d_iu\cdot d_ju=g_{ij}. \tag{2.13}$$

The existence of isometric immersions (resp. embeddings) of Riemannian manifolds into some Euclidean space is a classical problem, see Han-Hong [57]. Clearly, if the dimension of $dim\Sigma=n$, (2.13) consists of $s_n:=\frac{n(n+1)}{2}$ equations in N unknowns. A reasonable guess would therefore be that the system is solvable, at least locally, when $N=s_n$. In the first half of the twentieth century Janet, Cartan, and Burstin had proved the existence of local isometric embeddings in the case of analytic metrics (see [57]), precisely when $N=s_n$. For the very particular case of 2-dimensional spheres endowed with metrics of positive Gauss curvature, Weyl had raised the question of the existence of global isometric embeddings in R^3. Weyl's problem was solved by Lewy for analytic metrics, and Louis Nirenberg settled the case of smooth metrics in his PhD thesis in 1949. A different proof was given independently by Pogorelov around the same time.

A short embedding is C^1 embedding (immersion) u such that $u^\sharp(e)\le g$ as quadratic forms, or in local coordinates, $\partial_iu\cdot\partial_ju\le g_{ij}$. The celebrated Nash-Kuiper theorem [76,102] states that any short immersion or embedding $u:M^n\longrightarrow R^{n+1}$ can be uniformly approximated by C^1 isometric immersions/embeddings. As a particular case, for the classical Weyl problem, i.e. $(S^2,g)\longrightarrow R^3$ with positive Gauss curvature $K_g>0$ this result implies the existence of a vast set of noncongruent C^1 surfaces, each isometric to (S^2,g). This is in stark contrast with the situation for C^2 isometric embeddings: the famous rigidity theorem of Cohn-Vossen and Herglotz states that the C^2 isometric embedding $(S^2,g) \longrightarrow R^3$ is uniquely determined up to congruencies.

Nash [103] proves the more expected version of smooth isometric embeddings from M^n to a sufficiently high-dimensional Euclidean space. For C^∞ imbeddings the best such result was later proved by Gromov [53].

Theorem 2.2. (Nash (1956) [103] and Gromov (1970) [53]). *Any short embedding $u: M^n \longrightarrow R^q$ can be uniformly approximated by isometric embeddings (immersions) provided that $q \geq \frac{(n+2)(n+3)}{2}$.*

The question of what happens in between the rigid C^2 case and the highly nonrigid C^1 case on the Hölder scale $C^{1,\theta}$ has a long history. In the 1950s in a series of papers by Borisov [10], building upon the work of Pogorelov [107], showed that the rigidity of convex surfaces prevails for $\theta > 2/3$. More recently there has been intensive work on lowering the rigidity exponent [37], one conjecture (see [54]) being that some form of rigidity should hold for all $\theta > 1/2$.

Problem. Let $N = 3 = n+1$. Is there a threshold $\theta_0 \in (0,1)$ such that: $C^{1,\theta}$ solutions of the Weyl problem are rigid for $\theta > \theta_0$ and the Nash-Kuiper Theorem holds for $C^{1,\theta}$ immersions when $\theta < \theta_0$?

De Lellis and Székelyhidi (see Subsection 2.9) have pointed out the celebrated conjecture of Onsager in the theory of fully developed turbulence shares many similarities with this problem.

We should mention other questions:

1. For more readable partial differential equations proof in Pogorelov [107] theorem on the rigidity of convex surface;

2. Local embedding for variable curvature [57] in 2 dimensional Riemannian manifold.

2.5 Unique continuation

The classical technique to get the unique continuation for elliptic partial differential equations is Carleman estimates, one can see the reference from [49]. From 1980s, an important tool to study unique continuation, nodal sets of eigenfunctions and growth properties of solutions of elliptic PDEs is the so called **frequency function**. The idea goes back to works of Almgren and Agmon. It was developed further by Garofalo and Lin [51], Let $A(x)$ be a symmetric uniformly elliptic matrix with Lipschitz coefficients defined on some ball B_r centered at the origin and such that $A(0)=I$. Let further

$$\mu(x)=\frac{(A(x)x,x)}{|x|^2}, \quad \mu(0)=1, \quad \Lambda^{-1}\leq\mu(x)\leq\Lambda.$$

Let u be a solution to the equation $div(A(x)\nabla u(x))=0$. We consider weighted averages of $|u|^2$ over spheres:

$$H(r)=r^{1-d}\int_{\partial B_r}\mu(x)|u(x)|^2 ds(x).$$

And we define

$$I(r)=r^{1-d}\int_{B_r}(A\nabla u,\nabla u)dx,$$

then for the frequency function

$$N(r)=r\frac{I(r)}{H(r)}.$$

There exists C that depends only on the ellipticity and Lipschitz constants of the operator such that for any solution u to $div(A(x)\nabla u(x))=0$, the function $e^{Cr}N(r)$ is an increasing function of r.

A consequence of the monotonicity of the frequency function is the so called three sphere theorem. Its simplest version is the classical Hadamard three circle theorem for analytic functions. One can derive the three spheres from the properties of the frequency function following.

In recent Logunov and Malinnikova [93], they got an big improvement. Let Ω will be a bounded domain in R^n and u will denote a solution of an elliptic equation in the divergence form $div(A\nabla u)=0$ in Ω with Lipschitz coefficients Let E and K be subsets of Ω such that the distances from E and K to $\partial\Omega$ are positive. We assume that E has positive n-dimensional Lebesgue measure. They prove the following estimate

$$\sup_K|u|\leq C(\sup_E|u|)^\gamma(\sup_\Omega|u|)^{1-\gamma}, \tag{2.14}$$

where $C>0$ and $\gamma\in(0,1)$ are independent of u, but depend on Ω, A, the measure of E, and the distances from K and E to the boundary of Ω. This is the quantitative results on propagation of smallness for solutions of elliptic PDE.

On the way of proving (2.14) they obtain an interesting inequality for solutions of elliptic equations, which reminds the classical **Remez inequality** for polynomials, the role of the degree is now played by the doubling index. Let Q be a unit cube. Assume u is a solution to $div(A\nabla u)=0$ and the **doubling index** $N=\log\frac{\sup_{2Q}|u|}{\sup_Q|u|}$. Then

$$\sup_Q|u|\le C\sup_E|u|\Big(C\frac{|Q|}{|E|}\Big)^{CN}, \tag{2.15}$$

where C depends on A only, E is any subset of Q of a positive measure.

Mainly by the important work by Logunov and Malinnikova in recent five years, they introduce new combinatorial methods and obtain some deep theorems. We mention three well known conjecture.

Conjecture 1 (Yau). Let M be a smooth compact d-dimensional Riemannian manifold. There exist constants C_1 and C_2, which depend on M, such that

$$C_1\sqrt{\lambda}\le H^{d-1}(Z(\varphi_\lambda))\le C_2\sqrt{\lambda},$$

for any eigenfunction φ_λ satisfying $\Delta_M\varphi_\lambda+\lambda\varphi_\lambda=0$.

One can see the recent work by Logunov [91, 92] an he prove the following result

$$C_1\sqrt{\lambda}\le H^{d-1}(Z(\varphi_\lambda))\le C_2\lambda^\alpha,$$

and $\alpha\ge\frac{1}{2}$.

Conjecture 2 (Fang-Hua Lin). Let u be a non-zero harmonic function in the unit ball $B_1\subset R^n$, $n\ge3$. Consider $N=\log\frac{\sup_{B_1}|\nabla u|}{\sup_{B_{\frac{1}{2}}}|\nabla u|}$. Is it true that the Hausdorff measure

$$H^{n-2}(\{\nabla u=0\}\cap B_{\frac{1}{2}})\le C_nN^2$$

for some C_n depending only on the dimension?

As mentioned by Logunov and Malinnikova [93], it is connect to propagation of smallness for gradients of the harmonic function.

The last conjecture is connected to the quantitative version of the Cauchy uniqueness problem, it dates back to at least L. Bers. The two-dimensional case is not difficult due to connections with complex analysis. The fact that the question is open in higher dimensions shows that we still don't understand well the Cauchy uniqueness problem even for ordinary harmonic functions in the dimension three or higher.

Conjecture 3. Assume that u is a harmonic function in the unit ball $B_1\subset R^3$ and u is $C^{+\infty}$-smooth in the closed ball $\bar{B_1}$. Let $S\subset\partial B_1$ be any closed set with positive area. Is it true that $\nabla u=0$ on S implies $\nabla u\equiv0$?

For the class $C^{1+\varepsilon}(B_1)$ there is a striking counterexample [12], which however is not $C^{+\infty}$-smooth up to the boundary. The attempts to construct C^2-smooth counterexamples were not successful.

2.6 General relativity

A central mathematical problem in theory of General Relativity is to show the nonlinear stability of Kerr spacetimes whose metrics are axisymmetric, stationary, asymptotical flat solutions to the vacuum Einstein equations. From the experience of proving nonlinear stability of Minkowski spacetime [23], to show the nonlinear stability of Kerr spacetimes, one needs to start with a proper linearized system of vacuum Einstein equations and prove polynomial decay for linearized gravity.

As a special case, different approaches towards linear stability of the vacuum, spherically symmetric Schwarzschild spacetimes has been developed in [33, 60]. Both works start from Regge-Wheeler type equations where the technique of treating scalar wave equation can be applied. In particular, spherical symmetry is crucial in the latter approach. A nontrivial nonlinear stability result of Schwarzschild spacetimes under polarized axisymmetry is obtained in [74].

In the case of Kerr spacetimes, Teukolsky found in a seminal work [117] that the extreme curvature components are governed by a separable, decoupled wave equation – Teukolsky Master Equation (TME), and they fully determine the dynamics of linearized gravity up to linearized mass and angular momentum perturbations plus pure gauge solutions. Moreover, these extreme curvature components are gauge invariant, hence as a beginning step, one can treat them using TME without imposing any gauge choice. Energy and decay estimates for these extreme curvature components are shown in [34,94] for slowly rotating Kerr backgrounds ($|a|/M \ll 1$) by applying a suitable modification of *Chandrasekhar's transformations* [20]. Based on these estimates, the authors in [2] derived strong decay estimates for TME and obtained a linear stability result of slowly rotating Kerr metrics in an outgoing radiation gauge. See also [56] for a linear stability proof but using a microlocal approach.

2.7 Modulation method in heat flow

In many nonlinear evolution equations, the blow up of solutions is caused by the formation of bubbles, especially when the blow up if of Type II. For example, if $u:\mathbb{R}^2\times[0,T)\mapsto\mathbb{S}^2\subseteq\mathbb{R}^3$ solves the harmonic map heat flow

$$u_t-\Delta u=|\nabla u|^2u, \tag{2.16}$$

and T is the blow up time, under some technical assumptions, it is shown that (see Qing [109], Ding-Tian [42], Lin-Wang [84] and Qing-Tian [108]), as $t_n \to T$, there exist finitely many points $q_1,\cdots,q_k$ such that

$$u(x,t_n) \to u_*(x) + \sum_{i=1}^{k} \left[W_i\left(\frac{x-q_i^n}{\lambda_i^n}\right) - W_i(\infty)\right] \quad \text{in } H^1,$$

where u_* is the trace of u at $t=T$, $q_i^n \to q_i$, $0<\lambda_i^n \to 0$, and W_i are *bubbles*, i.e. finite energy solutions of

$$-\Delta W_i = |\nabla W_i|^2 W_i \quad \text{in } \mathbb{R}^2.$$

One crucial point behind this bubbling phenomena is the scaling invariance of this problem. This induces the concentration of energy, which is an important class of the loss of compactness phenomena, a topic studied intensively in calculus of variations during the last two decades.

The construction of blow up solutions is of great interest, especially for the understanding of Type II blow up. For harmonic map heat flow, Chang, Ding and Ye [21] first constructed solutions exhibiting finite time blow up, by using a special ansatz and then reducing the equation to a simpler one. However, this does not give too much information about the blow up behavior. In [111, 112], Raphaël and Schweyer rigorously constructed blow up solutions by using *the modulation method*. By this method they get a precise control on the blow up rate, blow up profile as well as the stability of this blow up mechanism.

In the modulation method, by noting that bubbles are the main order term when approaching the blow up time, one modulates the standard bubbles to get an approximation to the original solution. For example, for (2.16) one takes the decomposition

$$u(x,t) = u_*(x) + \sum_{i=1}^{k} \left[W_i\left(\frac{x-q_i(t)}{\lambda_i(t)}\right) - W_i(\infty)\right] + \mathcal{R}.$$

Here $q_i(t)$ and $\lambda_i(t)$ are modulation parameters, $\mathcal{R}$ is a small error term. By a suitable choice of modulation parameters, one obtains some good estimates on $\mathcal{R}$. (This is because $\mathcal{R}$ satisfies a linearized equation.) Furthermore, the modulation parameters also satisfy a modulation equation (here an ODE), which can be used to derive the asymptotic behavior of the blow up rate (that is, $\lambda_i(t)$ in the above equation) as $t \to T$.

The modulation method is applicable to many other problems, for example,

- mass critical Schrödinger equations (Merle [99]);

- energy critical Schrödinger equations (Merle, Raphaël and Schweyer [100]);
- nonlinear wave equations and Yang-Mills equation (Raphaël and Schweyer [110], Rodniansk and Sterbenz [113]);
- and energy supercritical Schrödinger equations (Merle, Raphaël and Rodnianski [101]).

Recently, in a series of work (see e.g. [32,35,36]), Del Pino, Wei and their collaborators, based on their earlier works in various elliptic equations (e.g. [105,106]), introduced the *inner-outer gluing method* for the construction of blow up solutions. Here one takes a further decomposition of the error term $\mathcal{R}$ according to the scale, one on the scale of bubbles, which forms the inner problem, and the other one on the original scale, which forms the outer problem. With this decomposition they can treat the bubbling formation at multiple points and without any symmetry.

The modulation method is also useful in the study of "Soliton resolution Conjecture" (see Subsection 2.1), especially when the solution is near the ground state, see e.g. [24,95].

2.8 The Navier-Stokes equation (uniqueness/non-uniqueness, finite-time blowup)

d (2 or 3) dimensional incompressible Navier-Stokes equations can be written as follows:

$$\begin{cases} \partial_t u + u\cdot\nabla u - \nu\Delta u = -\nabla p + f, & (t,x)\in\mathbb{R}^+\times\mathbb{R}^d, \\ div u = 0, & \\ u|_{t=0} = u_0, \quad \text{with} \quad div u_0 = 0, & \end{cases} \tag{2.17}$$

where $u=(u_1,\cdots,u_d)$ denotes the fluid velocity, $u\cdot\nabla u=\sum_{j=1}^d u_j\partial_j u$, p represents the scalar pressure function, ν the viscous coefficient and f the force term.

For any initial data u_0 of finite kinetic energy, Leray [79] proved in 1934 that (2.17) has at least one global in time finite energy weak solution, which satisfies the energy inequality

$$\frac{1}{2}\|u(t)\|_{L^2}^2+\nu\|u\|_{L_t^2(L^2)}^2 \le \frac{1}{2}\|u_0\|_{L^2}^2. \tag{2.18}$$

This solution is unique and regular in two space dimension as long as the time $t>0$. Hopf [59] established a similar result for (2.17) posed on smooth bounded domain with Dirichlet boundary condition. Yet the question of uniqueness and regularity of Leray-Hopf weak solutions for 3D Navier-Stokes system remains to be one of the biggest open problems (see [48]) in mathematical fluid mechanics.

Fujita-Kato [50] constructed local in time unique solution to (2.17) with initial data in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$. Furthermore, if $\|u_0\|_{\dot{H}^{\frac{1}{2}}}$ is sufficiently small, then the solution exists globally in time (see [65] for similar result with initial data in $L^3(\mathbb{R}^3)$). This result was extended by Cannone, Meyer and Planchon [18] for initial data belonging to $\dot{B}_{p,\infty}^{-1+\frac{3}{p}}(\mathbb{R}^3)$ with $p\in(3,\infty)$. The end-point result in this direction is given by Koch and Tataru [75]. They proved that given initial data being sufficiently small in $\mathrm{BMO}^{-1}(\mathbb{R}^3)$, then (2.17) has a unique global solution. We remark that for $p\in(3,\infty)$, there holds

$$\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)\hookrightarrow L^3(\mathbb{R}^3)\hookrightarrow \dot{B}_{p,\infty}^{-1+\frac{3}{p}}(\mathbb{R}^3)\hookrightarrow \mathrm{BMO}^{-1}(\mathbb{R}^3)\hookrightarrow \dot{B}_{\infty,\infty}^{-1}(\mathbb{R}^3),$$

and the norms to the above spaces are scaling-invariant under the following transformation:

$$u_\lambda(t,x)=\lambda u(\lambda^2 t,\lambda x) \quad \text{and} \quad u_{0,\lambda}(x)=\lambda u_0(\lambda x). \tag{2.19}$$

We notice that for any solution u of (2.17) on $[0,T]$, u_λ determined by (2.19) is also a solution of (2.17) on $[0,T/\lambda^2]$. We remark that the largest space, which belongs to $\mathcal{S}'(\mathbb{R}^3)$ and the norm of which is scaling invariant under (2.19), is $\dot{B}_{\infty,\infty}^{-1}(\mathbb{R}^3)$ (see [98]). Moreover, Bourgain and Pavlović [11] proved that (2.17) is actually ill-posed with initial data in $\dot{B}_{\infty,\infty}^{-1}(\mathbb{R}^3)$. That is the reason why people call such kind of initial data, with the $\dot{B}_{\infty,\infty}^{-1}(\mathbb{R}^3)$ norm of which is large, as large initial data.

On the other hand, in [62, 63], Jia and Šverák proved the non-uniqueness of Leray-Hopf weak solutions in the regularity class $L_t^\infty(L^{3,\infty})$ if a certain spectral assumptions for a linearized Navier-Stokes operator holds. Very recently, Buckmaster and Vicol [14] proved the following very interesting result:

Theorem 2.3 (Theorem 1.2 of [14]). *There exits $\beta>0$, such that for any non-negative smooth function $e(t):[0,T]\to R^+$, (2.17) has a weak solution $u\in C^0([0,T];H^\beta(\mathbb{T}^3))$ such that $\int_{\mathbb{T}^3}|u(t,x)|^2dx=e(t)$ for all $t\in[0,T]$. Moreover, the associated vorticity $\nabla\times u$ lies in $C^0([0,T];L^1(\mathbb{T}^3))$.*

We remark that the weak solution of (2.17) constructed in Theorem 2.3 is still not Leray-Hopf solutions, which do not satisfy the energy inequality (2.18). The main idea of the proof is based on the pioneering papers [38, 39] by De Lellis and Székelyhidi.

2.9 The Onsager conjecture

In 1949, Onsager [104] conjectured that there exists a threshold regularity for the conservation of energy to the incompressible Euler equations (2.10). In particular, for the weak solution u, which is Hölder continuous in space, i.e.

$$|u(t,x)-u(t,y)|\leq C|x-y|^{\beta} \quad \text{for all } \ t\in[0,T] \tag{2.20}$$

for some constant C that is independent of time t. He asserted that for any Hölder continuous weak solution u of (2.10) with exponent $\beta>\frac{1}{3}$, the energy should be conserved; whereas for any weak solution with smaller exponent, there are solutions that do not conserve energy. The first assertion was proved by Constantin, E and Titi in [25]. Concerning the second assertion, the first proof of the existence of a square integrable weak solution that do not preserve the energy is due to Scheffer in his pioneering paper [114] and a different proof was given by Shnirelman in [115].

On the other hand, De Lellis and Székelyhidi observed in [38] that the techniques from the theory of differential inclusions could applied to construct bounded weak solutions that violate the energy conservation. After a series of important partial results improving the threshold, Isett [61] has been able to finally reach the Onsager exponent $\frac{1}{3}$. One may check [13] for recent progress in this direction.

2.10 Control of Navier-Stokes system

In the late 1980's, Jacques-Louis Lions introduced in [86] (see also [87–89]) the question of the controllability of fluid flows in the sense of how the Navier-Stokes system can be driven by a control of the flow on a part of the boundary to a wished plausible state, say a vanishing velocity. Lions' problem has been solved in [30] by Coron, Marbach and Sueur in the particular case of the Navier slip-with-friction boundary condition. In its original statement with the no-slip Dirichlet boundary condition, it is still an important open problem in fluid controllability.

In the special case when the space dimension is two and the geometric domain

$$\Omega:=(0,L)\times(-1,1),$$

where $L>0$ is the length of the domain. Inside this domain, a fluid evolves under the Navier-Stokes equation, that is, the velocity field u satisfies:

$$\begin{cases} \partial_t u+(u\cdot\nabla)u+\nabla p-\Delta u=f_g, \\ \operatorname{div} u=0, \end{cases} \tag{2.21}$$

in Ω, where p denotes the fluid pressure and f_g a force term. One may think of this domain as a river or a tube and assume that we can act on the fluid flow at both end boundaries:

$$\Gamma_0 := \{0\} \times (-1,1) \quad \text{and} \quad \Gamma_L := \{L\} \times (-1,1).$$

On the remaining parts of the boundary,

$$\Gamma_\pm := (0,L) \times \{\pm 1\},$$

we assume that we cannot control the fluid flow and that it satisfies null Dirichlet boundary conditions:

$$u = 0 \quad \text{on } \Gamma_\pm. \tag{2.22}$$

Let $L^2_{\text{div}}(\Omega)$ be the space of divergence free vector fields in $L^2(\Omega)$, which are tangential to the boundaries $\Gamma_\pm$. Lately, Coron, Mabach, Sueur and Zhang proved the following result in [31]:

Theorem 2.4 ([31]). *Let $T>0$ and u_* be in $L^2_{\text{div}}(\Omega)$. For any $k \in \mathbb{N}$ and for any $\eta > 0$, there exists a force $f_g \in L^1((0,T); H^k(\Omega))$ satisfying*

$$\|f_g\|_{L^1((0,T);H^k(\Omega))} \leq \eta \tag{2.23}$$

and an associated weak Leray solution $u \in C^0([0,T];\, L^2_{\text{div}}(\Omega)) \cap L^2((0,T);\, H^1(\Omega))$ to (2.21) and (2.22) satisfying $u(0) = u_*$ and $u(T) = 0$.

We remark that Lions' problem in this special situation corresponds to $f_g = 0$ in (2.21). Hence even for this particular case, Lions' problem has still not been solved completely.

2.11 Complex fluid system and MHD system

One of the common origins and manifestations of anomalous phenomena in complex fluids are different "elastic" effects. Most complex fluids are indeed viscoelastic. It is the interaction between the elastic properties and the fluid motions that gives not only the complicated rheological phenomena, but also formidable challenges in analysis, modeling and numerical simulations. For a general viscoelastic fluids, we have the following equation for conservation of momentum:

$$\rho(\partial_t u + u \cdot \nabla u) = \nabla \cdot \tau, \tag{2.24}$$

where τ is the total stress.

In the case of linear elasticity, (2.24) writes

$$\begin{cases} \partial_t u+u\cdot\nabla u+\nabla p=\mu\Delta u+\nabla\cdot\left(FF^T\right) \quad \text{in } \mathbb{R}_+\times\mathbb{R}^d, \\ \operatorname{div} u=0, \\ F_t+u\cdot\nabla F=\nabla u F, \\ (u,F)|_{t=0}=(u_0,F_0), \end{cases} \tag{2.25}$$

where u denotes the fluid velocity and F the deformation matrix.

Mathematically, Lin et al. [82] first proved the global existence of smooth solutions to (2.25) in two space dimension when the initial data (u_0,F_0) is a small perturbation of the trivial state $\left(0,Id_{2\times 2}\right)$. Then a similar result was generalized to three space dimension in [22,77]. In general, the global existence or finite time singularity of solutions, even the global existence of weak solution to (2.25), is still open except the special case in [90].

In the case when the deformation matrix F in (2.25) is reduced to a vector-valued function, the equation becomes incompressible MHD system without resistance:

$$\begin{cases} \partial_t u+u\cdot\nabla u+\nabla p=\mu\Delta u+b\cdot\nabla b \quad \text{in } \mathbb{R}_+\times\mathbb{R}^d, \\ \operatorname{div} u=\operatorname{div} b=0, \\ \partial_t b+u\cdot\nabla b=b\cdot\nabla u, \\ (u,b)|_{t=0}=(u_0,b_0), \end{cases} \tag{2.26}$$

where b denotes the magnetic field. Starting with [85] for a modified model of (2.26), Lin et al. proved the global existence of solutions to (2.26) with initial velocity being sufficiently small and the initial magnetic field being sufficiently close to non-zero constant vector. The same result for (2.26) was proved in [1] and an optimal decay rates for such solutions was obtained in [41]. He et al. proved the vanishing viscosity limit of the viscous MHD system in [58] (see also [16,118]).

We remark that hydrodynamical and rheological properties of complex fluids depend intimately on their molecular conformation and configurations. The hydrodynamics of these materials are described by the coupled micro-macro models. A model system is given by

$$\begin{cases} \partial_t u+u\cdot\nabla u+\nabla p=\mu\Delta u+\nabla\cdot\tau & \text{in } \mathbb{R}_+\times\Omega, \\ \operatorname{div} u=0, \\ \partial_t f+u\cdot\nabla f=\Delta_q f-\nabla_q\cdot\left(\nabla u q f-\nabla_q U f\right) & \text{in } \mathbb{R}_+\times\Omega\times B(0,R), \\ (u,f)|_{t=0}=(u_0,f_0), \end{cases} \tag{2.27}$$

where $U(q)=U(|q|^2)$ and $\tau=\int_{B(0,R)}\nabla_q U\otimes q f dq$.

Lin et al. [83] proved the global existence of smooth solutions to (2.27) for initial data near equilibrium. For the so-called polymeric fluids, the local existence of smooth solutions was established in [45, 64, 96]. Masmoudi proved the global existence of weak solution in [97].

One may refer to the survey paper [81] by Lin for more details in this direction.

2.12 SQG equations

The surface quasi-geostrophic (SQG) equation is an important model in geophysical fluid dynamics. It was proposed as a two dimensional model to study the singularity formation of the inviscid incompressible flows [26]. The SQG equation with the dissipation takes

$$\partial_t\theta+u\cdot\nabla\theta+\kappa(-\Delta)^{\alpha}\theta=0. \tag{2.28}$$

Here $\alpha\in[0,\frac{1}{2}],\kappa>0$ is the dissipative coefficient, $\theta(t,x)$ is a real-valued function of t and x. The function θ represents the potential temperature, the fluid velocity u is determined from θ by a stream function ψ

$$(u_1,u_2)=\Big(-\frac{\partial\psi}{\partial x_2},\frac{\partial\psi}{\partial x_1}\Big),\quad (-\Delta)^{\frac{1}{2}}\psi=-\theta. \tag{2.29}$$

The fractional Laplacian $(-\Delta)^{\alpha}$ is defined by

$$\widehat{(-\Delta)^{\alpha}f}(\xi)=|\xi|^{2\alpha}\hat{f}(\xi).$$

Due to maximum principle, $\|\theta(t)\|_{L^\infty}\le\|\theta(0)\|_{L^\infty}$. Formally, nonlinear term $u\cdot\nabla\theta$ behaves as $|D|\theta^2$. Thus, in the case of $\alpha>\frac{1}{2}$, the linear diffusion term $(-\Delta)^{\alpha}$ will dominate nonlinear term so that (2.28) has a global in time smooth solution for smooth data [27]. The SQG equation with $\alpha=\frac{1}{2}$ is called the critical SQG. The global well-posedness of the critical SQG was independently proved by Caffarelli and Vasseur [15], and Kiselev, Nazarov and Volberg [73]. The work [15] used the harmonic extension and the De Georgi method, and the work [73] used the method of constructing the moduli of continuity. Recently, Constantin and Vicol gave a new proof via the so-called nonlinear maximum principles [28]. Global regularity or finite time singularity for the super critical SQG with $\alpha\in[0,1)$ remains a challenging problem. Let us refer to [19, 29] for some important progress in this direction.

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References

[1] H. Abidi and P. Zhang, On the global well-posedness of 3-D MHD system with initial data near the equilibrium state, *Comm. Pure Appl. Math.*, **70** (2017), 1509-1561.

[2] L. Andersson, T. Bäckdahl, P. Blue and S. Ma, *Stability for linearized gravity on the Kerr spacetime*, arXiv preprint arXiv:1903.03859 (2019).

[3] M. Avellaneda and F. Lin, Compactness methods in the theory of homogenization, *Comm. Pure Appl. Math.*, **40** (1987), 803–847.

[4] M. Avellaneda and F. Lin, Homogenization of elliptic problems with L^p boundary data, *Appl. Math. Optim.*, **15** (1987), 93–107.

[5] M. Avellaneda and F. Lin, Compactness methods in the theory of homogenization. ii. equations in nondivergence form, *Comm. Pure Appl. Math.*, **42** (1989), 139–172.

[6] M. Avellaneda and F. Lin, Homogenization of Poisson's kernel and applications to boundary control, *J. Math. Pures Appl.*, **68** (1989), 1–29.

[7] M. Avellaneda and F. Lin, L^p bounds on singular integrals in homogenization, *Comm. Pure Appl. Math.*, **44** (1991), 897–910.

[8] J. T. Beale, T. Kato and A. Majda, Remarks on the breakdown of smooth solutions for the 3-D Euler equations, *Comm. Math. Phys.*, **94** (1984), 61-66.

[9] A. Bensoussan, J.-L. Lions, and G. C. Papanicolaou, *Asymptotic Analysis for Periodic Structures*. North Holland, Amsterdam, 1978.

[10] Y. F. Borisov, On the connection bewteen the spatial form of smooth surfaces and their intrinsic geometry, *Vestnik Leningrad. Univ.*, **14** (1959), 20-26.

[11] J. Bourgain and N. Pavlović, Ill-posedness of the Navier-Stokes equations in a critical space in 3D, *J. Funct. Anal.*, **255** (2008), 2233-2247.

[12] J. Bourgain and T. Wolff, A remark on gradients of harmonic functions in dimension ≥ 3, *Colloq. Math.*, **60/61** (1990), 253-260.

[13] T. Buckmaster, C. De Lellis, L. Jr. Székelyhidi and V. Vicol, Onsager's conjecture for admissible weak solutions, *Comm. Pure Appl. Math.*, **72** (2019), 229-274.

[14] T. Buckmaster and V. Vicol, Nonuniqueness of weak solutions to the Navier-Stokes equation, *Ann. of Math. (2)*, **189** (2019), 101-144.

[15] L. Caffarelli and A. Vasseur, Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation, *Ann. Math.*, **171** (2010), 1903-1930.

[16] Y. Cai and Z. Lei, Global well-posedness of the incompressible magnetohydrodynamics, *Arch. Ration. Mech. Anal.*, **228** (2018), 969-993.

[17] A. P. Calderón, *Boundary value problems for the Laplace equation in Lipschitzian domains,* Recent progress in Fourier analysis, North-Holland Math. Stud., 111:33-48,1983.

[18] M. Cannone, Y. Meyer and F. Planchon, Solutions autosimilaires des équations de Navier-Stokes, Séminaire "Équations aux Dérivées Partielles" de l'École polytechnique, Exposé VIII, 1993–1994.

[19] A. Castro, D. Cordoba and J. Gomez-Serrano, Global smooth solutions for the inviscid SQG equation, arXiv:1603.03325.

[20] S. Chandrasekhar, *On the equations governing the perturbations of the Schwarzschild black hole*, Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences, vol. 343, The Royal Society, 1975, pp. 289–298.

[21] K.-C. Chang, W.-Y. Ding and R. Ye, Finite-time blow-up of the heat flow of harmonic maps from surfaces, *J. Differ. Geom.*, **36** (1992), 507-515.

[22] Y. Chen and P. Zhang, The global existence of small solutions to the incompressible viscoelastic fluid System in general space dimensions, *Comm. P. D. E.*, **31** (2006), 1793-1810.

[23] D. Christodoulou and S. Klainerman, *The global nonlinear stability of the Minkowski space*, Princeton University Press, 1993.

[24] C. Collot, F. Merle and P. Raphaël, Dynamics near the ground state for the energy critical nonlinear heat equation in large dimensions, *Comm. Math. Phys.*, **352** (2017), 215-285.

[25] P. Constantin, W. E and E. S. Titi, Onsager's conjecture on the energy conservation for solutions of Euler's equation, *Comm. Math. Phys.*, **165** (1994), 207-209.

[26] P. Constantin, A. Majda and E. Tabak, Formation of strong fronts in the 2-D quasi-geostrophic thermal active scalar, *Nonlinearity*, **7** (1994), 1495-1533.

[27] P. Constantin and J. Wu, Behavior of solutions of 2D quasi-geostrophic equations, *SIAM J. Math. Anal.* , **30** (1999), 937-948.

[28] P. Constantin and V. Vicol, Nonlinear maximum principles for dissipative linear nonlocal operators and applications, *GAFA*, **22** (2012), 1289-1321.

[29] D. Cordoba, Nonexistence of simple hyperbolic blow-up for the quasi-geostrophic equation, *Ann. of Math.* (2), **148** (1998), 1135-1152.

[30] J.-M. Coron, F. Marbach and F. Sueur, Small-time global exact controllability of the Navier-Stokes equation with Navier slip-with-friction boundary conditions, *J. Eur. Math. Soc.*, (2016) (In press)

[31] J.-M. Coron, F. Marbach, F. Sueur and P. Zhang, Controllability of the Navier-Stokes equation in a rectangle, *Ann. PDE*, **5** (2019), no. 2, Art. 17, 49 pp.

[32] C. Cortzar, M. del Pino and M. Musso, Green's function and infinite-time bubbling in the critical nonlinear heat equation, *J. Eur. Math. Soc. (JEMS)*, **22** (2020), 283-344.

[33] M. Dafermos, G. Holzegel and I. Rodnianski, *The linear stability of the Schwarzschild solution to gravitational perturbations*, arXiv preprint arXiv:1601.06467 (2016).

[34] M. Dafermos, G. Holzegel and I. Rodnianski, Boundedness and decay for the teukolsky equation on Kerr spacetimes I: The case $|a| \ll m$, *Annals of PDE*, **5** (2019), no. 1, 2.

[35] P. Daskalopoulos, M. del Pino and N. Sesum, Type II ancient compact solutions to the Yamabe flow, *J. Reine Angew. Math.*, **738** (2018), 1-71.

[36] J. Davila, M. del Pino and J. Wei, Singularity formation for the two-dimensional harmonic map flow into $\mathbb{S}^2$, to appear in *Invent. Math.*.

[37] C. De Lellis, D. Inauen and L. Székelyhidi, Jr. A Nash-Kuiper theorem for $C^{1,1/5-\delta}$ immersions on surfaces in 3 dimensions, *Rev. Mat. Iberoamericana*, **34** (2018), 1119-1152.

[38] C. De Lellis and L. Jr. Székelyhidi, The Euler equations as a differential inclusion, *Ann. of Math.*, (2) **170** (2009), 1417-1436.

[39] C. De Lellis and L. Jr. Székelyhidi, Dissipative continuous Euler flows, *Invent. Math.*, **193** (2013), 377-407.

[40] G. De Philippis and F. Rindler, On the structure of A-free measures and applications, *Ann. of Math.*, (2) **184** (2016), 1017-1039.

[41] W. Deng and P. Zhang, Large time behavior of solutions to 3-D MHD system with initial data near equilibrium, *Arch. Ration. Mech. Anal.*, **230** (2018), 1017-1102.

[42] W.-Y. Ding and G. Tian, Energy identity for a class of approximate harmonic maps from surfaces, *Comm. Anal. Geom.*, **3** (1995), 543-554.

[43] T. Duyckaerts, C. Kenig and F. Merle, Classification of the radial solutions of the focusing, energy-critical wave equation, *Camb. J. Math.*, (2013), 75-144.

[44] T. Duyckaerts, C. Kenig and F. Merle, Soliton resolution for the radial critical wave equation in all odd, space dimensions, *arxiv 1912.07664*

[45] W. E, T. Li and P. W. Zhang, Well-posedness for the dumbbell model of polymeric fluids, *Comm. Math. Phys.*, **248** (2004), 409-427.

[46] T. M. Elgindi, Finite-time singularity formation for $C^{1,\alpha}$ solutions to the incompressible Euler equations on $\mathbb{R}^3$. arXiv:1904.04795 [math.AP]

[47] T. M. Elgindi and N. Masmoudi, L^∞ ill-posedness for a class of equaitons arising in hydrodynamics. To appear in *Arch. Rat. Mech. Anal.*

[48] C. L. Fefferman, Existence and smoothness of the Navier-Stokes equation. The mil-

lennium prize problems, 57-67, Clay Math. Inst., Cambridge, MA, 2006.

[49] X. Y. Fu, Q. L*ü* and X. Zhang, *Carleman Estimates for Second Order Partial Differential Operators and Applications: A Unified Approach*, Springer-Verlag, 2019.

[50] H. Fujita and T. Kato, On the Navier-Stokes initial value problem I. *Arch. Ration. Mech. Anal.*, **16** (1964), 269-315.

[51] N. Garofalo and F. Lin, Monotonicity properties of variational integrals, A_p weights and unique continuation, *Indiana Univ. Math. J.*, **35** (1986), 245-268.

[52] P. G*é*rard, Patrick Microlocal defect measures, *Comm. Partial Differential Equations*, **16** (1991), 1761-1794.

[53] M. L. Gromov, Isometric imbeddings and immersions (Russian), *Dokl. Akad. Nauk SSSR*, **192** (1970), 1206-1209.

[54] M. Gromov, Geometric, algebraic, and analytic descendants of Nash isometric embedding theorems, *Bull. Amer. Math. Soc.*, **54** (2017), 173-245.

[55] N. Gunther, On the motion of fluid in a moving container, *Izvestia Akad. Nauk USSR, Ser. Fiz.-Mat.*, **20** (1927), 1323-1348.

[56] D. Häfner, P. Hintz, and A. Vasy, *Linear stability of slowly rotating Kerr black holes*, arXiv preprint arXiv:1906.00860 (2019).

[57] Q. Han and J. X. Hong, *Isometric embedding of Riemannian manifolds in Euclidean s-paces,* American Mathematical Society, Providence, RI, **2006**.

[58] L. He, L. Xu and P. Yu, On global dynamics of three dimensional magnetohydrodynamics: nonlinear stability of Alfvén waves, *Ann. PDE*, **4** (2018), no. 1, Art. 5, 105 pp.

[59] E. Hopf, *Ü*ber die Anfangswertaufgabe f*ü*r die hydrodynamischen Grundgleichungen, (German) *Math. Nachr.*, **4** (1951), 213-231.

[60] P. K. Hung, J. Keller and M. T. Wang, *Linear stability of Schwarzschild spacetime: The Cauchy problem of metric coefficients*, arXiv preprint arXiv:1702.02843 (2017).

[61] P. Isett, A proof of Onsager's conjecture, *Ann. of Math. (2)*, **188** (2018), 871-963.

[62] H. Jia and V. Šverák, Local-in-space estimates near initial time for weak solutions of the Navier-Stokes equations and forward self-similar solutions, *Invent. Math.*, **196** (2014), 233-265.

[63] H. Jia and V. Šverák, Are the incompressible 3d Navier-Stokes equations locally ill-posed in the natural energy space? *J. Funct. Anal.*, **268** (2015), 3734-3766.

[64] B. Jourdain, T. Leliévre and C. Le Bris, Existence of solution for a micro-macro model of polymeric fluid: the FENE model, *J. Funct. Anal.*, **209** (2004), 162-193.

[65] T. Kato, Strong L^p-solutions of the Navier-Stokes equation in $\mathbb{R}^m$ with applications to weak solutions, *Math. Z.*, **187** (1984), 471-480.

[66] C. Kenig, *Harmonic Analysis Techniques for Second Order Elliptic Boundary Value Problems*, CBMS Regional Conference Series in Math., vol. 83, AMS, Providence, RI, 1994.

[67] C. Kenig, F. Lin and Z. Shen, Convergence rates in L^2 for elliptic homogenization problems, *Arch. Ration. Mech. Anal.*, **203** (2012), 1009–1036.

[68] C. Kenig, F. Lin and Z. Shen, Estimates of eigenvalues and eigenfunctions in periodic homogenization, *J. Eur. Math. Soc.*, **15** (2013), 1901–1925.

[69] C. Kenig, F. Lin and Z. Shen, Homogenization of elliptic systems with Neumann boundary conditions, *J. Amer. Math. Soc.*, **26** (2013), 901–937.

[70] C. Kenig, F. Lin and Z. Shen, Periodic homogenization of Green and Neumann functions. *Comm. Pure Appl. Math.*, **67** (2014), 1219–1262.

[71] C. Kenig and J. Pipher, The oblique derivative problem on Lipschitz domains with L^p data. *Amer. J. Math.* , **4** (1988), 715-737.

[72] C. Kenig and Z. Shen, Layer potential methods for elliptic homogenization problems. *Comm. Pure Appl. Math.*, **64** (2011), 1–44.

[73] A. Kiselev, F. Nazarov and A. Volberg, Global well-posedness for the critical 2D dissipative quasi-geostrophic equation, *Invent. Math.*, **167** (2007), 445-453.

[74] S. Klainerman and Jérémie Szeftel, *Global nonlinear stability of Schwarzschild spacetime under polarized perturbations*, arXiv preprint arXiv:1711.07597 (2017).

[75] H. Koch and D. Tataru, Well-posedness for the Navier-Stokes equations, *Adv. Math.*, **157** (2001), 22-35.

[76] N. Kuiper, On C^1 isometric imbeddings i,ii., *Proc. Kon. Acad. Wet. Amsterdam A*, **58** (1955), 545-556, 683-689.

[77] Z. Lei, C. Liu and Y. Zhou, Global solutions for incompressible viscoelastic fluids, *Arch. Ration. Mech. Anal.*, **188** (2008), 371-398.

[78] P. G. Lemarié-Rieusset, *The Navier-Stokes problem in the 21st century.* CRC Press, Boca Raton, FL, 2016.

[79] J. Leray, Sur le mouvement d'un liquide visqueux emplissant l'espace, *Acta Math.*, **63** (1934), 193-248.

[80] L. Lichtenstein, Über einige Existenz probleme der Hydrodynamik, *Mat. Zeit. Phys.*, **23** (1925), 89-154.

[81] F. Lin, Some analytical issues for elastic complex fluids, *Comm. Pure Appl. Math.*, **65** (2012), 893-919.

[82] F. Lin, C. Liu and P. Zhang, On the hydrodynamics of visco-elasicity, *Comm. Pure Appl. Math.*, **58** (2005), 1437-1471.

[83] F. Lin, C. Liu and P. Zhang, On the multi-scal Dumbbell model in the polymer fluids near equilibrium, *Comm. Pure Appl. Math.*, **60** (2007), 838-866.

[84] F. Lin and C. Y. Wang, Energy identity of harmonic map flows from surfaces at finite singular time, *Calc. Var. Partial Differ. Equ.*, **6** (1998), 369-380.

[85] F. Lin and P. Zhang, Global small solutions to MHD type system (I): 3-D case, *Comm. Pure. Appl. Math.*, **67** (2014), 531-580.

[86] J.-L. Lions, Exact controllability for distributed systems. Some trends and some problems. In: Applied and industrial mathematics (Venice, 1989), vol. 56 of Math. Appl., pp. 59-84. Kluwer Academic Publications, Dordrecht (1991).

[87] J.-L. Lions, On the controllability of distributed systems, *Proc. Nat. Acad. Sci. USA*,

94 (10) (1997), 4828-4835.

[88] J.-L. Lions, Remarks on the control of everything. In: *European Congress on Computational Methods in Applied Sciences and Engineering*, Barcelona, 11-14 September. ECCOMAS (2000)

[89] J.-L. Lions, Sur le contrôle des équations de Navier-Stokes. In: *Jean Leray '99 Conference Proceedings*, vol. **24**, Mathematical Physics Studies, pp. 543-558. Kluwer Academic Publications, Dordrecht (2003).

[90] P. L. Lions and N. Masmoudi, Global solutions for some Oldroyd models of non-Newtonian flows, *Chinese Ann. Math. Ser. B*, **21** (2000), 131-146.

[91] A. Logunov. Nodal sets of Laplace eigenfunctions: polynomial upper estimates of the Hausdorff measure, *Ann. of Math.* (2), **187** (2018), 221-239.

[92] A. Logunov. Nodal sets of Laplace eigenfunctions: proof of Nadirashvili's conjecture and of the lower bound in Yau's conjecture, *Ann. of Math.* (2), **187** (2018), 241-262.

[93] A. Logunov and E. Malinnikova, *Quantitative propogation of smallness for solutions of elliptic equations.* In Proceedings of the International Congress of Mathematicians-Rio de Janeiro 2018, volume 2 of ICM, pages 2357-2378. World Scientific, 2018.

[94] S. Y. Ma, *Uniform energy bound and Morawetz estimate for extreme components of spin fields in the exterior of a slowly rotating Kerr black hole II: linearized gravity*, arXiv preprint arXiv:1708.07385 (2017).

[95] Y. Martel, F. Merle and P. Raphaël, Blow up for the critical generalized Korteweg-de Vries equation. I: Dynamics near the soliton, *Acta Math.*, **212** (2014), 59-140.

[96] N. Masmoudi, Well-posedness for the FENE dumbbell model of polymeric flows, *Comm. Pure Appl. Math.*, **61** (2008), 1685-1714.

[97] N. Masmoudi, Global existence of weak solutions to the FENE dumbbell model of polymeric flows, *Invent. Math.*, **191** (2013), 427-500.

[98] Y. Meyer. Wavelets, Paraproducts and Navier–Stokes. Current Developments in Mathematics, International Press, Cambridge, Massachussets, 1996.

[99] F. Merle, Construction of solutions with exactly k blow-up points for the Schrödinger equation with critical nonlinearity, *Comm. Math. Phys.*, **129** (1990), 223-240.

[100] F. Merle, P. Raphaël and R. Schweyer, Blowup dynamics for smooth data equivariant solutions to the critical Schrödinger map problem, *Invent. Math.*, **193** (2013), 249-365.

[101] F. Merle, P. Raphaël and I. Rodnianski, Type II blow up for the energy supercritical NLS, *Camb. J. Math.*, **3** (2015), 439-617.

[102] J. Nash, C^1 isometric imbeddings, *Ann. of Math.*, **60** (1954), 383-396.

[103] J. Nash, The imbedding problem for Riemannian manifolds, *Ann. of Math.* (2), **63** (1956), 20-63.

[104] L. Onsager, *Statistical hydrodynamics. Nuovo Cimento (9)* **6**, (1949). Supplemento, no.

2 (Convegno Internazionale di Meccanica Statistica), 279-287.

[105] M. del Pino, M. Kowalczyk and J. Wei, Concentration on curves for nonlinear Schrödinger equations, *Commun. Pure Appl. Math.*, **60** (2007), 113-146.

[106] M. del Pino, M. Kowalczyk and J. Wei, On De Giorgi's conjecture in dimension $N=9$, *Ann. of Math.*, **174** (2011), 1485-1569.

[107] A. V. Pogorelov, *Extrinsic geometry of convex surfaces (Transl. Math. Monogr. Vol. 35)*, Providence, RI, Am.Math.Soc., **1973**.

[108] J. Qing and G. Tian, Bubbling of the heat flows for harmonic maps from surfaces, *Comm. Pure Appl. Math.*, **50** (1997), 295-310.

[109] J. Qing, On singularities of the heat flow for harmonic maps from surfaces into spheres, *Commun. Anal. Geom.*, **3** (1995), 297-315.

[110] P. Raphaël and R. Schweyer, Stable blow up dynamics for the critical co-rotational wave maps and equivariant Yang-Mills problems, *Publ. Math. IHES.*, **115** (2012), 1-122.

[111] P. Raphaël and R. Schweyer, Stable blow up dynamics for the 1-corotational energy critical harmonic heat flow, *Commun. Pure Appl. Math.*, **66** (2013), 414-480.

[112] P. Raphaël and R. Schweyer, Quantized slow blow-up dynamics for the corotational energy critical harmonic heat flow, *Anal. PDE*, **7** (2014), 1713-1805.

[113] I. Rodniansk and J. Sterbenz, On the formation of singularities in the critical $O(3)$-model, *Ann. of Math.*, **172** (2010), 187-242.

[114] V. Scheffer, An inviscid flow with compact support in space-time, *J. Geom. Anal.*, **3** (1993), 343-401.

[115] A. Shnirelman, On the nonuniqueness of weak solution of the Euler equation, *Comm. Pure Appl. Math.*, **50** (1997), 1261-1286.

[116] L. Tartar, H-measures, a new approach for studying homogenisation, oscillations and concentration effects in partial differential equations, *Proc. Roy. Soc. Edinburgh Sect. A*, **115** (1990), 193-230.

[117] S. A. Teukolsky, Rotating black holes: Separable wave equations for gravitational and electromagnetic perturbations, *Physical Review Letters* **29** (1972), no. 16, 1114.

[118] D. Wei and Z. Zhang, Global well-posedness of the MHD equations in a homogeneous magnetic field, *Anal. PDE*, **10** (2017), 1361-1406.