

REVIEW ARTICLE

On Current Developments in Partial Differential Equations

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Received 11 February 2020; Accepted 27 February 2020

Abstract. The article is resulted from a round table discussion. The first part of it contains an overview on some important recent developments in the field of analysis and partial differential equations. It is a slightly expanded version of a presentation given by the author at the round table and a briefly summary of speeches by several others. The second part is a supplement of various details written by some researchers in China working on related topics.

AMS subject classifications: 35Q30

Key words: Partial differential equations, harmonic analysis, geometric measure theory.

1 An overview

During the period of January 5–15, 2020, there was a special program on the analysis and partial differential equations at the Southern University of Science and Technology of China organized by Professor Tao Tang, Xiao-Ming Wang, Linlin Su and myself. In the first seven days, there were special lecture series on the theory of quantitative periodic and stochastic homogenizations and its applications. The two principle lecturers are Professor T. Kuusi and Professor Zhongwei Shen. It is followed by a conference “On the recent trend in the analysis of partial differential equations”. The program also included a special Round Table discussion on some important current developments in the field of analysis and partial

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differential equations; and to have a discussion on the research and education programs for this field in Chinese universities.

I am very happy that we had such a round table discussions. Many friends and colleagues working in the field got together having wonderful conversations and sharing stories about old and new events. Naturally, many recent research works in field were also brought into our attention. At such a round table, each of us can open his/her mind to express freely their individual and particular views on the past and current developments and to provide thoughts for the future if that would be possible. Thinking back, it's truly worth to have such an occasion, once a while. Last year was also special in some sense at my personal level. I arrived at 60, and it's a natural time to have a reflection. I left China in 1981 for a graduate study in the USA and I have since stayed and worked here. My first returning to China after my Ph.D was in 1989, so it is also the 30th anniversary for me to be participated in and sometime to co-organize some of academic activities in China during my academic breaks or sabbatical leaves. There are a lot of great memories and a lot of wonderful old and young friends over all these years. Many have also become scientific collaborators. It has become an important part of life and it may be why I (and may be almost all of us) love mathematics.

At the round table discussions, many of us have noticed that our generations have probably not done that well as one would have hoped; and as we shall believe, history may not necessary be so friendly. For our teacher's generation 40 years ago, China is at a completely different state. Excellent education and research programs are always difficult to build, and particularly so during the school years of our generation. For now, we no longer have such excuses for not catching up forefronts of research and not to do high quality and more fundamental researches. We have also no excuses not to do high quality teachings for next generations. One can take a look at a simple statistics in China: there are thousands of researchers in the field of Analysis & PDE, and in each year (over the past decades) there have been also in thousands of research papers got published. On the other hand, it is evident over the years that the impacts and visibilities of these works are relatively limited. It is not about the lack of communications or supports in academic activities. In fact, it is not something that one can say that it is normal. One should do something about it strategically. At least, one needs to know why? What and how one can do about it?

Looking outside the field of analysis and PDEs, the field of computational and applied math seems doing much better in China via comparisons. It may be also for several other subjects such as differential geometry /geometric analysis. In fact, a very noticeable advances and sharp improvements have been made in certain parts of pure and abstract mathematics. So we could ask: does the field

of Analysis & PDE loss its historical importance and relevance? Of course, no one could be certain about the future. But one may have good feelings as what it would likely be in the next few, 5 or even 10 years by looking closely at what are going on now. It may be one of the goals that we run this round table to discuss the current developments in the field of Analysis & PDEs. In particular we have discussed the following issues for the field of analysis/PDEs:

- (1) List some of (you think they are) most outstanding works with high impacts over the past 10-15 years.
- (2) What you think are important directions of researches with exciting developments over the last decade or two?
- (3) What you think are important and representative works done by researchers in China over the past 10-20 years?

As a result of these discussions, all of us are deeply impressed by the Italian School of Analysis (since the time of De Giorgi, Stampachia and others). It has re-emerged to be a leading force which drives the developments in the field of Calculus of Variations, Geometric Measure Theory and Nonlinear PDEs with such top mathematicians as: L. Ambrosio, C. De Lellis, A. Figalli, G. De Philippis,... We have observed again the tradition and the strength of the French school of analysis which has brought in remarkably high impact researches. Many institutions in Europe are (as they have been) attractive to top talents with rising prestige. The mathematics at the ETH-Zurich, IHES are examples. The field of analysis and PDE is strongly influenced also by the top harmonic analysts in USA and their remarkable and fundamental contributions in dispersive equations (non-linear wave, Schrödinger,...). It is also a striking phenomena that, once a while, some young genius coming out from the former Soviet Union. It was pointed out in discussions that in several major directions with exciting and important developments, there are unfortunately not many researchers in China being involved in and, often, there are not many visible research works being done in such directions. One can be convinced that it is not because the field of analysis/PDEs has lost its relevance. It has become more evident over last decades the fundamental importance of the field of Analysis and PDEs in the both pure and applied mathematics. Indeed, the field has grown strikingly fast with many remarkably deep and important new developments.

In the discussions, some (which are major developments from my own view too) of these important recent works have been listed:

- (1) Works around Onsager Conjecture and turbulent solutions of classical fluid equations (the idea goes back to Nash's isometric embedding, see Subsec-

tion 2.4 for a detailed explanation). It started with works by C. De Lellis and L. Székelyhidi on the 3D incompressible Euler equations and some very weak solutions (modeling turbulence) and have now been extended to the incompressible Navier Stokes equations and other classical compressible fluids by T. Buckmaster and V. Vicol, P. Isset and others. One believes that the technique (from Nash) may be applied in many problems, much beyond those from physics and geometry (see Subsection 2.9 for more details).

- (2) Recent resolution of the Nadirashvili and Yau conjectures by A. Logunov [91, 92] and difficult new results by A. Logunov and E. Malinnikova [93] on the propagation of smallness and Remez estimates for solutions of the second order elliptic equations are truly remarkable.
- (3) Recent important breakthrough on the finite time blow up of $C^{1,\alpha}$ solutions of 3D incompressible Euler equations by T. Elgindi et al and its extensions, in particular the works by J. Chen and T. Hou and others. This combines with positive results by Seregin, Severak, H. Jia and others have opened a new door to this outstanding problem (see Subsection 2.3).
- (4) D. Christodolou have written two big papers (which resulted in two books of total pages close to 2000). One is on 3D compressible relativistic fluids and how smooth solutions develop shocks, the other is about compressible Euler and maximal Cauchy developments (singularities). One of these work have very recently been greatly simplified and improved by F. Merle, I. Rodinaski et al). And the other simplification with a much direct approach is done by T. Buckmaster and V. Vicol. These two recent work used two completely different approaches (see Subsection 2.6).

One observes that three of the above major developments are in some sense motivated by the millennium problem on the 3-D Navire-Stokes and Euler equations (see Subsection 2.8 for more details). While the idea in (1) uses Nash's iterative construction and higher and higher oscillations along with deep understandings of special flows and use them as building blocks. The work in (3) (it may also be used in part too in the work of Buckmaster-Vicol) uses the so called modulation method. The method in the work of F. Merle, I. Rodinanski and others (related to some earlier work by German physicists) involves studies of very classical and difficult autonomous ODE systems with rational nonlinearity. The delicate phase-plane analysis and the bifurcation-analysis involving infinite dimensions (parameters) are truly remarkable.

We have to say something about the modulation method (in some sense it is a very classical method) which has first used very successfully in the works by F.

Merle (over last 25 years) on KDV, Nonlinear Wave, Schrödinger equations. It has been applied for blow up analysis of solutions near critical masses. The method has now been used in works on the so-called Soliton-Resolution Conjecture (see Subsection 2.1). It is also used in the study of phase transitions, De Giorgi conjecture and related geometric problems (in particular, works by J. C. Wei and his collaborators), the finite time blow ups in the 2D harmonic map of heat flows (by J. C. Wei et al), and the finite time singularities in 2D-liquid crystal flows. It would be truly remarkable if one can extend these studies for constructions of finite time blow ups for the Navier-Stokes equations, we refer to the Subsection 2.7 for more explanations on this method.

At the round table, various discussions are made on several other directions that obviously worth much more and further studies:

- (a) Free boundary problems, linear and nonlinear, local and nonlocal (like SQG equation in Subsection 2.12), of the elliptic, parabolic and even hyperbolic (for example, the aforementioned work by Christodolou in some sense is to study shocks/free boundaries) nature. Also there is a large class of problems involving phase transitions in vector setting with more complexity and nonlinearity, and even with nonlinear constraints.
- (b) Control and Inverse problems often involve illposed nature and a lot of deep harmonic, microlocal and PDE analysis(see for instance Subsection 2.10 for the Lions' problem on the control of Navier-Stokes equations). One believes that such problems may be also relevant in applications in geometry, data analysis and nonlinear partial differential equations,..., particularly for the case that solutions may possess singularities.
- (c) Complex fluids and fluid + object (which can be a map such as for liquid crystal flows; it can be a vector like that for MHD; and it can be a deformation like that in nonlinear elasticity or viscoelastic fluids). In general, it can be a fluid + a geometric object of various dimensions, and its coupled flows can be of various type with applications in immersed boundary problems, evolution of active particles, neurons in electric-magnetic fields, phonetic dynamics.... we refer to Subsection 2.11 for more details.
- (d) Quantum physics and nonlinear Schrödinger equations and other equations from classical and quantum field theory are fundamentally important and mathematically fascinating. These problems also include studies of various focusing and oscillating phenomena. For examples, knotted-solitons, wave localizations or de-localizations, and their interactions and hydrodynamics are fascinating issues from both analysis and applications.

- (e) Microlocal analysis and geometric nature of PDE constrained measures. Recent geometric study of Tartar's H-measure [116] (or Gerard's microlocal measures [52]) by G. De Philippis et al [40], and some study on shock structure in general via GMT seem to be an interesting new direction that worth much further researches.

The above list is far from being exclusive. The only purpose to make this list of scientific issues and to have some discussions about is to attract attentions of our colleagues and friends in China to a larger picture of some of current developments. Based on these we also made following suggestions.

- (i) Beside one's own research interests (and dig deep in some particular problems which are very important), one must stand on a high ground with a very far and globally view. One can't miss the large picture and not to understand the relevance of one's own research in this large picture.
- (ii) The value of a research work at end must either solve a problem or to rise a new interesting aspect of it including that to develop a new theory. For example, for old problems one needs to ask new questions, and look at truly new challenges. Works are always better to have their intrinsic values. For example, many have looked at compressible 3D flows, however, recent major developments as described in item (3) tell us that one has to ask important and more fundamental questions.
- (iii) Training students and young researchers are most critical for changing the current status. It needs to start at the undergraduate and graduate educations. To provide excellent trainings from the undergraduate levels and all the way to the postdoctoral studies are most important for all of us as teachers and mentors.
- (iv) It is truly useful to organize in groups of students and post-doctors to study most important and high impact works with strikingly new ideas/results. Such organized working seminars have proved again and again in the past (for examples, the Chen and Su seminars at the old Zhejiang University in 1930s and 1940s, and its tradition had been preserved by the old Hangzhou University in 1980s and 1990s; and extended and improved by various top institutions in China these days) to be one of the most effective and valuable ways to guide our students. It is truly worth to invest our time and energy into this key task as it could be the most critical in changing the current status of our research field in China.

2 Some challenging problems

In this section, we shall present more details on some particular problems. Some have been mentioned in the previous section. For more concrete and precise presentations, I have invited: Chenjie Fan from University of Chicago to write Subsection 2.1 on the soliton resolution conjecture; Jun Geng from Lanzhou University to write Subsection 2.2 on homogenization; Yanlin Liu from The Chinese Academy of Sciences to write Subsection 2.3 on the blow-up solutions of 3-D Euler equations; Xinan Ma from University of Sciences and Technology of China to write Subsections 2.4 and 2.5; Shuang Miao from Wuhan University to write Subsection 2.6 on general relativity; Kelei Wang from Wuhan University to write Subsection 2.7 concerning modulation method in heat flow; Ping Zhang from The Chinese Academy of Sciences to write Subsections 2.8, 2.9, 2.10 and 2.11; Zhifei Zhang from Beijing University to write Subsection 2.12 concerning the well-posedness of SQG equation. I thank all of them for their contributions.

2.1 Dispersive equation: Soliton resolution Conjecture

Soliton resolution Conjecture is one major conjecture in the field of dispersive PDEs and predicts the generic behaviors for the so-called Type-II solutions for focusing nonlinear dispersive equations. Here, Type-II means certain critical Sobolev norm of the solution stays bounded within the evolution. Focusing means the nonlinearity competes with the linear part, and blow up behaviors are expected in such models. The conjecture suggests that all such solutions will asymptotically decouple to several solitary waves living at different scales plus a regular radiation term. This conjecture is widely open and particularly hard when the model is not integrable. Typical models covered by this conjecture include energy critical nonlinear wave equations, generalized KdV, mass critical nonlinear Schrödinger equations, and energy critical nonlinear Schrödinger equations.

For example, let u solves (focusing) energy critical nonlinear Schrödinger equations in \mathbb{R}^d

$$iu_t - \Delta u = |u|^{\frac{4}{d-2}} u. \quad (2.1)$$

Assuming u blows up at finite time T , and $\limsup_{t \rightarrow T} \|u(t, \cdot)\|_{\dot{H}^1} < \infty$, then as t approaches T , Soliton resolution Conjecture predicts that

$$u(t, x) = \sum_{j=1}^J \frac{1}{\lambda_j(t)^{\frac{d}{2}-1}} P_j \left(\frac{x - x_j(t)}{\lambda_j(t)} \right) e^{i\gamma_j(t)} + v(x) + r(t, x), \quad (2.2)$$

for some J in \mathbb{N} , $x_j(t) \in \mathbb{R}^d$, $\gamma_i(t) \in \mathbb{R}$, $\lambda_j(t) > 0$, $v(x) \in \dot{H}^1$ and $\lim_{t \rightarrow T} \|r(t, \cdot)\|_{\dot{H}^1} = 0$. And each $P_i \in \dot{H}^1$, and solves the following stationary energy critical NLS

$$-\Delta f = |f|^{\frac{4}{d-2}} f, \quad (2.3)$$

and

$$\frac{|x_j - x_{j'}|}{\lambda_j} + \left| \ln \frac{\lambda_j}{\lambda_{j'}} \right| \xrightarrow{t \rightarrow T} \infty, \quad \forall j \neq j'. \quad (2.4)$$

This is wide open even in the radial case and dimension $d = 3$. It should be noted that, the radial case are more approachable since there is essentially only one \dot{H}^1 solution to (2.3) and is explicit. Meanwhile, (2.3) admits infinite many non-radial solutions and they are yet to be classified. There is also a parallel conjecture for solutions to (2.1) which are global, where the $v(x)$ in (2.2) should be replaced by a solution $v(t, x)$ to linear Schrödinger equations.

The parallel conjecture for (nonradial) wave is of same spirits but slightly more complicated to state due to Lorentz symmetry. But we will state a precise version for radial wave below. Indeed, one of the most complete results in Soliton resolution Conjecture is on radial energy critical wave equations in odd dimensions, where this conjecture is confirmed.

Let u solves radial (focusing) energy critical nonlinear wave equation in \mathbb{R}^d

$$u_{tt} - \Delta u = |u|^{\frac{4}{d-2}} u, \quad (2.5)$$

and let W be the unique (up to scaling) positive radial \dot{H}^1 solutions to (2.3). Assuming u blows up at finite time T , and $\limsup_{t \rightarrow T} (\|u(t, \cdot)\|_{\dot{H}^1} + \|\partial_t u(t, \cdot)\|_{L_x^2}) < \infty$, then as t approaches T , the Soliton resolution Conjecture predicts that

$$\begin{aligned} u(t, x) &= \sum_{j=1}^J \frac{1}{\lambda_j^{\frac{d}{2}-1}(t)} \ell_j W\left(\frac{x}{\lambda_j(t)}\right) + v_0(x) + r_0(t, x), \\ \partial_t u(t, x) &= v_1(x) + r_1(t, x). \end{aligned} \quad (2.6)$$

For some J in \mathbb{N} , $\ell_j = \pm 1$, $\lambda_j(t) > 0$, $v(x) \in \dot{H}^1$, $v_1(x) \in L^2$ and $\lim_{t \rightarrow T} (\|r_0(t, \cdot)\|_{\dot{H}^1} + \|r_1(t, \cdot)\|_{L^2}) = 0$. and

$$\left| \ln \frac{\lambda_j}{\lambda_{j'}} \right| \xrightarrow{t \rightarrow T} \infty, \quad \forall j \neq j'. \quad (2.7)$$

This conjecture was proved first when $d = 3$, by Duyckaerts, Kenig and Merle, in [43]. And very recently, in the end of December 2019, Duyckaerts, Kenig and

Merle proved this conjecture for all $d \geq 3$ which is odd, in [44]. A parallel result for global solutions was also proved in [43,44]. Note that only for $d \geq 3$ can one make sense of energy critical. One may also refer to the introduction of those two papers for a lot of related works.

2.2 Homogenization

Partial differential equations and systems with rapidly oscillating coefficients are used to model various physical phenomena in inhomogeneous or heterogeneous media, such as composite and perforated materials. For a family of second-order linear operators in divergence form with rapidly oscillating periodic coefficients

$$\mathcal{L}_\uparrow = -\frac{\partial}{\partial x_i} \left[a_{ij}^{\alpha\beta} \left(\frac{x}{\varepsilon} \right) \frac{\partial}{\partial x_j} \right] = -\operatorname{div} \left[A \left(\frac{x}{\varepsilon} \right) \nabla \right], \quad \varepsilon > 0, \tag{2.8}$$

where the coefficient matrix $A(y) = (a_{ij}^{\alpha\beta}(y))$ is real, elliptic and periodic (i.e. $A(y+z) = A(y)$, $z \in \mathbb{Z}^d$, $y \in \mathbb{R}^d$). Under these assumptions, the operators \mathcal{L}_ε converges as $\varepsilon \rightarrow 0$ to a “homogenized” constant coefficient elliptic operator $\mathcal{L}_0 = -\operatorname{div}(\hat{A} \nabla)$ and the constant coefficient \hat{A} in fact can be explicitly computed [9]. During 1987-1991, in a series of papers [3–7], with extra assumption that A verifies the regularity hypothesis $\|A\|_{C^\alpha(\mathbb{R}^d)} \leq C$, Avellaneda and Lin obtained the uniform estimates such as boundary Lipschitz estimate and $W^{1,p}$ estimate etc. for the Dirichlet problem

$$\mathcal{L}_\uparrow(u_\varepsilon) = F \quad \text{in } \Omega, \quad u_\varepsilon = g \quad \text{on } \partial\Omega \tag{DP}$$

in $C^{1,\alpha}$ domain Ω . In particular, if $F=0$, they also proved that the non-tangential maximal function estimates

$$\|(u_\varepsilon)^*\|_{L^p(\partial\Omega)} \leq C \|g\|_{L^p(\partial\Omega)} \tag{2.9}$$

holds for any $1 < p < \infty$ with C independent of ε . In 1994, Kenig [66] conjectured that (2.9) holds for strongly elliptic, periodic, C^α systems and $p=2$ for arbitrary bounded Lipschitz domains $\Omega \subset \mathbb{R}^d$. And this problem was solved by Kenig and Shen [72] by using the layer potential method. One may notice that the Neumann problem

$$\mathcal{L}_\uparrow(u_\varepsilon) = F \quad \text{in } \Omega, \quad \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = g \quad \text{on } \partial\Omega \tag{NP}$$

is more difficult than the Dirichlet problem (DP) since the boundary conditions in (NP) are ε -dependent. Extending quantitative theory to homogenization of

elliptic systems with Neumann boundary conditions has been a longstanding open problem for nearly thirty years. In 2013, this was solved by Kenig, Lin and Shen [67–70] and [72].

Once the qualitative and quantitative homogenization of (DP) and (NP) are established, it is natural for us to be interested in the qualitative and quantitative homogenization theory for oblique derivative problem

$$\mathcal{L}_\uparrow(u_\varepsilon) = F \quad \text{in } \Omega, \quad \nabla u_\varepsilon \cdot \nu_\varepsilon = g \quad \text{on } \partial\Omega, \quad (\text{ODP})$$

where $\nu_\varepsilon(x)$ is nowhere tangential to the boundary $\partial\Omega$ and $\mathcal{L}_\uparrow = -a_{ij}(\frac{x}{\varepsilon}) \frac{\partial^2}{\partial x_i \partial x_j}$ is a second-order linear operators in non-divergence form with rapidly oscillating periodic coefficients. However, to my best knowledge, even the energy estimates for the oblique derivative problem is not trivial even when \mathcal{L}_\uparrow is the Laplacian Δ . Thus interesting challenges arise in dealing with (ODP). The natural question is: can one have the boundary Lipschitz estimates, boundary Hölder estimates, and $W^{1,p}$ estimates as well as the nontangential maximal function estimates (2.9)? One may see [17] and [71] for the nontangential maximal function estimates for oblique derivative problem of Laplace equation in Lipschitz domains.

2.3 $C^{1,\alpha}$ blow-up solutions of 3D incompressible Euler equations

Consider the incompressible Euler equations on \mathbb{R}^3 :

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = 0, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0, \end{cases} \quad (2.10)$$

where $u: [0, \infty) \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ stands for the velocity of the fluid flow, f for the external force, and p designates scalar pressure function, which guarantees the divergence free condition of the velocity field. The incompressibility ensures that any smooth solution u of (2.10) on $[0, T) \times \mathbb{R}^3$ satisfies

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |u(t, x)|^2 dx = \int_{\mathbb{R}^3} u(t, x) \cdot f(t, x) dx, \quad \forall t \in [0, T). \quad (2.11)$$

The question of global regularity for the solutions to (2.10) has been studied by numerous mathematicians and is considered to be one of the biggest open problems in the field of mathematical fluid mechanics. It dates back to 1920's that Gunther [55] and Lichtenstein [80] showed that if $u_0 \in C^{1,\alpha}(\mathbb{R}^3)$ for some $0 < \alpha < 1$ and the initial vorticity decays sufficiently rapidly at infinity, then there exists a

time $T > 0$ so that (2.10) has a unique solution $u \in C^{1,\alpha}([0, T] \times \mathbb{R}^3)$. We will call this solution “classical solution”. Later, similar results were established by many authors in Sobolev spaces, Besov spaces, etc.

The well-known Beale-Kato-Majda criterion [8] tells us that a classical solution loses its regularity at some finite time T if and only if

$$\lim_{t \rightarrow T} \int_0^t \|\omega(s, \cdot)\|_{L^\infty} ds = \infty, \tag{2.12}$$

where $\omega \stackrel{\text{def}}{=} \text{curl} u$ is the vorticity of the fluid, which satisfies

$$\partial_t \omega + u \cdot \nabla \omega = \omega \cdot \nabla u + \text{curl} f.$$

It is easy to use this criterion to show that the classical solution of 2-D Euler equation cannot develop singularity at finite time. Indeed, for 2-D case, the vortex stretching term $\omega \cdot \nabla u \equiv 0$, thus we have the following *a priori* bound

$$\|\omega(t, \cdot)\|_{L^\infty} \leq \int_0^t \|\text{curl} f(s, \cdot)\|_{L^\infty} ds.$$

However, this bound is not available for 3-D case, see [47]. What’s worse, recently Elgindi [46] showed that the condition (2.12) can be satisfied for some well-designed classical solutions. Precisely, Elgindi proved that:

Theorem 2.1. *There exists some sufficiently small $\alpha > 0$, a divergence-free and odd[†] $u_0 \in C^{1,\alpha}(\mathbb{R}^3)$ with $|\omega_0(x)| \leq \frac{C}{1+|x|^\alpha}$ for some constant $C > 0$, so that the unique local odd solution to (2.10) (with $f = 0$) belongs to $C^{1,\alpha}([0, 1] \times \mathbb{R}^3)$ and satisfies*

$$\lim_{t \rightarrow 1} \int_0^t \|\omega(s, \cdot)\|_{L^\infty} ds = \infty.$$

Elgindi’s work [46] is a very important progress in studying the global regularity for solutions to Euler equations. However, there are still several pitfalls. First, the solutions in Theorem 2.1 have infinite energy and do not satisfy the energy equality (2.11). But if one allows a uniformly $C^{1,\alpha}$ external force, then his construction can exactly give some finite energy solutions which blow up at finite time. Second, although the blow-up solutions here have $C^{1,\alpha}$ regularity, but this α has to be sufficiently small, so the following basic problem still remains open:

Open Problem: Given a solution $u \in C^\infty([0, T] \times \mathbb{R}^3)$ to (2.10) satisfying the energy equality (2.11) and the external force $f \in (C^\infty \cap L^2)([0, T] \times \mathbb{R}^3)$, is it possible that

$$\limsup_{t \rightarrow T} \|\nabla u(t, \cdot)\|_{L^\infty} = \infty?$$

[†]Here we say a vector field $u: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is odd if u_i is odd in x_i and even in the other two variables for each $1 \leq i \leq 3$.

2.4 Nash isometric imbedding

The problem of isometric embeddings of Riemannian manifolds consists of a global topological condition (being an embedding) and a system of partial differential equations (being an isometry). For concreteness let us consider a smooth n -dimensional Riemannian manifold (Σ^n, g) . A continuous map $u : \Sigma \rightarrow R^N$ is isometric if it preserves the length of curves, namely if

$$l_g(\gamma) = l_e(u(\gamma))$$

for any C^1 curve $\gamma \subset \Sigma$, where $g(\gamma)$ denotes the length of γ with respect to the metric g :

$$g(\gamma) = \int \sqrt{g(\gamma(t))[\dot{\gamma}(t), \dot{\gamma}(t)]} dt.$$

As customary, in local coordinates we can express the metric tensor g as $g = g_{ij} dx_i \otimes dx_j$. Then, if u is C^1 , is equivalent to a system of partial differential equations, which in local coordinates takes the following form:

$$d_i u \cdot d_j u = g_{ij}. \quad (2.13)$$

The existence of isometric immersions (resp. embeddings) of Riemannian manifolds into some Euclidean space is a classical problem, see Han-Hong [57]. Clearly, if the dimension of $\dim \Sigma = n$, (2.13) consists of $s_n := \frac{n(n+1)}{2}$ equations in N unknowns. A reasonable guess would therefore be that the system is solvable, at least locally, when $N = s_n$. In the first half of the twentieth century Janet, Cartan, and Burstin had proved the existence of local isometric embeddings in the case of analytic metrics (see [57]), precisely when $N = s_n$. For the very particular case of 2-dimensional spheres endowed with metrics of positive Gauss curvature, Weyl had raised the question of the existence of global isometric embeddings in R^3 . Weyl's problem was solved by Lewy for analytic metrics, and Louis Nirenberg settled the case of smooth metrics in his PhD thesis in 1949. A different proof was given independently by Pogorelov around the same time.

A short embedding is C^1 embedding (immersion) u such that $u^\sharp(e) \leq g$ as quadratic forms, or in local coordinates, $\partial_i u \cdot \partial_j u \leq g_{ij}$. The celebrated Nash-Kuiper theorem [76, 102] states that any short immersion or embedding $u : M^n \rightarrow R^{n+1}$ can be uniformly approximated by C^1 isometric immersions/embeddings. As a particular case, for the classical Weyl problem, i.e. $(S^2, g) \rightarrow R^3$ with positive Gauss curvature $K_g > 0$ this result implies the existence of a vast set of non-congruent C^1 surfaces, each isometric to (S^2, g) . This is in stark contrast with the situation for C^2 isometric embeddings: the famous rigidity theorem of

Cohn-Vossen and Herglotz states that the C^2 isometric embedding $(S^2, g) \rightarrow R^3$ is uniquely determined up to congruencies.

Nash [103] proves the more expected version of smooth isometric embeddings from M^n to a sufficiently high-dimensional Euclidean space. For C^∞ imbeddings the best such result was later proved by Gromov [53].

Theorem 2.2. (Nash (1956) [103] and Gromov (1970) [53]). *Any short embedding $u : M^n \rightarrow R^q$ can be uniformly approximated by isometric embeddings (immersions) provided that $q \geq \frac{(n+2)(n+3)}{2}$.*

The question of what happens in between the rigid C^2 case and the highly nonrigid C^1 case on the Hölder scale $C^{1,\theta}$ has a long history. In the 1950s in a series of papers by Borisov [10], building upon the work of Pogorelov [107], showed that the rigidity of convex surfaces prevails for $\theta > 2/3$. More recently there has been intensive work on lowering the rigidity exponent [37], one conjecture (see [54]) being that some form of rigidity should hold for all $\theta > 1/2$.

Problem. Let $N=3=n+1$. Is there a threshold $\theta_0 \in (0,1)$ such that: $C^{1,\theta}$ solutions of the Weyl problem are rigid for $\theta > \theta_0$ and the Nash-Kuiper Theorem holds for $C^{1,\theta}$ immersions when $\theta < \theta_0$?

De Lellis and Székelyhidi (see Subsection 2.9) have pointed out the celebrated conjecture of Onsager in the theory of fully developed turbulence shares many similarities with this problem.

We should mention other questions:

1. For more readable partial differential equations proof in Pogorelov [107] theorem on the rigidity of convex surface;
2. Local embedding for variable curvature [57] in 2 dimensional Riemannian manifold.

2.5 Unique continuation

The classical technique to get the unique continuation for elliptic partial differential equations is Carleman estimates, one can see the reference from [49]. From 1980s, an important tool to study unique continuation, nodal sets of eigenfunctions and growth properties of solutions of elliptic PDEs is the so called **frequency function**. The idea goes back to works of Almgren and Agmon. It was developed further by Garofalo and Lin [51], Let $A(x)$ be a symmetric uniformly

elliptic matrix with Lipschitz coefficients defined on some ball B_r centered at the origin and such that $A(0) = I$. Let further

$$\mu(x) = \frac{(A(x)x, x)}{|x|^2}, \quad \mu(0) = 1, \quad \Lambda^{-1} \leq \mu(x) \leq \Lambda.$$

Let u be a solution to the equation $\operatorname{div}(A(x)\nabla u(x)) = 0$. We consider weighted averages of $|u|^2$ over spheres:

$$H(r) = r^{1-d} \int_{\partial B_r} \mu(x) |u(x)|^2 ds(x).$$

And we define

$$I(r) = r^{1-d} \int_{B_r} (A\nabla u, \nabla u) dx,$$

then for the frequency function

$$N(r) = r \frac{I(r)}{H(r)}.$$

There exists C that depends only on the ellipticity and Lipschitz constants of the operator such that for any solution u to $\operatorname{div}(A(x)\nabla u(x)) = 0$, the function $e^{Cr}N(r)$ is an increasing function of r .

A consequence of the monotonicity of the frequency function is the so called three sphere theorem. Its simplest version is the classical Hadamard three circle theorem for analytic functions. One can derive the three spheres from the properties of the frequency function following.

In recent Logunov and Malinnikova [93], they got an big improvement. Let Ω will be a bounded domain in R^n and u will denote a solution of an elliptic equation in the divergence form $\operatorname{div}(A\nabla u) = 0$ in Ω with Lipschitz coefficients. Let E and K be subsets of Ω such that the distances from E and K to $\partial\Omega$ are positive. We assume that E has positive n -dimensional Lebesgue measure. They prove the following estimate

$$\sup_K |u| \leq C (\sup_E |u|)^\gamma (\sup_\Omega |u|)^{1-\gamma}, \quad (2.14)$$

where $C > 0$ and $\gamma \in (0, 1)$ are independent of u , but depend on Ω , A , the measure of E , and the distances from K and E to the boundary of Ω . This is the quantitative results on propagation of smallness for solutions of elliptic PDE.

On the way of proving (2.14) they obtain an interesting inequality for solutions of elliptic equations, which reminds the classical **Remez inequality** for polynomials, the role of the degree is now played by the doubling index. Let Q be

a unit cube. Assume u is a solution to $\operatorname{div}(A\nabla u) = 0$ and the **doubling index** $N = \log \frac{\sup_{2Q} |u|}{\sup_Q |u|}$. Then

$$\sup_Q |u| \leq C \sup_E |u| \left(C \frac{|Q|}{|E|} \right)^{CN}, \tag{2.15}$$

where C depends on A only, E is any subset of Q of a positive measure.

Mainly by the important work by Logunov and Malinnikova in recent five years, they introduce new combinatorial methods and obtain some deep theorems. We mention three well known conjecture.

Conjecture 1 (Yau). Let M be a smooth compact d -dimensional Riemannian manifold. There exist constants C_1 and C_2 , which depend on M , such that

$$C_1 \sqrt{\lambda} \leq H^{d-1}(Z(\varphi_\lambda)) \leq C_2 \sqrt{\lambda},$$

for any eigenfunction φ_λ satisfying $\Delta_M \varphi_\lambda + \lambda \varphi_\lambda = 0$.

One can see the recent work by Logunov [91, 92] an he prove the following result

$$C_1 \sqrt{\lambda} \leq H^{d-1}(Z(\varphi_\lambda)) \leq C_2 \lambda^\alpha,$$

and $\alpha \geq \frac{1}{2}$.

Conjecture 2 (Fang-Hua Lin). Let u be a non-zero harmonic function in the unit ball $B_1 \subset R^n, n \geq 3$. Consider $N = \log \frac{\sup_{B_1} |\nabla u|}{\sup_{B_{\frac{1}{2}}} |\nabla u|}$. Is it true that the Hausdorff measure

$$H^{n-2}(\{\nabla u = 0\} \cap B_{\frac{1}{2}}) \leq C_n N^2$$

for some C_n depending only on the dimension?

As mentioned by Logunov and Malinnikova [93], it is connect to propagation of smallness for gradients of the harmonic function.

The last conjecture is connected to the quantitative version of the Cauchy uniqueness problem, it dates back to at least L. Bers. The two-dimensional case is not difficult due to connections with complex analysis. The fact that the question is open in higher dimensions shows that we still don't understand well the Cauchy uniqueness problem even for ordinary harmonic functions in the dimension three or higher.

Conjecture 3. Assume that u is a harmonic function in the unit ball $B_1 \subset R^3$ and u is $C^{+\infty}$ -smooth in the closed ball \bar{B}_1 . Let $S \subset \partial B_1$ be any closed set with positive area. Is it true that $\nabla u = 0$ on S implies $\nabla u \equiv 0$?

For the class $C^{1+\varepsilon}(B_1)$ there is a striking counterexample [12], which however is not $C^{+\infty}$ -smooth up to the boundary. The attempts to construct C^2 -smooth counterexamples were not successful.

2.6 General relativity

A central mathematical problem in theory of General Relativity is to show the nonlinear stability of Kerr spacetimes whose metrics are axisymmetric, stationary, asymptotical flat solutions to the vacuum Einstein equations. From the experience of proving nonlinear stability of Minkowski spacetime [23], to show the nonlinear stability of Kerr spacetimes, one needs to start with a proper linearized system of vacuum Einstein equations and prove polynomial decay for linearized gravity.

As a special case, different approaches towards linear stability of the vacuum, spherically symmetric Schwarzschild spacetimes has been developed in [33, 60]. Both works start from Regge-Wheeler type equations where the technique of treating scalar wave equation can be applied. In particular, spherical symmetry is crucial in the latter approach. A nontrivial nonlinear stability result of Schwarzschild spacetimes under polarized axisymmetry is obtained in [74].

In the case of Kerr spacetimes, Teukolsky found in a seminal work [117] that the extreme curvature components are governed by a separable, decoupled wave equation – Teukolsky Master Equation (TME), and they fully determine the dynamics of linearized gravity up to linearized mass and angular momentum perturbations plus pure gauge solutions. Moreover, these extreme curvature components are gauge invariant, hence as a beginning step, one can treat them using TME without imposing any gauge choice. Energy and decay estimates for these extreme curvature components are shown in [34, 94] for slowly rotating Kerr backgrounds ($|a|/M \ll 1$) by applying a suitable modification of *Chandrasekhar's transformations* [20]. Based on these estimates, the authors in [2] derived strong decay estimates for TME and obtained a linear stability result of slowly rotating Kerr metrics in an outgoing radiation gauge. See also [56] for a linear stability proof but using a microlocal approach.

2.7 Modulation method in heat flow

In many nonlinear evolution equations, the blow up of solutions is caused by the formation of bubbles, especially when the blow up is of Type II. For example, if $u: \mathbb{R}^2 \times [0, T) \mapsto \mathbb{S}^2 \subseteq \mathbb{R}^3$ solves the harmonic map heat flow

$$u_t - \Delta u = |\nabla u|^2 u, \quad (2.16)$$

and T is the blow up time, under some technical assumptions, it is shown that (see Qing [109], Ding-Tian [42], Lin-Wang [84] and Qing-Tian [108]), as $t_n \rightarrow T$, there exist finitely many points q_1, \dots, q_k such that

$$u(x, t_n) \rightarrow u_*(x) + \sum_{i=1}^k \left[W_i \left(\frac{x - q_i^n}{\lambda_i^n} \right) - W_i(\infty) \right] \quad \text{in } H^1,$$

where u_* is the trace of u at $t = T$, $q_i^n \rightarrow q_i$, $0 < \lambda_i^n \rightarrow 0$, and W_i are *bubbles*, i.e. finite energy solutions of

$$-\Delta W_i = |\nabla W_i|^2 W_i \quad \text{in } \mathbb{R}^2.$$

One crucial point behind this bubbling phenomena is the scaling invariance of this problem. This induces the concentration of energy, which is an important class of the loss of compactness phenomena, a topic studied intensively in calculus of variations during the last two decades.

The construction of blow up solutions is of great interest, especially for the understanding of Type II blow up. For harmonic map heat flow, Chang, Ding and Ye [21] first constructed solutions exhibiting finite time blow up, by using a special ansatz and then reducing the equation to a simpler one. However, this does not give too much information about the blow up behavior. In [111, 112], Raphaël and Schweyer rigorously constructed blow up solutions by using *the modulation method*. By this method they get a precise control on the blow up rate, blow up profile as well as the stability of this blow up mechanism.

In the modulation method, by noting that bubbles are the main order term when approaching the blow up time, one modulates the standard bubbles to get an approximation to the original solution. For example, for (2.16) one takes the decomposition

$$u(x, t) = u_*(x) + \sum_{i=1}^k \left[W_i \left(\frac{x - q_i(t)}{\lambda_i(t)} \right) - W_i(\infty) \right] + \mathcal{R}.$$

Here $q_i(t)$ and $\lambda_i(t)$ are modulation parameters, \mathcal{R} is a small error term. By a suitable choice of modulation parameters, one obtains some good estimates on \mathcal{R} . (This is because \mathcal{R} satisfies a linearized equation.) Furthermore, the modulation parameters also satisfy a modulation equation (here an ODE), which can be used to derive the asymptotic behavior of the blow up rate (that is, $\lambda_i(t)$ in the above equation) as $t \rightarrow T$.

The modulation method is applicable to many other problems, for example,

- mass critical Schrödinger equations (Merle [99]);

- energy critical Schrödinger equations (Merle, Raphaël and Schweyer [100]);
- nonlinear wave equations and Yang-Mills equation (Raphaël and Schweyer [110], Rodnianski and Sterbenz [113]);
- and energy supercritical Schrödinger equations (Merle, Raphaël and Rodnianski [101]).

Recently, in a series of work (see e.g. [32,35,36]), Del Pino, Wei and their collaborators, based on their earlier works in various elliptic equations (e.g. [105,106]), introduced the *inner-outer gluing method* for the construction of blow up solutions. Here one takes a further decomposition of the error term \mathcal{R} according to the scale, one on the scale of bubbles, which forms the inner problem, and the other one on the original scale, which forms the outer problem. With this decomposition they can treat the bubbling formation at multiple points and without any symmetry.

The modulation method is also useful in the study of “Soliton resolution Conjecture” (see Subsection 2.1), especially when the solution is near the ground state, see e.g. [24,95].

2.8 The Navier-Stokes equation (uniqueness/non-uniqueness, finite-time blowup)

d (2 or 3) dimensional incompressible Navier-Stokes equations can be written as follows:

$$\begin{cases} \partial_t u + u \cdot \nabla u - \nu \Delta u = -\nabla p + f, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^d, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0, \quad \text{with} \quad \operatorname{div} u_0 = 0, \end{cases} \quad (2.17)$$

where $u = (u_1, \dots, u_d)$ denotes the fluid velocity, $u \cdot \nabla u = \sum_{j=1}^d u_j \partial_j u$, p represents the scalar pressure function, ν the viscous coefficient and f the force term.

For any initial data u_0 of finite kinetic energy, Leray [79] proved in 1934 that (2.17) has at least one global in time finite energy weak solution, which satisfies the energy inequality

$$\frac{1}{2} \|u(t)\|_{L^2}^2 + \nu \|u\|_{L_t^2(L^2)}^2 \leq \frac{1}{2} \|u_0\|_{L^2}^2. \quad (2.18)$$

This solution is unique and regular in two space dimension as long as the time $t > 0$. Hopf [59] established a similar result for (2.17) posed on smooth bounded domain with Dirichlet boundary condition. Yet the question of uniqueness and

regularity of Leray-Hopf weak solutions for 3D Navier-Stokes system remains to be one of the biggest open problems (see [48]) in mathematical fluid mechanics.

Fujita-Kato [50] constructed local in time unique solution to (2.17) with initial data in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$. Furthermore, if $\|u_0\|_{\dot{H}^{\frac{1}{2}}}$ is sufficiently small, then the solution exists globally in time (see [65] for similar result with initial data in $L^3(\mathbb{R}^3)$). This result was extended by Cannone, Meyer and Planchon [18] for initial data belonging to $\dot{B}_{p,\infty}^{-1+\frac{3}{p}}(\mathbb{R}^3)$ with $p \in (3, \infty)$. The end-point result in this direction is given by Koch and Tataru [75]. They proved that given initial data being sufficiently small in $\text{BMO}^{-1}(\mathbb{R}^3)$, then (2.17) has a unique global solution. We remark that for $p \in (3, \infty)$, there holds

$$\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3) \hookrightarrow \dot{B}_{p,\infty}^{-1+\frac{3}{p}}(\mathbb{R}^3) \hookrightarrow \text{BMO}^{-1}(\mathbb{R}^3) \hookrightarrow \dot{B}_{\infty,\infty}^{-1}(\mathbb{R}^3),$$

and the norms to the above spaces are scaling-invariant under the following transformation:

$$u_\lambda(t, x) = \lambda u(\lambda^2 t, \lambda x) \quad \text{and} \quad u_{0,\lambda}(x) = \lambda u_0(\lambda x). \quad (2.19)$$

We notice that for any solution u of (2.17) on $[0, T]$, u_λ determined by (2.19) is also a solution of (2.17) on $[0, T/\lambda^2]$. We remark that the largest space, which belongs to $\mathcal{S}'(\mathbb{R}^3)$ and the norm of which is scaling invariant under (2.19), is $\dot{B}_{\infty,\infty}^{-1}(\mathbb{R}^3)$ (see [98]). Moreover, Bourgain and Pavlović [11] proved that (2.17) is actually ill-posed with initial data in $\dot{B}_{\infty,\infty}^{-1}(\mathbb{R}^3)$. That is the reason why people call such kind of initial data, with the $\dot{B}_{\infty,\infty}^{-1}(\mathbb{R}^3)$ norm of which is large, as large initial data.

On the other hand, in [62, 63], Jia and Šverák proved the non-uniqueness of Leray-Hopf weak solutions in the regularity class $L_t^\infty(L^{3,\infty})$ if a certain spectral assumptions for a linearized Navier-Stokes operator holds. Very recently, Buckmaster and Vicol [14] proved the following very interesting result:

Theorem 2.3 (Theorem 1.2 of [14]). *There exists $\beta > 0$, such that for any non-negative smooth function $e(t): [0, T] \rightarrow \mathbb{R}^+$, (2.17) has a weak solution $u \in C^0([0, T]; H^\beta(\mathbb{T}^3))$ such that $\int_{\mathbb{T}^3} |u(t, x)|^2 dx = e(t)$ for all $t \in [0, T]$. Moreover, the associated vorticity $\nabla \times u$ lies in $C^0([0, T]; L^1(\mathbb{T}^3))$.*

We remark that the weak solution of (2.17) constructed in Theorem 2.3 is still not Leray-Hopf solutions, which do not satisfy the energy inequality (2.18). The main idea of the proof is based on the pioneering papers [38, 39] by De Lellis and Székelyhidi.

2.9 The Onsager conjecture

In 1949, Onsager [104] conjectured that there exists a threshold regularity for the conservation of energy to the incompressible Euler equations (2.10). In particular, for the weak solution u , which is Hölder continuous in space, i.e.

$$|u(t,x) - u(t,y)| \leq C|x-y|^\beta \quad \text{for all } t \in [0, T] \quad (2.20)$$

for some constant C that is independent of time t . He asserted that for any Hölder continuous weak solution u of (2.10) with exponent $\beta > \frac{1}{3}$, the energy should be conserved; whereas for any weak solution with smaller exponent, there are solutions that do not conserve energy. The first assertion was proved by Constantin, E and Titi in [25]. Concerning the second assertion, the first proof of the existence of a square integrable weak solution that do not preserve the energy is due to Scheffer in his pioneering paper [114] and a different proof was given by Shnirelman in [115].

On the other hand, De Lellis and Székelyhidi observed in [38] that the techniques from the theory of differential inclusions could be applied to construct bounded weak solutions that violate the energy conservation. After a series of important partial results improving the threshold, Isett [61] has been able to finally reach the Onsager exponent $\frac{1}{3}$. One may check [13] for recent progress in this direction.

2.10 Control of Navier-Stokes system

In the late 1980's, Jacques-Louis Lions introduced in [86] (see also [87–89]) the question of the controllability of fluid flows in the sense of how the Navier-Stokes system can be driven by a control of the flow on a part of the boundary to a wished plausible state, say a vanishing velocity. Lions' problem has been solved in [30] by Coron, Marbach and Sueur in the particular case of the Navier slip-with-friction boundary condition. In its original statement with the no-slip Dirichlet boundary condition, it is still an important open problem in fluid controllability.

In the special case when the space dimension is two and the geometric domain

$$\Omega := (0, L) \times (-1, 1),$$

where $L > 0$ is the length of the domain. Inside this domain, a fluid evolves under the Navier-Stokes equation, that is, the velocity field u satisfies:

$$\begin{cases} \partial_t u + (u \cdot \nabla) u + \nabla p - \Delta u = f_g, \\ \operatorname{div} u = 0, \end{cases} \quad (2.21)$$

in Ω , where p denotes the fluid pressure and f_g a force term. One may think of this domain as a river or a tube and assume that we can act on the fluid flow at both end boundaries:

$$\Gamma_0 := \{0\} \times (-1,1) \quad \text{and} \quad \Gamma_L := \{L\} \times (-1,1).$$

On the remaining parts of the boundary,

$$\Gamma_{\pm} := (0,L) \times \{\pm 1\},$$

we assume that we cannot control the fluid flow and that it satisfies null Dirichlet boundary conditions:

$$u = 0 \quad \text{on} \quad \Gamma_{\pm}. \tag{2.22}$$

Let $L^2_{\text{div}}(\Omega)$ be the space of divergence free vector fields in $L^2(\Omega)$, which are tangential to the boundaries Γ_{\pm} . Lately, Coron, Mabach, Sueur and Zhang proved the following result in [31]:

Theorem 2.4 ([31]). *Let $T > 0$ and u_* be in $L^2_{\text{div}}(\Omega)$. For any $k \in \mathbb{N}$ and for any $\eta > 0$, there exists a force $f_g \in L^1((0,T); H^k(\Omega))$ satisfying*

$$\|f_g\|_{L^1((0,T); H^k(\Omega))} \leq \eta \tag{2.23}$$

and an associated weak Leray solution $u \in C^0([0,T]; L^2_{\text{div}}(\Omega)) \cap L^2((0,T); H^1(\Omega))$ to (2.21) and (2.22) satisfying $u(0) = u_*$ and $u(T) = 0$.

We remark that Lions' problem in this special situation corresponds to $f_g = 0$ in (2.21). Hence even for this particular case, Lions' problem has still not been solved completely.

2.11 Complex fluid system and MHD system

One of the common origins and manifestations of anomalous phenomena in complex fluids are different "elastic" effects. Most complex fluids are indeed viscoelastic. It is the interaction between the elastic properties and the fluid motions that gives not only the complicated rheological phenomena, but also formidable challenges in analysis, modeling and numerical simulations. For a general viscoelastic fluids, we have the following equation for conservation of momentum:

$$\rho(\partial_t u + u \cdot \nabla u) = \nabla \cdot \tau, \tag{2.24}$$

where τ is the total stress.

In the case of linear elasticity, (2.24) writes

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \mu \Delta u + \nabla \cdot (FF^T) & \text{in } \mathbb{R}_+ \times \mathbb{R}^d, \\ \operatorname{div} u = 0, \\ F_t + u \cdot \nabla F = \nabla u F, \\ (u, F)|_{t=0} = (u_0, F_0), \end{cases} \quad (2.25)$$

where u denotes the fluid velocity and F the deformation matrix.

Mathematically, Lin et al. [82] first proved the global existence of smooth solutions to (2.25) in two space dimension when the initial data (u_0, F_0) is a small perturbation of the trivial state $(0, Id_{2 \times 2})$. Then a similar result was generalized to three space dimension in [22, 77]. In general, the global existence or finite time singularity of solutions, even the global existence of weak solution to (2.25), is still open except the special case in [90].

In the case when the deformation matrix F in (2.25) is reduced to a vector-valued function, the equation becomes incompressible MHD system without resistance:

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \mu \Delta u + b \cdot \nabla b & \text{in } \mathbb{R}_+ \times \mathbb{R}^d, \\ \operatorname{div} u = \operatorname{div} b = 0, \\ \partial_t b + u \cdot \nabla b = b \cdot \nabla u, \\ (u, b)|_{t=0} = (u_0, b_0), \end{cases} \quad (2.26)$$

where b denotes the magnetic field. Starting with [85] for a modified model of (2.26), Lin et al. proved the global existence of solutions to (2.26) with initial velocity being sufficiently small and the initial magnetic field being sufficiently close to non-zero constant vector. The same result for (2.26) was proved in [1] and an optimal decay rates for such solutions was obtained in [41]. He et al. proved the vanishing viscosity limit of the viscous MHD system in [58] (see also [16, 118]).

We remark that hydrodynamical and rheological properties of complex fluids depend intimately on their molecular conformation and configurations. The hydrodynamics of these materials are described by the coupled micro-macro models. A model system is given by

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \mu \Delta u + \nabla \cdot \tau & \text{in } \mathbb{R}_+ \times \Omega, \\ \operatorname{div} u = 0, \\ \partial_t f + u \cdot \nabla f = \Delta_q f - \nabla_q \cdot (\nabla u q f - \nabla_q U f) & \text{in } \mathbb{R}_+ \times \Omega \times B(0, R), \\ (u, f)|_{t=0} = (u_0, f_0), \end{cases} \quad (2.27)$$

where $U(q) = U(|q|^2)$ and $\tau = \int_{B(0, R)} \nabla_q U \otimes q f dq$.

Lin et al. [83] proved the global existence of smooth solutions to (2.27) for initial data near equilibrium. For the so-called polymeric fluids, the local existence of smooth solutions was established in [45, 64, 96]. Masmoudi proved the global existence of weak solution in [97].

One may refer to the survey paper [81] by Lin for more details in this direction.

2.12 SQG equations

The surface quasi-geostrophic (SQG) equation is an important model in geophysical fluid dynamics. It was proposed as a two dimensional model to study the singularity formation of the inviscid incompressible flows [26]. The SQG equation with the dissipation takes

$$\partial_t \theta + u \cdot \nabla \theta + \kappa (-\Delta)^\alpha \theta = 0. \tag{2.28}$$

Here $\alpha \in [0, \frac{1}{2}]$, $\kappa > 0$ is the dissipative coefficient, $\theta(t, x)$ is a real-valued function of t and x . The function θ represents the potential temperature, the fluid velocity u is determined from θ by a stream function ψ

$$(u_1, u_2) = \left(-\frac{\partial \psi}{\partial x_2}, \frac{\partial \psi}{\partial x_1} \right), \quad (-\Delta)^{\frac{1}{2}} \psi = -\theta. \tag{2.29}$$

The fractional Laplacian $(-\Delta)^\alpha$ is defined by

$$\widehat{(-\Delta)^\alpha f(\xi)} = |\xi|^{2\alpha} \hat{f}(\xi).$$

Due to maximum principle, $\|\theta(t)\|_{L^\infty} \leq \|\theta(0)\|_{L^\infty}$. Formally, nonlinear term $u \cdot \nabla \theta$ behaves as $|D|\theta^2$. Thus, in the case of $\alpha > \frac{1}{2}$, the linear diffusion term $(-\Delta)^\alpha$ will dominate nonlinear term so that (2.28) has a global in time smooth solution for smooth data [27]. The SQG equation with $\alpha = \frac{1}{2}$ is called the critical SQG. The global well-posedness of the critical SQG was independently proved by Caffarelli and Vasseur [15], and Kiselev, Nazarov and Volberg [73]. The work [15] used the harmonic extension and the De Giorgi method, and the work [73] used the method of constructing the moduli of continuity. Recently, Constantin and Vicol gave a new proof via the so-called nonlinear maximum principles [28]. Global regularity or finite time singularity for the super critical SQG with $\alpha \in [0, 1)$ remains a challenging problem. Let us refer to [19, 29] for some important progress in this direction.

Acknowledgments

This article is initiated during a round table discussion at International Center for Mathematics in Southern University of Science and Technology (SUSTech) held in the January 2020. I am grateful to the wonderful organization and hospitality provided by Professors Tao Tang, Xiaoming Wang and Linlin Su. In the course of the preparation of this work, several colleagues have contributed to the scientific part of this review article. Here I would like to thank particularly: Chenjie Fan from University of Chicago, Jun Geng from Lanzhou University, Yanlin Liu from The Chinese Academy of Sciences, Xinan Ma from University of Sciences and Technology of China, Shuang Miao and Kelei Wang from Wuhan University, Ping Zhang from The Chinese Academy of Sciences and Zhifei Zhang from Beijing University. Their enthusiasm and passion have greatly improved the presentation and the quality of this work.

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