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Gradient Flow of the L_β-Functional

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Abstract. In this paper, we start to study the gradient flow of the functional L_β introduced by Han-Li-Sun in [8]. As a first step, we show that if the initial surface is symplectic in a Kähler surface, then the symplectic property is preserved along the gradient flow. Then we show that the singularity of the flow is characterized by the maximal norm of the second fundamental form. When $\beta=1$, we derive a monotonicity formula for the flow. As applications, we show that the λ-tangent cone of the flow consists of the finite flat planes.

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1 Introduction

Suppose that M is a Kähler surface. Let ω be the Kähler form on M and let J be a complex structure compatible with ω. The Riemannian metric $\langle\cdot,\cdot\rangle$ on M is defined by

$$\langle U,V\rangle=\omega(U,JV).$$

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For a compact oriented real surface Σ which is smoothly immersed in M, one defines, following [5], the Kähler angle α of Σ in M by

$$\omega|_{\Sigma}=\cos\alpha d\mu_{\Sigma}, \tag{1.1}$$

where $d\mu_{\Sigma}$ is the area element of Σ of the induced metric from $\langle\cdot,\cdot\rangle$. We say that Σ is a holomorphic curve if $\cos\alpha\equiv 1$, Σ is a Lagrangian surface if $\cos\alpha\equiv 0$ and Σ is a symplectic surface if $\cos\alpha>0$.

The existence of holomorphic curves in a Kähler surface is a fundamental problem in differential geometry. Since holomorphic curves are always area-minimizing in its homological class due to the Wirtinger inequality, we see that holomorphic curves are all stable symplectic minimal surfaces. Wolfson [18] showed that a symplectic minimal surface in a Kähler-Einstein surface with non-negative scalar curvature must be holomorphic. Thus, we can look for the holomorphic curves by finding the symplectic minimal surfaces in this case.

Furthermore, Chen-Li [3] and Wang [17] showed that symplectic property is preserved along the mean curvature flow. Therefore, an idea approaching the existence of holomorphic curves is to looking for symplectic minimal surfaces using the mean curvature flow starting from a symplectic surface, which we call symplectic mean curvature flow. There are some interesting results on the study of symplectic mean curvature flow. For instance, Chen-Li [3] and Wang [17] showed that there is no Type I singularities for such a flow at the finite time. However, since the flow is of codimension two and the normal bundle is much more complex, it is hard to clear out all singularities. On the other hand, C. Arezzo [2] constructed examples which shows that a strictly stable minimal surface in a Kähler-Einstein surface with negative scalar curvature may not be holomorphic.

For this reason, we introduce a new idea to approach the existence of holomorphic curves using variational method combined with the continuity method. More precisely, we consider a sequence of functionals [8]

$$L_\beta=\int_\Sigma \frac{1}{\cos^\beta\alpha}d\mu, \tag{1.2}$$

where $\beta\geq 0$. The functional L_1 was introduce by Han-Li in [7]. The critical point of the functionals L_β in the class of symplectic surfaces in a Kähler surface is called a β-symplectic critical surface. We have proved that (cf., [8]) the Euler-Lagrange equation of the functional L_β is

$$\cos^3\alpha\mathbf{H}-\beta\big(J(J\nabla\cos\alpha)^\top\big)^\perp=0, \tag{1.3}$$

where $\mathbf{H}$ is the mean curvature vector of Σ in M, and $(\cdot)^\top$ means tangential components of $(\cdot)$, $(\cdot)^\perp$ means the normal components of $(\cdot)$. It is clear that holomorphic curves are β-symplectic critical surfaces for each β. When $\beta=0$, the functional is exactly the area functional, and 0-symplectic critical surface is exactly minimal surface. We aim to deform, from 0-symplectic critical surface (i.e., a minimal surface) to a holomorphic curve when β tends to infinity. We showed the openness (cf., [8]) and partial results on the compactness (cf., [9]) of the program.

The first step is about the existence of β-symplectic critical surface for each fixed β. An natural idea is to consider the negative gradient flow of the functional L_β. For this purpose, let us recall the first variation formula for L_β.

Theorem 1.1 ([8]). *Let M be a Kähler surface. The first variational formula of the functional L_β is, for any smooth normal vector field $\boldsymbol{X}$ on Σ,*

$$\delta_X L_\beta = -(\beta+1)\int_\Sigma \frac{\boldsymbol{X}\cdot\boldsymbol{H}}{\cos^\beta\alpha}d\mu + \beta(\beta+1)\int_\Sigma \frac{\boldsymbol{X}\cdot\big(J(J\nabla\cos\alpha)^\top\big)^\perp}{\cos^{\beta+3}\alpha}d\mu, \tag{1.4}$$

where $\boldsymbol{H}$ is the mean curvature vector of Σ in M.

Now, we will consider the negative gradient flow of L_β, i.e.,

$$\frac{dF}{dt} = \cos^2\alpha\mathbf{H} - \frac{\beta}{\cos\alpha}\big(J(J\nabla\cos\alpha)^\top\big)^\perp. \tag{1.5}$$

We set

$$\mathbf{f} = \cos^2\alpha\mathbf{H} - \frac{\beta}{\cos\alpha}\big(J(J\nabla\cos\alpha)^\top\big)^\perp.$$

It is clear that $\mathbf{f}\equiv 0$ if and only if Σ is a β-symplectic critical surface.

By the first variation formula, we see that along the flow (1.5),

$$\begin{aligned}\frac{dL_\beta}{dt} &= -(\beta+1)\int_\Sigma \frac{1}{\cos^{\beta+2}\alpha}\left|\cos^2\alpha\mathbf{H} - \frac{\beta}{\cos\alpha}\big(J(J\nabla\cos\alpha)^\top\big)^\perp\right|^2 d\mu \\ &= -(\beta+1)\int_\Sigma \frac{1}{\cos^{\beta+2}\alpha}|\mathbf{f}|^2 d\mu. \end{aligned}\tag{1.6}$$

Thus the flow (1.5) is a gradient flow of the functional L_β. From [8, Proposition 3.1], we know that the flow (1.5) is a parabolic system, the short time existence can be shown by a standard argument. The first step to study the flow (1.5) is to show that symplectic property is preserved along this flow. This is exactly the first result in this paper.

Theorem 1.2. *Symplectic property is preserved along the flow* (1.5).

Since we aim to use the flow (1.5) to derive β-symplectic critical surfaces, one natural question is when the solution can be extended. For mean curvature flow

$$\frac{dF}{dt}=\mathbf{H}, \tag{1.7}$$

Huisken [10] showed that the flow can be extended if the second fundamental form of Σ in M is uniformly bounded. We can show that the same conclusion holds for the flow (1.5) (see Theorem 3.1).

During the study of the geometric flows, singularity analysis is an important subject. An important tool to study the singularity is the monotonicity formula. In this paper, we derive the monotonicity formula for the case $\beta=1$. For general β, the parabolic operator associated with the flow (1.5) is $\frac{\partial}{\partial t}-(\cos^2\alpha+\beta\sin^2\alpha)\Delta$, which reduces to $\frac{\partial}{\partial t}-\Delta$ when $\beta=1$. This case is easier to be handled. Although we prove the monotonicity formula for $\beta=1$, we believe that the monotonicity formula also holds for general β.

When the singularity occurs, we can rescale the flow near the singular point and obtain some limiting model in some sense, which we call λ-tangent cones. Understanding the behaviours of the tangent cones is crucial to study the flow. For mean curvature flow, from Huisken's monotonicity formula [11], we know that the tangent flow are self-shrinkers. There are many important works on classification of self-shrinkers for mean curvature flow (cf., [6,12]). For Lagrangian mean curvature flow, we can obtain more information due to the extra geometric condition (cf., [4,15]).

As a consequence of our monotonicity formula, we can show that the λ-tangent cone consists of finite many union of flat planes if it is nonempty.

The advantage of our flow is that we need not to assume that the ambient Kähler surface to be Einstein.

2 Preserving the Symplectic Property

In this section, we will prove our first result. To start, we set $\Sigma_t=F(\Sigma,t)$ with $\Sigma_0=\Sigma$. In the following, we will choose a frame $\{e_1,e_2,v_3,v_4\}$ at a fixed point $p\in\Sigma$ so that $\{e_1,e_2\}$ spans $T\Sigma$, $\{v_3,v_4\}$ spans $N\Sigma$ and the complex structure takes the form

$$J=\begin{pmatrix} 0 & \cos\alpha & \sin\alpha & 0 \\ -\cos\alpha & 0 & 0 & -\sin\alpha \\ -\sin\alpha & 0 & 0 & \cos\alpha \\ 0 & \sin\alpha & -\cos\alpha & 0 \end{pmatrix}. \tag{2.1}$$

It is easy to see that (cf., [7]) if we set $\mathbf{V}=\partial_2\alpha v_3+\partial_1\alpha v_4$, then (1.5) can be rewritten as

$$\frac{dF}{dt}=\cos^2\alpha\mathbf{H}-\beta\sin^2\alpha\mathbf{V}\equiv\mathbf{f}. \tag{2.2}$$

The evolution of the area form along the flow (1.5) is given in the next lemma:

Lemma 2.1.

$$\frac{d}{dt}d\mu_t=\frac{1}{2}\mathrm{tr}_g\frac{\partial g}{\partial t}d\mu_t=\left(-\cos^2\alpha|\mathbf{H}|^2+\beta\sin^2\alpha\mathbf{V}\cdot\mathbf{H}\right)d\mu_t. \tag{2.3}$$

Proof. We compute it in local coordinates.

$$\begin{aligned}\frac{\partial}{\partial t}g_{ij}&=\frac{\partial}{\partial t}\left\langle\frac{\partial F}{\partial x^i},\frac{\partial F}{\partial x^j}\right\rangle=\left\langle\frac{\partial \mathbf{f}}{\partial x^i},\frac{\partial F}{\partial x^j}\right\rangle+\left\langle\frac{\partial F}{\partial x^i},\frac{\partial \mathbf{f}}{\partial x^j}\right\rangle\\&=-2\left\langle\mathbf{f},\frac{\partial F}{\partial x^i\partial x^j}\right\rangle=-2\left\langle\mathbf{f},h_{ij}^{\alpha}v_\alpha\right\rangle,\end{aligned}$$

thus

$$\frac{d}{dt}d\mu_t=\frac{1}{2}\mathrm{tr}_g\frac{\partial g}{\partial t}d\mu_t=-\mathbf{f}\cdot\mathbf{H}d\mu_t=(-\cos^2\alpha|\mathbf{H}|^2+\beta\sin^2\alpha\mathbf{V}\cdot\mathbf{H})d\mu_t.$$

The proof is complete. □

We also recall the following elliptic equation of the Kähler angle:

Proposition 2.1 ([7]). *If Σ is a closed symplectic surface which is smoothly immersed in M with the Kähler angle α, then α satisfies the following equation:*

$$\begin{aligned}\Delta\cos\alpha=&\cos\alpha\left(-\left|h_{1k}^3-h_{2k}^4\right|^2-\left|h_{1k}^4+h_{2k}^3\right|^2\right)\\&+\sin\alpha\left(H_{,1}^4+H_{,2}^3\right)-\sin^2\alpha Ric(Je_1,e_2),\end{aligned} \tag{2.4}$$

where K is the curvature operator of M and $H_{,i}^{\alpha}=\langle\overline{\nabla}_{e_i}^N\boldsymbol{H},v_\alpha\rangle$.

Now we can derive the evolution equation of $\cos\alpha$ along the flow (1.5).

Theorem 2.1. *Let M be a Kähler surface. Assume that α is the Kähler angle of Σ_t which evolves by the flow* (1.5). *Then* $\cos\alpha$ *satisfies the equation*

$$\begin{aligned}\frac{\partial}{\partial t}\cos\alpha=&\left(\cos^2\alpha+\beta\sin^2\alpha\right)\Delta\cos\alpha+\cos^3\alpha\left(\left|h_{1k}^3-h_{2k}^4\right|^2+\left|h_{2k}^3+h_{1k}^4\right|^2\right)\\&+\cos^2\alpha\sin^2\alpha Ric(Je_1,e_2)+\frac{1}{\beta}\cos\alpha\sin^2\alpha|\mathbf{H}|^2\\&-\beta\cos\alpha\sin^2\alpha\left|\mathbf{V}+\frac{1}{\beta}\mathbf{H}\right|^2,\end{aligned} \tag{2.5}$$

where $\{e_1,e_2,v_3,v_4\}$ is an orthonormal basis of T_pM such that J takes the form (2.1).

Proof. Using the fact that $\overline{\omega}=0$ and Lemma 2.1, we have

$$\begin{aligned}\frac{\partial}{\partial t}\cos\alpha&=\frac{\partial}{\partial t}\left(\frac{\omega(e_1,e_2)}{\sqrt{\det(g_t)}}\right)=\omega(\overline{\nabla}_{e_1}\mathbf{f},e_2)-\omega(\overline{\nabla}_{e_2}\mathbf{f},e_1)-\frac{1}{2}\cos\alpha\,\mathrm{tr}_g\frac{\partial g}{\partial t}\\&=\omega(\overline{\nabla}_{e_1}\mathbf{f},e_2)-\omega(\overline{\nabla}_{e_2}\mathbf{f},e_1)+\cos\alpha\,\mathbf{f}\cdot\mathbf{H}.\end{aligned}$$

By breaking $\overline{\nabla}_{e_1}\mathbf{f}$ and $\overline{\nabla}_{e_2}\mathbf{f}$ into the normal and tangential parts, we get

$$\begin{aligned}&\omega(\overline{\nabla}_{e_1}\mathbf{f},e_2)-\omega(\overline{\nabla}_{e_2}\mathbf{f},e_1)\\&=\omega(\overline{\nabla}^N_{e_1}\mathbf{f},e_2)-\omega(\overline{\nabla}^N_{e_2}\mathbf{f},e_1)+\omega(\overline{\nabla}^T_{e_1}\mathbf{f},e_2)-\omega(\overline{\nabla}^T_{e_2}\mathbf{f},e_1)\\&=\omega(\overline{\nabla}^N_{e_1}\mathbf{f},e_2)-\omega(\overline{\nabla}^N_{e_2}\mathbf{f},e_1)-\langle\overline{\nabla}^T_{e_1}\mathbf{f},Je_2\rangle+\langle\overline{\nabla}^T_{e_2}\mathbf{f},Je_1\rangle\\&=\omega(\overline{\nabla}^N_{e_1}\mathbf{f},e_2)-\omega(\overline{\nabla}^N_{e_2}\mathbf{f},e_1)+\cos\alpha\left(\langle\overline{\nabla}^T_{e_1}\mathbf{f},e_1\rangle+\langle\overline{\nabla}^T_{e_2}\mathbf{f},e_2\rangle\right)\\&=\omega(\overline{\nabla}^N_{e_1}\mathbf{f},e_2)-\omega(\overline{\nabla}^N_{e_2}\mathbf{f},e_1)-\cos\alpha\left(\langle\mathbf{f},\overline{\nabla}_{e_1}e_1+\overline{\nabla}_{e_2}e_2\rangle\right)\\&=\omega(\overline{\nabla}^N_{e_1}\mathbf{f},e_2)-\omega(\overline{\nabla}^N_{e_2}\mathbf{f},e_1)-\cos\alpha\,\mathbf{f}\cdot\mathbf{H}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\frac{\partial}{\partial t}\cos\alpha&=\omega(\overline{\nabla}^N_{e_1}\mathbf{f},e_2)-\omega(\overline{\nabla}^N_{e_2}\mathbf{f},e_1)\\&=\omega\left(\overline{\nabla}^N_{e_1}(\cos^2\alpha\mathbf{H}-\beta\sin^2\alpha\mathbf{V}),e_2\right)-\omega\left(\overline{\nabla}^N_{e_2}(\cos^2\alpha\mathbf{H}-\beta\sin^2\alpha\mathbf{V}),e_1\right)\\&=\left\langle\overline{\nabla}^N_{e_1}(\cos^2\alpha\mathbf{H}-\beta\sin^2\alpha\mathbf{V}),-Je_2\right\rangle+\left\langle\overline{\nabla}^N_{e_2}(\cos^2\alpha\mathbf{H}-\beta\sin^2\alpha\mathbf{V}),Je_1\right\rangle\\&=\sin\alpha\left[\left\langle\overline{\nabla}^N_{e_1}(\cos^2\alpha\mathbf{H}-\beta\sin^2\alpha\mathbf{V}),v_4\right\rangle+\left\langle\overline{\nabla}^N_{e_2}(\cos^2\alpha\mathbf{H}-\beta\sin^2\alpha\mathbf{V}),v_3\right\rangle\right]\\&=\sin\alpha\Big[-2H^4\sin\alpha\cos\alpha\partial_1\alpha+\cos^2\alpha H^4_{,1}-2H^3\sin\alpha\cos\alpha\partial_2\alpha+\cos^2\alpha H^3_{,2}\\&\qquad-\beta\left(2V^4\sin\alpha\cos\alpha\partial_1\alpha+\sin^2\alpha V^4_{,1}-2V^3\sin\alpha\cos\alpha\partial_2\alpha+\sin^2\alpha V^3_{,2}\right)\Big]\\&=\sin\alpha\left[-2\sin\alpha\cos\alpha(H^4\partial_1\alpha+H^3\partial_2\alpha)+\cos^2\alpha(H^4_{,1}+H^3_{,2})\right]\\&\quad-\beta\sin\alpha\left[2\sin\alpha\cos\alpha(V^4\partial_1\alpha+V^3\partial_2\alpha)+\sin^2\alpha(V^4_{,1}+V^3_{,2})\right]\\&:=I+II.\end{aligned}\tag{2.6}$$

Using Proposition 2.1 and the definition of $\mathbf{V}$, we see that

$$\begin{aligned}I=-2\sin^2\alpha\cos\alpha\mathbf{H}\cdot\mathbf{V}+\cos^2\alpha\Big[\Delta\cos\alpha+\cos\alpha\left(|h^3_{1k}-h^4_{2k}|^2+|h^4_{1k}+h^3_{2k}|^2\right)\\+\sin^2\alpha Ric(Je_1,e_2)\Big].\end{aligned}$$

We also have

$$
\begin{aligned}
II &= -\beta\sin\alpha\left[2\sin\alpha\cos\alpha|\mathbf{V}|^2+\sin^2\alpha\left(V^4_{,1}+V^3_{,2}\right)\right]\\
&= -\beta\sin\alpha\left[2\sin\alpha\cos\alpha|\mathbf{V}|^2+\sin^2\alpha\Delta\alpha\right]\\
&= -\beta\sin\alpha\left[2\sin\alpha\cos\alpha|\mathbf{V}|^2+\sin\alpha\left(-\Delta\cos\alpha-\cos\alpha|\nabla\alpha|^2\right)\right]\\
&= \beta\sin^2\alpha\Delta\cos\alpha-\beta\sin^2\alpha\cos\alpha|\mathbf{V}|^{\mathbf{2}}.
\end{aligned}
$$

Putting I and II into (2.6) yields

$$
\begin{aligned}
\frac{\partial}{\partial t}\cos\alpha &= \left(\cos^2\alpha+\beta\sin^2\alpha\right)\Delta\cos\alpha+\cos^3\alpha\left(\left|h^3_{1k}-h^4_{2k}\right|^2+\left|h^3_{2k}+h^4_{1k}\right|^2\right)\\
&\quad +\cos^2\alpha\sin^2\alpha Ric(Je_1,e_2)-2\sin^2\alpha\cos\alpha\mathbf{H}\cdot\mathbf{V}-\beta\sin^2\alpha\cos\alpha|\mathbf{V}|^{\mathbf{2}}\\
&= \left(\cos^2\alpha+\beta\sin^2\alpha\right)\Delta\cos\alpha+\cos^3\alpha\left(\left|h^3_{1k}-h^4_{2k}\right|^2+\left|h^3_{2k}+h^4_{1k}\right|^2\right)\\
&\quad +\cos^2\alpha\sin^2\alpha Ric(Je_1,e_2)+\frac{1}{\beta}\cos\alpha\sin^2\alpha|\mathbf{H}|^2-\beta\cos\alpha\sin^2\alpha\left|\mathbf{V}+\frac{1}{\beta}\mathbf{H}\right|^2.
\end{aligned}
$$

This proves the theorem. □

When the ambient manifold is a Kähler-Einstein surface, we have $Ric=\frac{K_0}{4}\overline{g}$ where $\overline{g}$ and K_0 are the Kähler metric and the scalar curvature of M, respectively.

Corollary 2.1. *Let M be a Kähler-Einstein surface with scalar curvature K_0. Assume that α is the Kähler angle of Σ_t which evolves by the flow* (1.5)*. Then* $\cos\alpha$ *satisfies the equation*

$$
\begin{aligned}
\frac{\partial}{\partial t}\cos\alpha &= \left(\cos^2\alpha+\beta\sin^2\alpha\right)\Delta\cos\alpha+\cos^3\alpha\left(\left|h^3_{1k}-h^4_{2k}\right|^2+\left|h^3_{2k}+h^4_{1k}\right|^2\right)\\
&\quad +\frac{K_0}{4}\cos^3\alpha\sin^2\alpha+\frac{1}{\beta}\cos\alpha\sin^2\alpha|\mathbf{H}|^2-\beta\cos\alpha\sin^2\alpha\left|\mathbf{V}+\frac{1}{\beta}\mathbf{H}\right|^2,
\end{aligned} \tag{2.7}
$$

where $\{e_1,e_2,v_3,v_4\}$ is an orthonormal basis of T_pM such that J takes the form (2.1)*. Consequently, if Σ_0 is symplectic, then along the flow* (1.5)*, at each time t, Σ_t is symplectic.*

The above theorem implies that symplectic property is preserved along the flow (1.5). Notice that we do not need M to be a Kähler-Einstein surface.

Corollary 2.2. *Let M be a compact Kähler surface and Σ be a closed symplectic surface in M. Then along the flow* (1.5)*, if Σ_0 is symplectic, then along the flow* (1.5)*, at each time t, Σ_t is symplectic. In particular, suppose that $|Ric_M|\leq K_1$, then we have that*

$$\min_{\Sigma_t}\cos\alpha\geq e^{-K_1 t}\min_{\Sigma_0}\cos\alpha \tag{2.8}$$

as long as the smooth solution exists. Furthermore, if M is a Kähler-Einstein surface with scalar curvature $K_0\geq 0$, then we have

$$\min_{\Sigma_t}\cos\alpha\geq\min_{\Sigma_0}\cos\alpha \tag{2.9}$$

as long as the smooth solution exists.

Proof. We can rewrite (2.5) as

$$\begin{aligned}\frac{\partial}{\partial t}\cos\alpha=&\left(\cos^2\alpha+\beta\sin^2\alpha\right)\Delta\cos\alpha+\cos^3\alpha\left(\left|h^3_{1k}-h^4_{2k}\right|^2+\left|h^3_{2k}+h^4_{1k}\right|^2\right)\\&+\cos^2\alpha\sin^2\alpha Ric(Je_1,e_2)-\cos\alpha\sin^2\alpha\langle\beta\mathbf{V}+2\mathbf{H},\mathbf{V}\rangle\\\geq&\left(\cos^2\alpha+\beta\sin^2\alpha\right)\Delta\cos\alpha+\cos^3\alpha\left(\left|h^3_{1k}-h^4_{2k}\right|^2+\left|h^3_{2k}+h^4_{1k}\right|^2\right)\\&-K_1\cos^3\alpha\sin^2\alpha-\cos\alpha\sin^2\alpha\langle\beta\mathbf{V}+2\mathbf{H},\mathbf{V}\rangle\\\geq&\left(\cos^2\alpha+\beta\sin^2\alpha\right)\Delta\cos\alpha+\cos^3\alpha\left(\left|h^3_{1k}-h^4_{2k}\right|^2+\left|h^3_{2k}+h^4_{1k}\right|^2\right)\\&-K_1\cos\alpha-\cos\alpha\sin^2\alpha\langle\beta\mathbf{V}+2\mathbf{H},\mathbf{V}\rangle.\end{aligned}$$

Notice that $\mathbf{V}=\partial_2\alpha v_3+\partial_1\alpha v_4$. Then (2.8) follows by the maximum principle. The proof of (2.9) is similar by using Corollary 2.1. $\square$

3 Extension of the flow

In this section, we will show that the singularity of the flow (1.5) is characterized by the maximal norm of the second fundamental form of Σ in M. More precisely, we prove that

Theorem 3.1. *Let M be a Kähler surface and Σ be a closed surface. Let $F:\Sigma\times[0,T)\to M$ be a smooth solution to the flow* (1.5)*. Set $\Sigma_t=F(\Sigma,t)$. If*

$$\max_{\Sigma_t}|A|^2\leq\Lambda \tag{3.1}$$

for all $t\in[0,T)$, then the flow can be extended smoothly to an interval $[0,T+\varepsilon)$ for some $\varepsilon>0$.

Proof. For simplicity, we will prove the theorem for the case $M=\mathbb{R}^4$. The proof for the general case is similar. By the assumption (3.1), we know that there exists a constant $r_0>0$, depending only on Λ, such that each connected component of $\Sigma_t\cap B_{p_t}(r)$, with $p_t\in\Sigma_t$ and $r<r_0$, can be graphed over $T_{p_t}\Sigma_t\cap B_{p_t}(r)$ by a pair of functions $(f(t),g(t))$. Furthermore, the bound in (3.1) also gives us a uniform bound on the $C^{1,\alpha}$-norm of $(f(t),g(t))$.

Next we will derive the evolution equations satisfied by f and g. First note that the flow (1.5) is equivalent to the following equation:

$$\left(\frac{dF}{dt}\right)^{\perp}=\cos^2\alpha\mathbf{H}-\frac{\beta}{\cos\alpha}\left(J(J\nabla\cos\alpha)^{\top}\right)^{\perp}. \tag{3.2}$$

For the immersion given by $F(x,y)=(x,y,f(x,y),g(x,y))$, we can choose

$$\begin{aligned}
e_1&=\frac{\partial F}{\partial x}=(1,0,f_x,g_x), \quad v_3=(-f_x,-g_x,1,0),\\
e_2&=\frac{\partial F}{\partial y}=(0,1,f_y,g_y), \quad v_4=(-f_y,-g_y,0,1),
\end{aligned}$$

so that $\{e_1,e_2\}$ spans $T\Sigma$ and $\{v_3,v_4\}$ spans $N\Sigma$. Then (3.2) is equivalent to that

$$\left\langle\frac{dF}{dt},v_3\right\rangle=\cos^2\alpha H^3-\frac{\beta}{\cos\alpha}\left\langle J(J\nabla\cos\alpha)^{\top},v_3\right\rangle, \tag{3.3}$$

$$\left\langle\frac{dF}{dt},v_4\right\rangle=\cos^2\alpha H^4-\frac{\beta}{\cos\alpha}\left\langle J(J\nabla\cos\alpha)^{\top},v_4\right\rangle. \tag{3.4}$$

Since $\frac{dF}{dt}=(0,0,f_t,g_t)$, we see that

$$\left\langle\frac{dF}{dt},v_3\right\rangle=f_t, \quad \left\langle\frac{dF}{dt},v_4\right\rangle=g_t.$$

On the other hand, from the proof of [7, Theorem 2.3], we know that the induced metric of Σ in the basis is

$$(g_{ij})_{1\le i,j\le 2}=\begin{pmatrix}1+f_x^2+g_x^2 & f_xf_y+g_xg_y\\ f_xf_y+g_xg_y & 1+f_y^2+g_y^2\end{pmatrix}$$

and the inverse matrix is

$$(g^{ij})_{1\le i,j\le 2}=\frac{1}{\det(g_{ij})}\begin{pmatrix}1+f_y^2+g_y^2 & -f_xf_y-g_xg_y\\ -f_xf_y-g_xg_y & 1+f_x^2+g_x^2\end{pmatrix}.$$

Moreover, the metric on the normal bundle $T^{\perp}\Sigma$ is

$$(g_{\alpha\beta})_{3\le\alpha,\beta\le4}=\begin{pmatrix}1+f_x^2+f_y^2 & f_xg_x+f_yg_y\\ f_xg_x+f_yg_y & 1+g_x^2+g_y^2\end{pmatrix}$$

and the inverse matrix is

$$(g^{\alpha\beta})_{3\le\alpha,\beta\le4}=\frac{1}{\det(g_{\alpha\beta})}\begin{pmatrix}1+g_x^2+g_y^2 & -f_xg_x-f_yg_y\\ -f_xg_x-f_yg_y & 1+f_x^2+f_y^2\end{pmatrix}.$$

We have

$$\det g:=\det(g_{ij})=\det(g_{\alpha\beta})=1+f_x^2+f_y^2+g_x^2+g_y^2+(f_xg_y-f_yg_x)^2,$$

and

$$\cos\alpha=\frac{1+f_xg_y-f_yg_x}{\sqrt{\det(g)}}.$$

Direct calculation shows that (see [7, Theorem 2.3] and [9, Theorem 2.1] for the elliptic case) (f,g) satisfies the following system:

$$\begin{aligned}f_t=\frac{1}{\det g}\Big\{&f_{xx}\Big[g^{11}g^{33}c^2-\beta\big(g^{11}g^{22}-g^{12}g^{12}\big)\big(g^{34}a+g^{33}b\big)\big(g_{12}a-g_{22}b\big)\Big]\\
&+f_{xy}\Big[2g^{33}g^{12}c^2-\beta\big(g^{11}g^{22}-g^{12}g^{12}\big)\big(g^{34}a+g^{33}b\big)\big(g_{12}b-g_{11}a\big)\\
&\qquad-\beta\big(g^{11}g^{22}-g^{12}g^{12}\big)\big(g^{33}a-g^{34}b\big)\big(g_{12}a-g_{22}b\big)\Big]\\
&+f_{yy}\Big[g^{22}g^{33}c^2-\beta\big(g^{11}g^{22}-g^{12}g^{12}\big)\big(g^{33}a-g^{34}b\big)\big(-g_{11}a+g_{12}b\big)\Big]\\
&+g_{xx}\Big[g^{11}g^{34}c^2-\beta\big(g^{11}g^{22}-g^{12}g^{12}\big)\big(g^{34}a+g^{33}b\big)\big(-g_{22}a-g_{12}b\big)\Big]\\
&+g_{xy}\Big[2g^{34}g^{12}c^2-\beta\big(g^{11}g^{22}-g^{12}g^{12}\big)\big(g^{34}a+g^{33}b\big)\big(g_{11}b+g_{12}a\big)\\
&\qquad-\beta\big(g^{11}g^{22}-g^{12}g^{12}\big)\big(g^{33}a-g^{34}b\big)\big(-g_{22}a-g_{12}b\big)\Big]\\
&+g_{yy}\Big[g^{22}g^{34}c^2-\beta\big(g^{11}g^{22}-g^{12}g^{12}\big)\big(g^{33}a-g^{34}b\big)\big(g_{11}b+g_{12}a\big)\Big]\Big\},\end{aligned}\tag{3.5}$$

$$\begin{aligned}g_t=\frac{1}{\det g}\Big\{&f_{xx}\Big[g^{11}g^{34}c^2-\beta\big(g^{11}g^{22}-g^{12}g^{12}\big)\big(g^{44}a+g^{34}b\big)\big(g_{12}a-g_{22}b\big)\Big]\\
&+f_{xy}\Big[2g^{34}g^{12}c^2-\beta\big(g^{11}g^{22}-g^{12}g^{12}\big)\big(g^{44}a+g^{34}b\big)\big(g_{12}b-g_{11}a\big)\\
&\qquad-\beta\big(g^{11}g^{22}-g^{12}g^{12}\big)\big(g^{34}a-g^{44}b\big)\big(g_{12}a-g_{22}b\big)\Big]\end{aligned}$$

$$
\begin{aligned}
&+f_{yy}\Big[g^{22}g^{34}c^2-\beta\big(g^{11}g^{22}-g^{12}g^{12}\big)\big(g^{34}a-g^{44}b\big)\big(-g_{11}a+g_{12}b\big)\Big]\\
&+g_{xx}\Big[g^{11}g^{44}c^2-\beta\big(g^{11}g^{22}-g^{12}g^{12}\big)\big(g^{44}a+g^{34}b\big)\big(-g_{22}a-g_{12}b\big)\Big]\\
&+g_{xy}\Big[2g^{44}g^{12}c^2-\beta\big(g^{11}g^{22}-g^{12}g^{12}\big)\big(g^{44}a+g^{34}b\big)\big(g_{11}b+g_{12}a\big)\\
&\qquad-\beta\big(g^{11}g^{22}-g^{12}g^{12}\big)\big(g^{34}a-g^{44}b\big)\big(-g_{22}a-g_{12}b\big)\Big]\\
&+g_{yy}\Big[g^{22}g^{44}c^2-\beta\big(g^{11}g^{22}-g^{12}g^{12}\big)\big(g^{34}a-g^{44}b\big)\big(g_{11}b+g_{12}a\big)\Big]\Big\}, \qquad (3.6)
\end{aligned}
$$

where

$$a=f_y+g_x,\quad b=f_x-g_y,\quad c=1+f_xg_y-f_yg_x=\sqrt{\det(g)}\cos\alpha.$$

Since we have uniform $C^{1,\alpha}$-estimates for f,g by [8, Proposition 3.1] (see also [7, Theorem 2.3]), we see that the system (3.5)-(3.6) is strictly parabolic, with parabolic constant depending on β and Λ. Schauder estimate for parabolic systems gives us the uniform $C^{k,\alpha}$ bound for $(f(t),g(t))$ for $t\in[0,T)$. Arzela-Ascoli theorem shows that we can have a smooth limit $F(T):\Sigma\to M$ when $t\to T$. Then the short time existence of the flow implies that the solution can be extended to $[0,T+\varepsilon)$ for some $\varepsilon>0$. This proves the theorem. □

4 Monotonicity formula

In this section, we will consider the monotonicity formula for the negative gradient flow of the functional L_1. Namely, we will consider the flow

$$\frac{dF}{dt}=\cos^2\alpha\mathbf{H}-\sin^2\alpha\mathbf{V}\equiv\mathbf{f}. \qquad (4.1)$$

We first consider the flow in $\mathbb{R}^4$. We have the following proposition:

Proposition 4.1. *Let $M=\mathbb{R}^4$. Then for $f(x)=e^{x^2}$, along the flow* (4.1)*, we have*

$$
\begin{aligned}
&\frac{d}{dt}\int_\Sigma f\left(\frac{1}{\cos\alpha}\right)\rho d\mu_t\\
\le&-\int_\Sigma\left\{\frac14\left|\frac{(F-X_0)^\perp}{t_0-t}+\mathbf{f}+\mathbf{H}\right|^2+\left(\frac12\cos^2\alpha-\frac14\cos^4\alpha\right)|\mathbf{H}+\mathbf{V}|^2\right.\\
&\left.+\frac74\left|\mathbf{H}+\frac47\left(-\frac{2}{\cos^2\alpha}+\frac{15}{4}\right)\mathbf{V}\right|^2+\frac14|\mathbf{H}|^2+\frac53|\mathbf{V}|^2\right\}f\rho d\mu_t. \qquad (4.2)
\end{aligned}
$$

Proof. By (2.5), we know that when $\beta=1$, $\cos\alpha$ satisfies the equation

$$\begin{aligned}\frac{\partial}{\partial t}\cos\alpha=&\Delta\cos\alpha+\cos^3\alpha\left(\left|h^3_{1k}-h^4_{2k}\right|^2+\left|h^3_{2k}+h^4_{1k}\right|^2\right)\\&+\cos\alpha\sin^2\alpha|\mathbf{H}|^2-\cos\alpha\sin^2\alpha|\mathbf{V}+\mathbf{H}|^2.\end{aligned}\tag{4.3}$$

In particular, we have that

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{1}{\cos\alpha}\right)=&\Delta\left(\frac{1}{\cos\alpha}\right)-\cos\alpha\left(\left|h^3_{1k}-h^4_{2k}\right|^2+\left|h^3_{2k}+h^4_{1k}\right|^2\right)\\&-\frac{\sin^2\alpha}{\cos\alpha}|\mathbf{H}|^2+\frac{\sin^2\alpha}{\cos\alpha}|\mathbf{V}+\mathbf{H}|^2-2\frac{\sin^2\alpha}{\cos^3\alpha}|\mathbf{V}|^2.\end{aligned}\tag{4.4}$$

Let f be a positive function defined on $\mathbb{R}^+$ to be determined later. Then we have that

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\Delta\right)f\left(\frac{1}{\cos\alpha}\right)=&f'\left(\frac{\partial}{\partial t}-\Delta\right)\frac{1}{\cos\alpha}-f''\left|\nabla\frac{1}{\cos\alpha}\right|^2\\=&f'\left[-\cos\alpha\left(\left|h^3_{1k}-h^4_{2k}\right|^2+\left|h^3_{2k}+h^4_{1k}\right|^2\right)-\frac{\sin^2\alpha}{\cos\alpha}|\mathbf{H}|^2\right.\\&\left.+\frac{\sin^2\alpha}{\cos\alpha}|\mathbf{V}+\mathbf{H}|^2-2\frac{\sin^2\alpha}{\cos^3\alpha}|\mathbf{V}|^2\right]-f''\frac{\sin^2\alpha}{\cos^4\alpha}|\mathbf{V}|^2.\end{aligned}\tag{4.5}$$

Now we define

$$\rho(X,t):=\frac{1}{4\pi(t_0-t)}e^{-\frac{|X-X_0|^2}{4(t_0-t)}}.$$

Then along the flow (4.1), we have that

$$\frac{\partial}{\partial t}\rho(X,t)=\left(\frac{1}{t_0-t}-\frac{\langle F-X_0,\mathbf{f}\rangle}{2(t_0-t)}-\frac{|X-X_0|^2}{4(t_0-t)^2}\right)\rho.$$

We also have that

$$\begin{aligned}&\nabla\rho(X,t)=-\rho\frac{\langle X-X_0,\nabla X\rangle}{2(t_0-t)},\\&\Delta\rho(X,t)=\left(\frac{|(X-X_0)^T|^2}{4(t_0-t)^2}-\frac{\langle F-X_0,\mathbf{H}\rangle}{2(t_0-t)}-\frac{1}{t_0-t}\right)\rho.\end{aligned}$$

Hence we have

$$\left(\frac{\partial}{\partial t}+\Delta\right)\rho(X,t)=-\left(\frac{\langle F-X_0,\mathbf{f}+\mathbf{H}\rangle}{2(t_0-t)}+\frac{|(X-X_0)^\perp|^2}{4(t_0-t)^2}\right)\rho.\tag{4.6}$$

Then we compute that

$$
\begin{aligned}
&\frac{d}{dt}\int_\Sigma f\left(\frac{1}{\cos\alpha}\right)\rho d\mu_t \\
=&\int_\Sigma\left(\frac{\partial}{\partial t}f\left(\frac{1}{\cos\alpha}\right)\right)\rho d\mu_t+\int_\Sigma f\left(\frac{\partial}{\partial t}\rho\right)d\mu_t-\int_\Sigma f\rho\langle\mathbf{f},\mathbf{H}\rangle d\mu_t \\
=&\int_\Sigma\left(\frac{\partial}{\partial t}-\Delta\right)f\left(\frac{1}{\cos\alpha}\right)\rho d\mu_t+\int_\Sigma f\left(\frac{\partial}{\partial t}+\Delta\right)\rho d\mu_t-\int_\Sigma f\rho\langle\mathbf{f},\mathbf{H}\rangle d\mu_t \\
=&-\int_\Sigma f'\left(\cos\alpha|\overline{\nabla}J_{\Sigma_t}|^2+\frac{\sin^2\alpha}{\cos\alpha}|\mathbf{H}|^2-\frac{\sin^2\alpha}{\cos\alpha}|\mathbf{V}+\mathbf{H}|^2+2\frac{\sin^2\alpha}{\cos^3\alpha}|\mathbf{V}|^2\right)\rho d\mu_t \\
&-\int_\Sigma f''\frac{\sin^2\alpha}{\cos^4\alpha}|\mathbf{V}|^2\rho d\mu-\int_\Sigma f\left(\frac{\langle F-X_0,\mathbf{f}+\mathbf{H}\rangle}{2(t_0-t)}+\frac{|(X-X_0)^\perp|^2}{4(t_0-t)^2}+\langle\mathbf{f},\mathbf{H}\rangle\right)\rho d\mu_t \\
=&-\int_\Sigma\left\{f'\left(\cos\alpha|\overline{\nabla}J_{\Sigma_t}|^2+\frac{\sin^2\alpha}{\cos\alpha}|\mathbf{H}|^2-\frac{\sin^2\alpha}{\cos\alpha}|\mathbf{V}+\mathbf{H}|^2+2\frac{\sin^2\alpha}{\cos^3\alpha}|\mathbf{V}|^2\right)\right. \\
&\qquad\left.+f''\frac{\sin^2\alpha}{\cos^4\alpha}|\mathbf{V}|^2+\frac{1}{4}\left|\frac{(F-X_0)^\perp}{t_0-t}+\mathbf{f}+\mathbf{H}\right|^2 f-\frac{|\mathbf{f}+\mathbf{H}|^2}{4}f+\langle\mathbf{f},\mathbf{H}\rangle f\right\}\rho d\mu_t. \qquad (4.7)
\end{aligned}
$$

Recall that

$$\mathbf{f}=\cos^2\alpha\mathbf{H}-\sin^2\alpha\mathbf{V},$$

and

$$|\overline{\nabla}J_{\Sigma_t}|^2=|\mathbf{H}|^2+2|\mathbf{V}|^2+2\langle\mathbf{H},\mathbf{V}\rangle.$$

We have that

$$
\begin{aligned}
&f'\left(\cos\alpha|\overline{\nabla}J_{\Sigma_t}|^2+\frac{\sin^2\alpha}{\cos\alpha}|\mathbf{H}|^2-\frac{\sin^2\alpha}{\cos\alpha}|\mathbf{V}+\mathbf{H}|^2+2\frac{\sin^2\alpha}{\cos^3\alpha}|\mathbf{V}|^2\right) \\
&+f''\frac{\sin^2\alpha}{\cos^4\alpha}|\mathbf{V}|^2-\frac{|\mathbf{f}+\mathbf{H}|^2}{4}f+\langle\mathbf{f},\mathbf{H}\rangle f \\
=&f'\cos\alpha\left(|\mathbf{H}|^2+2|\mathbf{V}|^2+2\langle\mathbf{H},\mathbf{V}\rangle\right)-f'\frac{\sin^2\alpha}{\cos\alpha}|\mathbf{V}|^2-2f'\frac{\sin^2\alpha}{\cos\alpha}\langle\mathbf{V},\mathbf{H}\rangle+2f'\frac{\sin^2\alpha}{\cos^3\alpha}|\mathbf{V}|^2 \\
&+f''\frac{\sin^2\alpha}{\cos^4\alpha}|\mathbf{V}|^2-\frac{1}{4}\left|(1+\cos^2\alpha)\mathbf{H}-\sin^2\alpha\mathbf{V}\right|^2 f+\left\langle\cos^2\alpha\mathbf{H}-\sin^2\alpha\mathbf{V},\mathbf{H}\right\rangle f \\
=&\left(f'\cos\alpha-\frac{1}{4}f+\frac{1}{2}f\cos^2\alpha-\frac{1}{4}f\cos^4\alpha\right)|\mathbf{H}|^2 \\
&+\left(f'\frac{2-3\cos^2\alpha+3\cos^4\alpha}{\cos^3\alpha}-\frac{1}{4}f\left(1-2\cos^2\alpha+\cos^4\alpha\right)+f''\frac{1-\cos^2\alpha}{\cos^4\alpha}\right)|\mathbf{V}|^2
\end{aligned}
$$

$$+\left(f'\frac{4\cos^2\alpha-2}{\cos\alpha}+f\left(-\frac{1}{2}+\cos^2\alpha-\frac{1}{2}\cos^4\alpha\right)\right)\langle\mathbf{H},\mathbf{V}\rangle.$$

Therefore, we have

$$\begin{aligned}
&\frac{d}{dt}\int_\Sigma f\left(\frac{1}{\cos\alpha}\right)\rho d\mu_t\\
=&-\int_\Sigma\left\{\frac{1}{4}\left|\frac{(F-X_0)^\perp}{t_0-t}+\mathbf{f}+\mathbf{H}\right|^2 f+\left(f'\cos\alpha-\frac{1}{4}f+\frac{1}{2}f\cos^2\alpha-\frac{1}{4}f\cos^4\alpha\right)|\mathbf{H}|^2\right.\\
&\qquad+\left(f'\frac{2-3\cos^2\alpha+3\cos^4\alpha}{\cos^3\alpha}-\frac{1}{4}f\left(1-2\cos^2\alpha+\cos^4\alpha\right)+f''\frac{1-\cos^2\alpha}{\cos^4\alpha}\right)|\mathbf{V}|^2\\
&\qquad\left.+\left(f'\frac{4\cos^2\alpha-2}{\cos\alpha}+f\left(-\frac{1}{2}+\cos^2\alpha-\frac{1}{2}\cos^4\alpha\right)\right)\langle\mathbf{H},\mathbf{V}\rangle\right\}\rho d\mu_t. \qquad (4.8)
\end{aligned}$$

Now we take $f(x)=e^{ax^2}$ with $a\geq 1$ to be determined, then

$$f'=2axf,\quad f''=(2a+4a^2x^2)f.$$

Hence, we have

$$\begin{aligned}
&\left(f'\cos\alpha-\frac{1}{4}f+\frac{1}{2}f\cos^2\alpha-\frac{1}{4}f\cos^4\alpha\right)|\mathbf{H}|^2\\
&+\left(f'\frac{2-3\cos^2\alpha+3\cos^4\alpha}{\cos^3\alpha}-\frac{1}{4}f\left(1-2\cos^2\alpha+\cos^4\alpha\right)+f''\frac{1-\cos^2\alpha}{\cos^4\alpha}\right)|\mathbf{V}|^2\\
&+\left(f'\frac{4\cos^2\alpha-2}{\cos\alpha}+f\left(-\frac{1}{2}+\cos^2\alpha-\frac{1}{2}\cos^4\alpha\right)\right)\langle\mathbf{H},\mathbf{V}\rangle\\
=&f\left(2a-\frac{1}{4}+\frac{1}{2}\cos^2\alpha-\frac{1}{4}\cos^4\alpha\right)|\mathbf{H}|^2\\
&+f\left(\frac{4a^2}{\cos^6\alpha}+\frac{6a-4a^2}{\cos^4\alpha}-\frac{8a}{\cos^2\alpha}+6a-\frac{1}{4}+\frac{1}{2}\cos^2\alpha-\frac{1}{4}\cos^4\alpha\right)|\mathbf{V}|^2\\
&+f\left(-\frac{4a}{\cos^2\alpha}+8a-\frac{1}{2}+\cos^2\alpha-\frac{1}{2}\cos^4\alpha\right)\langle\mathbf{H},\mathbf{V}\rangle\\
=&f\left(\frac{1}{2}\cos^2\alpha-\frac{1}{4}\cos^4\alpha\right)|\mathbf{H}+\mathbf{V}|^2+f\left(2a-\frac{1}{4}\right)|\mathbf{H}|^2\\
&+f\left(\frac{4a^2}{\cos^6\alpha}+\frac{6a-4a^2}{\cos^4\alpha}-\frac{8a}{\cos^2\alpha}+6a-\frac{1}{4}\right)|\mathbf{V}|^2+f\left(-\frac{4a}{\cos^2\alpha}+8a-\frac{1}{2}\right)\langle\mathbf{H},\mathbf{V}\rangle
\end{aligned}$$

$$
\begin{aligned}
&= f\left(\frac{1}{2}\cos^2\alpha-\frac{1}{4}\cos^4\alpha\right)|\mathbf{H}+\mathbf{V}|^2+\frac{1}{4}f|\mathbf{H}|^2\\
&\quad +f\left(2a-\frac{1}{2}\right)\left|\mathbf{H}+\frac{1}{2a-\frac{1}{2}}\left(-\frac{2a}{\cos^2\alpha}+4a-\frac{1}{4}\right)\mathbf{V}\right|^2\\
&\quad +f\left[\frac{4a^2}{\cos^6\alpha}+\frac{6a-4a^2}{\cos^4\alpha}-\frac{8a}{\cos^2\alpha}+6a-\frac{1}{4}-\frac{1}{2a-\frac{1}{2}}\left(-\frac{2a}{\cos^2\alpha}+4a-\frac{1}{4}\right)^2\right]|\mathbf{V}|^2\\
&\geq f\left(\frac{1}{2}\cos^2\alpha-\frac{1}{4}\cos^4\alpha\right)|\mathbf{H}+\mathbf{V}|^2+\frac{1}{4}f|\mathbf{H}|^2\\
&\quad +f\left(2a-\frac{1}{2}\right)\left|\mathbf{H}+\frac{1}{2a-\frac{1}{2}}\left(-\frac{2a}{\cos^2\alpha}+4a-\frac{1}{4}\right)\mathbf{V}\right|^2\\
&\quad +f\left[\frac{6a}{\cos^4\alpha}-\frac{8a}{\cos^2\alpha}+6a-\frac{1}{4}-\frac{4a^2}{2a-\frac{1}{2}}\frac{1}{\cos^4\alpha}+\frac{4a(4a-\frac{1}{4})}{2a-\frac{1}{2}}\frac{1}{\cos^2\alpha}-\frac{(4a-\frac{1}{4})^2}{2a-\frac{1}{2}}\right]|\mathbf{V}|^2\\
&= f\left(\frac{1}{2}\cos^2\alpha-\frac{1}{4}\cos^4\alpha\right)|\mathbf{H}+\mathbf{V}|^2+\frac{1}{4}f|\mathbf{H}|^2\\
&\quad +f\left(2a-\frac{1}{2}\right)\left|\mathbf{H}+\frac{1}{2a-\frac{1}{2}}\left(-\frac{2a}{\cos^2\alpha}+4a-\frac{1}{4}\right)\mathbf{V}\right|^2\\
&\quad +f\left(\frac{8a^2-3a}{2a-\frac{1}{2}}\frac{1}{\cos^4\alpha}+\frac{3a}{2a-\frac{1}{2}}\frac{1}{\cos^2\alpha}-\frac{4a^2+\frac{3}{2}a-\frac{1}{16}}{2a-\frac{1}{2}}\right)|\mathbf{V}|^2\\
&\geq f\left(\frac{1}{2}\cos^2\alpha-\frac{1}{4}\cos^4\alpha\right)|\mathbf{H}+\mathbf{V}|^2+\frac{1}{4}f|\mathbf{H}|^2\\
&\quad +f\left(2a-\frac{1}{4}\right)\left|\mathbf{H}+\frac{1}{2a-\frac{1}{4}}\left(-\frac{2a}{\cos^2\alpha}+4a-\frac{1}{4}\right)\mathbf{V}\right|^2\\
&\quad +f\left(\frac{4a^2-3a}{2a-\frac{1}{2}}\frac{1}{\cos^4\alpha}+\frac{\frac{3a}{2}}{2a-\frac{1}{2}}\frac{1}{\cos^2\alpha}+\frac{\frac{1}{16}}{2a-\frac{1}{2}}\right)|\mathbf{V}|^2.
\end{aligned}
$$

If we take $a=1$, then with $f(x)=e^{x^2}$, we have

$$
\begin{aligned}
&\frac{d}{dt}\int_\Sigma f\left(\frac{1}{\cos\alpha}\right)\rho d\mu_t\\
&\leq -\int_\Sigma\left\{\frac{1}{4}\left|\frac{(F-X_0)^\perp}{t_0-t}+\mathbf{f}+\mathbf{H}\right|^2+\left(\frac{1}{2}\cos^2\alpha-\frac{1}{4}\cos^4\alpha\right)|\mathbf{H}+\mathbf{V}|^2\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{7}{4}\left|\mathbf{H}+\frac{4}{7}\left(-\frac{2}{\cos^2\alpha}+\frac{15}{4}\right)\mathbf{V}\right|^2+\frac{1}{4}|\mathbf{H}|^2\\
&+\left(\frac{2}{3}\frac{1}{\cos^4\alpha}+\frac{1}{\cos^2\alpha}+\frac{1}{24}\right)|\mathbf{V}|^2\Bigg\}f\rho d\mu_t\\
\le&-\int_\Sigma\left\{\frac{1}{4}\left|\frac{(F-X_0)^\perp}{t_0-t}+\mathbf{f}+\mathbf{H}\right|^2+\left(\frac{1}{2}\cos^2\alpha-\frac{1}{4}\cos^4\alpha\right)|\mathbf{H}+\mathbf{V}|^2\right.\\
&\left.+\frac{7}{4}\left|\mathbf{H}+\frac{4}{7}\left(-\frac{2}{\cos^2\alpha}+\frac{15}{4}\right)\mathbf{V}\right|^2+\frac{1}{4}|\mathbf{H}|^2+\frac{5}{3}|\mathbf{V}|^2\right\}f\rho d\mu_t.
\end{aligned}
$$

This proves the proposition. □

Next, we will consider the monotonicity formula for the flow (4.1) in a Kähler surface M. Let i_M be the injectivity radius of M. We choose a cut-off function $\phi\in C_0^\infty(B_{2r}(X_0))$ with $\phi\equiv 1$ in $B_r(X_0)$, where $X_0\in M$, $0<2r<i_M$. Choose a normal coordinates in $B_{2r}(X_0)$ and express F using the coordinates (F^1,F^2,F^3,F^4) as a surface in $\mathbb{R}^4$. We define

$$
\Psi(X_0,t_0,t):=\int_{\Sigma_t}\phi(F)f\left(\frac{1}{\cos\alpha}\right)\rho d\mu_t,
$$

where $f(x)=e^{x^2}$. Then we have

Theorem 4.1. *Let M^4 be a compact Kähler surface. Then there are positive constants c_1 and c_2 depending only on M^4, F_0, r and t_0, such that along the flow* (4.1), *we have*

$$
\begin{aligned}
&\frac{d}{dt}\left(e^{c_1\sqrt{t_0-t}}\Psi(X_0,t_0,t)\right)\\
\le&-e^{c_1\sqrt{t_0-t}}\int_\Sigma\left\{\frac{1}{4}\left|\frac{(F-X_0)^\perp}{t_0-t}+\mathbf{f}+\mathbf{H}\right|^2+\left(\frac{1}{2}\cos^2\alpha-\frac{1}{4}\cos^4\alpha\right)|\mathbf{H}+\mathbf{V}|^2\right.\\
&\left.+\frac{7}{4}\left|\mathbf{H}+\frac{4}{7}\left(-\frac{2}{\cos^2\alpha}+\frac{15}{4}\right)\mathbf{V}\right|^2+\frac{1}{8}|\mathbf{H}|^2|+\mathbf{V}|^2\right\}f\rho d\mu_t\\
&+c_2e^{c_1\sqrt{t_0-t}}.
\end{aligned}
\tag{4.9}
$$

Proof. Note that

$$
\Delta F=\mathbf{H}+g^{ij}\Gamma_{ij}^\alpha v_\alpha,
$$

where $\{v_\alpha\}_{\alpha=3,4}$ is a basis of $N\Sigma_t$, g_{ij} is the induced metric on Σ, (g^{ij}) is the inverse of (g_{ij}) and Γ_{ij}^k is the Christoffel symbol on M. Then (4.6) reads

$$\left(\frac{\partial}{\partial t}+\Delta\right)\rho(X,t)=-\left(\frac{\langle F-X_0,\mathbf{f}+\mathbf{H}+g^{ij}\Gamma_{ij}^\alpha v_\alpha\rangle}{2(t_0-t)}+\frac{|(X-X_0)^\perp|^2}{4(t_0-t)^2}\right)\rho. \tag{4.10}$$

Using Theorem 2.1, we have for $\beta=1$ that

$$\begin{aligned}&\left(\frac{\partial}{\partial t}-\Delta\right)f\left(\frac{1}{\cos\alpha}\right)\\ &=f'\left(\frac{\partial}{\partial t}-\Delta\right)\frac{1}{\cos\alpha}-f''\left|\nabla\frac{1}{\cos\alpha}\right|^2\\ &=f'\bigg[-\cos\alpha\left(\left|h_{1k}^3-h_{2k}^4\right|^2+\left|h_{2k}^3+h_{1k}^4\right|^2\right)-\frac{\sin^2\alpha}{\cos\alpha}|\mathbf{H}|^2\\ &\qquad+\frac{\sin^2\alpha}{\cos\alpha}|\mathbf{V}+\mathbf{H}|^2-2\frac{\sin^2\alpha}{\cos^3\alpha}|\mathbf{V}|^2-\sin^2\alpha Ric(Je_1,e_2)\bigg]-f''\frac{\sin^2\alpha}{\cos^4\alpha}|\mathbf{V}|^2.\end{aligned} \tag{4.11}$$

We also have

$$\frac{\partial}{\partial t}\phi(F)=\langle D\phi,\mathbf{f}\rangle.$$

Hence, from the proof of Proposition 4.1, we have

$$\begin{aligned}\frac{d}{dt}\Psi(X_0,t_0,t)=&\frac{d}{dt}\int_{\Sigma_t}\phi(F)f\left(\frac{1}{\cos\alpha}\right)\rho d\mu_t\\ =&\int_\Sigma\phi\left(\frac{\partial}{\partial t}-\Delta\right)f\left(\frac{1}{\cos\alpha}\right)\rho d\mu_t+\int_\Sigma\phi f\left(\frac{\partial}{\partial t}+\Delta\right)\rho d\mu_t\\ &+\int_{\Sigma_t}\left(\frac{\partial}{\partial t}\phi(F)\right)f\rho d\mu_t-\int_\Sigma f\rho\langle\mathbf{f},\mathbf{H}\rangle d\mu_t\\ &+\int_{\Sigma_t}\phi\rho\Delta f\left(\frac{1}{\cos\alpha}\right)d\mu_t-\int_{\Sigma_t}\phi f\Delta\rho d\mu_t\\ \le&-\int_\Sigma\left\{\frac{1}{4}\left|\frac{(F-X_0)^\perp}{t_0-t}+\mathbf{f}+\mathbf{H}\right|^2+\left(\frac{1}{2}\cos^2\alpha-\frac{1}{4}\cos^4\alpha\right)|\mathbf{H}+\mathbf{V}|^2\right.\\ &\qquad\left.+\frac{7}{4}\left|\mathbf{H}+\frac{4}{7}\left(-\frac{2}{\cos^2\alpha}+\frac{15}{4}\right)\mathbf{V}\right|^2+\frac{1}{4}|\mathbf{H}|^2+\frac{5}{3}|\mathbf{V}|^2\right\}\phi f\rho d\mu_t\\ &+\int_{\Sigma_t}\langle D\phi,\mathbf{f}\rangle f\rho d\mu_t+\int_{\Sigma_t}f\rho\Delta\phi d\mu_t+2\int_{\Sigma_t}f\langle\nabla\rho,\nabla\phi\rangle d\mu_t\end{aligned}$$

$$+\int_{\Sigma_t}\phi f\rho\frac{\langle F-X_0,g^{ij}\Gamma_{ij}^{\alpha}v_\alpha\rangle}{2(t_0-t)}d\mu_t+K_1\int_{\Sigma_t}f'\phi\rho\sin^2\alpha\cos\alpha d\mu_t.$$

As in the proof of Proposition 2.1 in [4] (see [4, (13)]), we see that

$$\left|\frac{\langle F-X_0,g^{ij}\Gamma_{ij}^{\alpha}v_\alpha\rangle}{2(t_0-t)}\right|\rho\le c_1\frac{\rho(F,t)}{\sqrt{t_0-t}}+C. \tag{4.12}$$

Furthermore, since $\phi\in C_0^\infty(B_{2r}(X_0),\mathbb{R}^+)$, we have

$$\frac{|D\phi|^2}{\phi}\le 2\max_{\phi>0}\left|D^2\phi\right|^2.$$

We also have that

$$|\mathbf{f}|^2=\left|\cos^2\alpha\mathbf{H}-\sin^2\alpha\mathbf{V}\right|^2\le|\mathbf{H}|^2+|\mathbf{V}|^2.$$

Hence we have

$$|\langle D\phi,\mathbf{f}\rangle|f\rho\le\frac{1}{8}|\mathbf{f}|^2\phi f\rho+2\frac{|D\phi|^2}{\phi}f\rho\le\frac{1}{8}\big(|\mathbf{H}|^2+|\mathbf{V}|^2\big)\phi f\rho+2\frac{|D\phi|^2}{\phi}f\rho.$$

Note that $\nabla\phi=0$ in $B_r(X_0)$ so that $|\rho\Delta\phi|$ and $\langle\nabla\rho,\nabla\phi\rangle$ are bounded in $B_{2r}(X_0)$. We also note by the choice of f that

$$0\le f'\sin^2\alpha\cos\alpha=2f\sin^2\alpha\le 2f.$$

Hence we have

$$\begin{aligned}\frac{d}{dt}\Psi(X_0,t_0,t)\le&-\int_\Sigma\left\{\frac{1}{4}\left|\frac{(F-X_0)^\perp}{t_0-t}+\mathbf{f}+\mathbf{H}\right|^2+\left(\frac{1}{2}\cos^2\alpha-\frac{1}{4}\cos^4\alpha\right)|\mathbf{H}+\mathbf{V}|^2\right.\\&\left.+\frac{7}{4}\left|\mathbf{H}+\frac{4}{7}\left(-\frac{2}{\cos^2\alpha}+\frac{15}{4}\right)\mathbf{V}\right|^2+\frac{1}{8}|\mathbf{H}|^2+|\mathbf{V}|^2\right\}\phi f\rho d\mu_t\\&+\frac{c_1}{\sqrt{t_0-t}}\Psi+C\int_{\Sigma_t}fd\mu_t.\end{aligned} \tag{4.13}$$

Since (1.5) is the negative gradient flow of the functional $L_1=\int_\Sigma\frac{1}{\cos\alpha}d\mu_t$, we know that

$$\int_\Sigma\frac{1}{\cos\alpha}d\mu_t\le\int_\Sigma\frac{1}{\cos\alpha}d\mu_0 \tag{4.14}$$

for each $t\in[0,T)$. In particular,

$$Area(\Sigma_t)\leq\int_\Sigma\frac{1}{\cos\alpha}d\mu_t\leq\int_\Sigma\frac{1}{\cos\alpha}d\mu_0=L_1(\Sigma_0). \tag{4.15}$$

By (2.8), we know that $\cos\alpha\geq\delta>0$ on $[0,t_0]$ for some constant $\delta>0$ depending on Σ_0 and t_0 whenever the flow has a smooth solution on $[0,t_0]$. Hence we have that $f\leq e^{\frac{1}{\delta^2}}$ for $t\in[0,t_0]$ so that

$$\int_{\Sigma_t}fd\mu_t\leq e^{\frac{1}{\delta^2}}L_1(\Sigma_0). \tag{4.16}$$

Therefore, we have from (4.13) that

$$\begin{aligned}\frac{d}{dt}\Psi(X_0,t_0,t)\leq&-\int_\Sigma\left\{\frac{1}{4}\left|\frac{(F-X_0)^\perp}{t_0-t}+\mathbf{f}+\mathbf{H}\right|^2+\left(\frac{1}{2}\cos^2\alpha-\frac{1}{4}\cos^4\alpha\right)|\mathbf{H}+\mathbf{V}|^2\right.\\&\left.+\frac{7}{4}\left|\mathbf{H}+\frac{4}{7}\left(-\frac{2}{\cos^2\alpha}+\frac{15}{4}\right)\mathbf{V}\right|^2+\frac{1}{8}|\mathbf{H}|^2+|\mathbf{V}|^2\right\}\phi f\rho d\mu_t\\&+\frac{c_1}{\sqrt{t_0-t}}\Psi+c_2.\end{aligned}$$

This implies the desired estimate. □

5 Flatness of the λ-tangent cones

In this section, we will use the monotonicity formula obtained in the previous section to show that the λ-tangent cones of the flow (4.1) are union of flat planes.

Suppose that (X_0,T) is a singular point of the flow (4.1) where T is the first singular time. We now describe the rescaling process around (X_0,T). As in the previous section, we choose a normal coordinates centered at X_0 with radius r $(0<r<\frac{i_M}{2})$, using the exponential map. We express F in its coordinates functions. For any $t<0$, we set

$$F_\lambda(x,t)=\lambda\big(F(x,T+\lambda^{-2}t)-X_0\big),$$

where λ are positive constants which go to infinity. The scaled surface is denoted by $\Sigma_t^\lambda=F_\lambda(x,t)$ on which $d\mu_t^\lambda$ is the area element obtained from $d\mu_t$.

If g^λ is the metric on Σ_t^λ, it is clear that

$$g_{ij}^\lambda=\lambda^2g_{ij},\quad (g^\lambda)^{ij}=\lambda^{-2}g^{ij}.$$

It is easy to check that

$$\begin{aligned}&\frac{\partial F_\lambda}{\partial t}=\lambda^{-1}\frac{\partial F}{\partial t}, \quad \mathbf{H}_\lambda=\lambda^{-1}\mathbf{H},\\ &\mathbf{V}_\lambda=\lambda^{-1}\mathbf{V}, \qquad |\mathbf{A}_\lambda|^2=\lambda^{-2}|\mathbf{A}|^2.\end{aligned}$$

It follows that the scaled surface also evolves by the flow

$$\frac{\partial F_\lambda}{\partial t}=\cos^2\alpha_\lambda\mathbf{H}_\lambda-\sin^2\alpha_\lambda\mathbf{V}_\lambda\equiv\mathbf{f}_\lambda. \tag{5.1}$$

The weighted monotonicity formula leads to the following integral estimates.

Proposition 5.1. *Let M be a Kähler surface. If the initial compact surface is symplectic, then for any $R>0$ and any $-\infty<s_1<s_2<0$, we have*

$$\int_{s_1}^{s_2}\int_{\Sigma_t^\lambda\cap B_R(0)}|\mathbf{H}_\lambda|^2d\mu_t^\lambda dt\to 0 \quad \text{as} \quad \lambda\to\infty, \tag{5.2}$$

$$\int_{s_1}^{s_2}\int_{\Sigma_t^\lambda\cap B_R(0)}|\mathbf{V}_\lambda|^2d\mu_t^\lambda dt\to 0 \quad \text{as} \quad \lambda\to\infty, \tag{5.3}$$

$$\int_{s_1}^{s_2}\int_{\Sigma_t^\lambda\cap B_R(0)}|\mathbf{f}_\lambda|^2d\mu_t^\lambda dt\to 0 \quad \text{as} \quad \lambda\to\infty, \tag{5.4}$$

$$\int_{s_1}^{s_2}\int_{\Sigma_t^\lambda\cap B_R(0)}\left|F_\lambda^\perp\right|^2d\mu_t^\lambda dt\to 0 \quad \text{as} \quad \lambda\to\infty. \tag{5.5}$$

Proof. For any $R>0$, we choose a cut-off function $\phi_R\in C_0^\infty(B_{2R}(0))$ with $\phi_R\equiv 1$ in $B_R(0)$, where $B_\rho(0)$ is the metric ball centered at 0 with radius ρ in $\mathbf{R}^4$. For any fixed $t<0$, the flow (4.1) has a smooth solution near $T+\lambda^{-2}t<T$ for sufficiently large λ, since $T>0$ is the first blow-up time of the flow. Set

$$f_\lambda=e^{\frac{1}{\cos^2\alpha_\lambda}}.$$

It is clear

$$\begin{aligned}&\int_{\Sigma_t^\lambda}f_\lambda\frac{1}{0-t}\phi_R(F_\lambda)\exp\left(-\frac{|F_\lambda|^2}{4(0-t)}\right)d\mu_t^\lambda\\ =&\int_{\Sigma_{T+\lambda^{-2}t}}f_\lambda\phi(F_\lambda)\frac{1}{T-(T+\lambda^{-2}t)}\exp\left(-\frac{|F(x,T+\lambda^{-2}t)-X_0|^2}{4(T-(T+\lambda^{-2}t))}\right)d\mu_t,\end{aligned}$$

where ϕ is the function defined in the definition of Ψ. Note that $T+\lambda^{-2}t\to T$ for any fixed t as $\lambda\to\infty$. By (4.9)

$$\frac{\partial}{\partial t}\left(e^{c_1\sqrt{t_0-t}}\Psi\right)\le c_2 e^{c_1\sqrt{t_0-t}},$$

and it then follows that $\lim_{t\to t_0} e^{c_1\sqrt{t_0-t}}\Psi$ exists. This implies, by taking $t_0=T$ and $t=T+\lambda^{-2}s$, that for any fixed s_1 and s_2 with $-\infty<s_1<s_2<0$,

$$\begin{aligned}&e^{c_1\sqrt{T-(T+\lambda^{-2}s_2)}}\int_{\Sigma^\lambda_{s_2}} f_\lambda\phi_R\frac{1}{0-s_2}\exp\left(-\frac{|F_\lambda|^2}{4(0-s_2)}\right)d\mu^\lambda_{s_2}\\ &\quad -e^{c_1\sqrt{T-(T+\lambda^{-2}s_1)}}\int_{\Sigma^\lambda_{s_1}} f_\lambda\phi_R\frac{1}{0-s_1}\exp\left(-\frac{|F_\lambda|^2}{4(0-s_1)}\right)d\mu^\lambda_{s_1}\to 0\quad\text{as}\quad\lambda\to\infty.\end{aligned}\tag{5.6}$$

Integrating (4.9) from s_1 to s_2 yields

$$\begin{aligned}&-e^{c_1\sqrt{-\lambda^{-2}s_2}}\int_{\Sigma^\lambda_{s_2}} f_\lambda\phi_R\frac{1}{0-s_2}\exp\left(-\frac{|F_\lambda|^2}{4(0-s_2)}\right)d\mu^\lambda_{s_2}\\ &+e^{c_1\sqrt{-\lambda^{-2}s_1}}\int_{\Sigma^\lambda_{s_1}} f_\lambda\phi_R\frac{1}{0-s_1}\exp\left(-\frac{|F_\lambda|^2}{4(0-s_1)}\right)d\mu^\lambda_{s_1}\\ \ge&\int_{s_1}^{s_2}e^{c_1\sqrt{-\lambda^{-2}t}}\int_{\Sigma^\lambda_t} f_\lambda\phi_R\rho(F_\lambda,t)\left|\frac{(F_\lambda)^\perp}{t_0-t}+\mathbf{f}_\lambda+\mathbf{H}_\lambda\right|^2 d\mu^\lambda_t dt\\ &+\int_{s_1}^{s_2}e^{c_1\sqrt{-\lambda^{-2}t}}\int_{\Sigma^\lambda_t} f_\lambda\phi_R\rho(F_k,t)\left(\frac{1}{2}\cos^2\alpha_\lambda-\frac{1}{4}\cos^4\alpha_\lambda\right)|\mathbf{H}_\lambda+\mathbf{V}_\lambda|^2 d\mu^\lambda_t dt\\ &+\frac{7}{4}\int_{s_1}^{s_2}e^{c_1\sqrt{-\lambda^{-2}t}}\int_{\Sigma^\lambda_t} f_\lambda\phi_R\rho(F_k,t)\left|\mathbf{H}_\lambda+\frac{4}{7}\left(-\frac{2}{\cos^2\alpha_\lambda}+\frac{15}{4}\right)\mathbf{V}_\lambda\right|^2 d\mu^\lambda_t dt\\ &+\int_{s_1}^{s_2}e^{c_1\sqrt{-\lambda^{-2}t}}\int_{\Sigma^\lambda_t}\left(\frac{1}{8}|\mathbf{H}_\lambda|^2+|\mathbf{V}_\lambda|^2\right)f_\lambda\phi_R\rho(F_\lambda,t)d\mu^\lambda_t dt\\ &-c_2\lambda^{-2}(s_2-s_1)e^{c_1\lambda^{-1}\sqrt{-s_1}}.\end{aligned}\tag{5.7}$$

Putting (5.6) and (5.7) together, we have

$$\begin{aligned}&\lim_{\lambda\to\infty}\int_{s_1}^{s_2}\int_{\Sigma^\lambda_t}\phi_R\rho(F_k,t)|\mathbf{H}_\lambda|^2 d\mu^\lambda_t dt=0,\\ &\lim_{\lambda\to\infty}\int_{s_1}^{s_2}\int_{\Sigma^\lambda_t}\phi_R\rho(F_k,t)|\mathbf{V}_\lambda|^2 d\mu^\lambda_t dt=0,\end{aligned}$$

which yield (5.2) and (5.3) respectively, and

$$\lim_{\lambda\to\infty}\int_{s_1}^{s_2}\int_{\Sigma_t^\lambda}\phi_R\rho(F_\lambda,t)\left|\frac{(F_\lambda)^\perp}{t_0-t}+\mathbf{f}_\lambda+\mathbf{H}_\lambda\right|^2 d\mu_t^\lambda=0. \tag{5.8}$$

Recall that

$$\mathbf{f}_\lambda+\mathbf{H}_\lambda=\left(1+\cos^2\alpha_\lambda\right)\mathbf{H}_\lambda-\sin^2\alpha_\lambda\mathbf{V}_\lambda.$$

Hence (5.2) and (5.3) imply (5.5). (5.4) is a consequence of (5.2) and (5.3). □

Lemma 5.1. *For any $R>0$ and any $t<0$, for sufficiently large λ,*

$$\mu_t^\lambda\left(\Sigma_t^\lambda\cap B_R(0)\right)\le CR^2, \tag{5.9}$$

where $B_R(0)$ is a metric ball in $\mathbf{R}^4$ and $C>0$ is independent of λ.

Proof. We shall first prove the inequality (5.9). We shall use C below for uniform positive constants which are independent of R and λ. Straightforward computation shows

$$\begin{aligned}\mu_t^\lambda\left(\Sigma_t^\lambda\cap B_R(0)\right)&=\lambda^2\int_{\Sigma_{T+\lambda^{-2}t}\cap B_{\lambda^{-1}R}(X_0)}d\mu_t\\&=R^2\left(\lambda^{-1}R\right)^{-2}\int_{\Sigma_{T+\lambda^{-2}t}\cap B_{\lambda^{-1}R}(X_0)}d\mu_t\\&\le CR^2\int_{\Sigma_{T+\lambda^{-2}t}\cap B_{\lambda^{-1}R}(X_0)}f\frac{1}{4\pi(\lambda^{-1}R)^2}e^{-\frac{|X-X_0|^2}{4(\lambda^{-1}R)^2}}d\mu_t\\&=CR^2\Psi\left(X_0,T+(\lambda^{-1}R)^2+\lambda^{-2}t,T+\lambda^{-2}t\right).\end{aligned}$$

By the monotonicity inequality (4.9), we have

$$\begin{aligned}\mu_t^\lambda\left(\Sigma_t^\lambda\cap B_R(0)\right)&\le CR^2\left(\Psi\left(X_0,T+(\lambda^{-1}R)^2+\lambda^{-2}t,\frac{T}{2}\right)+C\right)\\&\le C\frac{R^2}{T}\left(\int_{\Sigma_{\frac{T}{2}}}fd\mu_{\frac{T}{2}}+C\right)\\&\le C\frac{R^2}{T}\left(e^{\frac{1}{\delta^2}}L_1(\Sigma_0)+C\right)\le CR^2,\end{aligned}$$

where we have used (4.16). □

Fixed $t_0<0$. By (5.9), for any $R>0$, we see that the total measure of $(\Sigma_{t_0}^{\lambda}\cap B_R(0),\mu_{t_0}^{\lambda})$ is bounded from above by CR^2, the compactness theorem of the measures (cf., [16, 4.4]) implies that there is a subsequence $\lambda_i(R)\to\infty$ of λ such that,

$$\left(\Sigma_{t_0}^{\lambda_i(R)}\cap B_R(0),\mu_{t_0}^{\lambda_i(R)}\right)\to\left(\Sigma_{t_0}^{\infty}\cap B_R(0),\mu_{t_0}^{\infty}\right)$$

in the sense of measure. Using a diagonal subsequence argument, we conclude that, there is a subsequence $\lambda_k\to\infty$ such that $(\Sigma_{t_0}^{\lambda_k},\mu_{t_0}^{\lambda_k})\to(\Sigma_{t_0}^{\infty},\mu_{t_0}^{\infty})$ in the sense of measures.

We now show that, for any $t<0$, the subsequence λ_k which we have chosen above satisfies $(\Sigma_t^{\lambda_k},\mu_t^{\lambda_k})\to(\Sigma_{t_0}^{\infty},\mu_{t_0}^{\infty})$ in the sense of measure. And consequently the limiting surface $(\Sigma_{t_0}^{\infty},\mu_{t_0}^{\infty})$ is independent of t_0.

Lemma 5.2. *For any $t<0$, the sequence $\lambda_k\to\infty$ we chosen above satisfies that $(\Sigma_t^{\lambda_k},\mu_t^{\lambda_k})\to(\Sigma^{\infty},\mu^{\infty})$ in the sense of measure, where $(\Sigma^{\infty},\mu^{\infty})$ is independent of t. The multiplicity of Σ^{∞} is finite.*

Proof. Note that the following standard formula for mean curvature flows

$$\frac{d}{dt}\int_{\Sigma_t^{\lambda}}\phi d\mu_t^{\lambda}=-\int_{\Sigma_t^{\lambda}}\left(\phi|\mathbf{H}_{\lambda}|^2+\nabla\phi\cdot\mathbf{f}_{\lambda}\right)d\mu_t^{\lambda} \tag{5.10}$$

is valid for any test function $\phi\in C_0^{\infty}(M)$ (cf., (1) in [14, Section 6]).

Then for any given $t<0$ integrating (5.10) yields

$$\int_{\Sigma_t^{\lambda_k}}\phi d\mu_t^{\lambda_k}-\int_{\Sigma_{t_0}^{\lambda_k}}\phi d\mu_{t_0}^{\lambda_k}=\int_t^{t_0}\int_{\Sigma_t^{\lambda_k}}\left(\phi|\mathbf{H}_{\lambda_k}|^2+\nabla\phi\cdot\mathbf{f}_{\lambda_k}\right)d\mu_t^{\lambda_k}dt\to 0\quad\text{as}\quad k\to\infty$$
$$\text{by}\quad(5.2)\quad\text{and}\quad(5.4).$$

So, for any fixed $t<0$, $(\Sigma_t^{\lambda_k},\mu_t^{\lambda_k})\to(\Sigma_{t_0}^{\infty},\mu_{t_0}^{\infty})$ in the sense of measures as $k\to\infty$. We denote $(\Sigma_{t_0}^{\infty},\mu_{t_0}^{\infty})$ by $(\Sigma^{\infty},\mu^{\infty})$, which is independent of t_0.

The inequality (5.9) yields a uniform upper bound on $R^{-2}\mu_t^{\lambda_k}(\Sigma_t^{\lambda_k}\cap B_R(0))$, which yields finiteness of the multiplicity of Σ^{∞}. □

Definition 5.1. *Let (X_0,T) be a singular point of the flow* (4.1) *of a closed symplectic surface Σ_0 in a compact Kähler surface M. We call $(\Sigma^{\infty},d\mu^{\infty})$ obtained in Lemma* 5.2 *a λ-tangent cone of the flow* (4.1) *at (X_0,T).*

In the remaining part of this section, we prove that the λ-tangent cones are flat. And for simplicity in notation, we write $\Sigma_t^{\lambda_k}$ as Σ_t^k.

A k-varifold is a Radon measure on $G^k(M)$, where $G^k(M)$ is the Grassmann bundle of all k-planes tangent to M. Allard's compactness theorem for rectifiable varifolds (see [1, 6.4], [14, 1.9] and [16, Theorem 42.7]) can be stated as follows.

Theorem 5.1 (Allard's compactness theorem)**.** *Let (V_i,μ_i) be a sequence of rectifiable k-varifolds in M with*

$$\sup_{i\geq 1}\big(\mu_i(U)+|\delta V_i|(U)\big)<\infty \quad \text{for each} \quad U\subset\subset M.$$

Then there is a rectifiable varifold (V,μ) of locally bounded first variation and a subsequence, which we also denote by (V_i,μ_i), such that

(i) Convergence of measures: $\mu_i\to\mu$ as Radon measures on M.

(ii) Convergence of tangent planes: $V_i\to V$ as Radon measures on $G^k(M)$.

(iii) Convergence of first variations: $\delta V_i\to\delta V$ as TM-valued Radon measures.

(iv) Lower semi-continuity of total first variations: $|\delta V|\leq \liminf_{i\to\infty}|\delta V_i|$ as Radon measures.

We first show that the λ-tangent cone is rectifiable and stationary. The proof is similar to that of [4, Proposition 3.1].

Proposition 5.2. *Let M be a compact Kähler surface. If the initial compact surface is symplectic, then the λ tangent cone Σ^∞ is rectifiable and stationary.*

Proof. We set

$$A_R=\left\{t\in(-\infty,0)\,\middle|\, \liminf_{k\to\infty}\int_{\Sigma_t^k\cap B_R(0)}\big(|\mathbf{H}_k|^2+|\mathbf{V}_k|^2\big)d\mu_t^k\neq 0\right\},$$

and

$$A=\bigcup_{R>0}A_R.$$

Denote the measures of A_R and A by $|A_R|$ and $|A|$, respectively. It is clear from (5.3), (5.4) and (5.2) that $|A_R|=0$ for any $R>0$. So $|A|=0$.

Choose $t\notin A$. Let V_t^k be the varifold defined by Σ_t^k. It is explained in the previous section, that V_t^k is well defined in $B_R(0)\subset \mathbf{R}^4$ for any $R>0$ when k sufficiently large. By the definition of varifolds, we have

$$V_t^k(\psi)=\int_{\Sigma_t^k}\psi(x,T\Sigma_t^k)d\mu_t^k$$

for any $\psi\in C_0^0(G^2(\mathbf{R}^4),R)$, where $G^2(\mathbf{R}^4)$ is the Grassmanian bundle of all 2-planes tangent to Σ_t^∞ in $\mathbf{R}^4$. For each smooth surface Σ_t^k, the first variation δV_t^k of V_t^k (cf., [1], [16, (39.4)] and [14, (1.7)]) is that, for any smooth vector field X with support in $B_R(0)$,

$$\delta V_t^k(X)=-\int_{\Sigma_t^k\cap B_R(0)}X\cdot\mathbf{H}_k d\mu_t^k,$$

so by the area upper bound (5.9)

$$\left|\delta V_t^k(X)\right|\le CR\|X\|_{L^\infty(B_R(0))}\left(\int_{\Sigma_t^k\cap B_R(0)}|\mathbf{H}_k|^2d\mu_t^k\right)^{\frac{1}{2}}. \tag{5.11}$$

We therefore have that, for any $R>0$

$$\mu_t^k(B_R(0))+\delta V_t^k(B_R(0))\le C(R) \tag{5.12}$$

by Allard's compactness theorem, there exists a subsequence which we also denote by (V_t^k,μ_t^k) such that $(V_t^k,\mu_t^k)\to(V_t^\infty,\mu_t^\infty)$ with the conclusions in Theorem 5.1 hold in $B_R(0)$. By a diagonal subsequence argument, there exists a subsequence which we also denote by (V_t^k,μ_t^k) such that $(V_t^k,\mu_t^k)\to(V_t^\infty,\mu_t^\infty)$ and satisfies (*i*) - (*iv*) in Theorem 5.1 in $\mathbf{R}^4$.

Because $t\notin A$, by (5.11), we see that $\delta V_t^k\to 0$ at t as $k\to\infty$ and Σ^∞ is rectifiable by applying Theorem 5.1. Furthermore by (iii) in Theorem 5.1, we have that

$$-\mu^\infty\lfloor\mathbf{H}_\infty=\delta V^\infty=\lim_{k\to\infty}\delta V_t^k=0.$$

Therefore Σ^∞ is stationary. □

Theorem 5.2. *Let M be a compact Kähler surface. If the initial compact surface is symplectic and $T>0$ is the first blow-up time of the flow* (4.1)*, then the λ-tangent cone Σ^∞ of the flow* (4.1) *at (X_0,T) is a finite union of planes if it is not empty-set.*

Proof. Since Σ^∞ is not empty, without loss of any generality, we may assume $0\in\Sigma^\infty$ where 0 is the origin of $\mathbf{R}^4$. There is a sequence of points $X_k\in\Sigma_t^k$ satisfying

$X_k\to 0$ as $k\to\infty$. By Proposition 5.1, for any s_1 and s_2 with $-\infty<s_1<s_2<0$ and any $R>0$, we have

$$\int_{s_1}^{s_2}\int_{\Sigma_t^k\cap B_R(0)}|F_k^\perp|^2 d\mu_t^k dt\to 0 \quad \text{as} \quad k\to\infty.$$

Thus, by (5.9)

$$\begin{aligned}&\lim_{k\to\infty}\int_{s_1}^{s_2}\int_{\Sigma_t^k\cap B_R(0)}\big|(F_k-X_k)^\perp\big|^2 d\mu_t^k dt\\ \le& 2\lim_{k\to\infty}\int_{s_1}^{s_2}\int_{\Sigma_t^k\cap B_R(0)}|F_k^\perp|^2 d\mu_t^k dt+C(s_2-s_1)R^2\lim_{k\to\infty}|X_k|^2=0.\end{aligned}$$

Let us denote the tangent spaces of Σ_t^k at the point $F_k(x,t)$ and of Σ^∞ at the point $F^\infty(x,t)$ by $T\Sigma_t^k$ and $T\Sigma^\infty$ respectively. It is clear that

$$\begin{aligned}&(F_k-X_k)^\perp=\mathrm{dist}\big(X_k,T\Sigma_t^k\big),\\ &(F_\infty)^\perp=\mathrm{dist}\big(0,T\Sigma^\infty\big).\end{aligned}$$

By Allard's compactness theorem, i.e. Theorem 5.1 (ii), we have

$$\begin{aligned}&\int_{s_1}^{s_2}\int_{\Sigma^\infty\cap B_R(0)}\big|(F_\infty)^\perp\big|^2 d\mu^\infty dt\\ =&\int_{s_1}^{s_2}\int_{\Sigma^\infty\cap B_R(0)}\big|\mathrm{dist}(0,T\Sigma^\infty)\big|^2 d\mu^\infty dt\\ =&\lim_{k\to\infty}\int_{s_1}^{s_2}\int_{\Sigma_t^k\cap B_R(0)}\big|\mathrm{dist}\ (X_k,T\Sigma_t^k)\big|^2 d\mu_t^k dt\\ =&\lim_{k\to\infty}\int_{s_1}^{s_2}\int_{\Sigma_t^k\cap B_R(0)}\big|(F_k-X_k)^\perp\big|^2 d\mu_t^k dt=0.\end{aligned}$$

By [13, Theorem 1], we know that Σ^∞ is smooth outside a discrete set of points $\mathcal{S}$. So outside $\mathcal{S}$, we have

$$\langle F_\infty,v_\alpha\rangle=0.$$

Note that the above inner product is taken in $\mathbf{R}^4$, and differentiating in $\mathbf{R}^4$ then yields

$$0=\langle\partial_i F_\infty,v_\alpha\rangle+\langle F_\infty,\partial_i v_\alpha\rangle=\langle F_\infty,\partial_i v_\alpha\rangle.$$

Because $\partial_i F_\infty$ is tangential to Σ^∞, by Weingarten's equation we observe

$$(h_\infty)_{ij}^\alpha\langle F_\infty,e_j\rangle=0 \quad \text{for all} \quad \alpha, \quad i=1,2.$$

So for $\alpha=1,2$, we have

$$\det\big((h_\infty)^\alpha_{ij}\big)=0.$$

Since $\mathbf{H}=0$, for $\alpha=1,2$ we also have

$$tr\big((h_\infty)^\alpha_{ij}\big)=0.$$

It then follows immediately that the symmetric matrix $((h_\infty)^\alpha_{ij})$ is in fact the zero matrix, for all $i,j,\alpha=1,2$, which obviously yields $|\mathbf{A}_\infty|\equiv 0$. By Lemma 5.1, the tangent cone consists of finitely many planes. This completes the proof of Theorem 5.2. □

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