Global Well-Posedness of the Inviscid Heat-Conductive Resistive Compressible MHD in a Strip Domain

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Abstract. This paper concerns the inviscid, heat-conductive and resistive compressible MHD system in a horizontally periodic flat strip domain. The global well-posedness of the problem around an equilibrium with the positive constant density and temperature and a uniform non-horizontal magnetic field is established, and the solution decays to the equilibrium almost exponentially. Our result reveals the strong stabilizing effect of the transversal magnetic field and resistivity as the global well-posedness of compressible inviscid heat-conductive flows in multi-D is unknown.

1 Introduction

When the viscosity is neglected whereas the heat conduction and magnetic diffusion are taken into account, the dynamics of compressible electrically conducting

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fluids interacting with magnetic fields can be described by the following magnetohydrodynamic system (MHD) [7,11]:

$$\begin{cases} \partial_t \rho + \text{div}(\rho u) = 0, \\ \partial_t(\rho u) + \text{div}(\rho u \otimes u) + \nabla p = \text{curl}\, B \times B, \\ \partial_t(\rho e) + \text{div}(\rho u e) + p\,\text{div}\, u - \mu \Delta \theta = \kappa |\text{curl}\, B|^2, \\ \partial_t B = \text{curl}\, E, \quad E = u \times B - \kappa\, \text{curl}\, B, \\ \text{div}\, B = 0, \end{cases} \tag{1.1}$$

where ρ, u, θ and B denote the density, velocity, temperature and magnetic field, respectively, and E is the electric field. The fluid is assumed to obey the ideal polytropic law, so the pressure $p = R\rho\theta$ and the internal energy $e = c_v \theta$ with constants $R, c_v > 0$. $\mu > 0$ is the heat conduction coefficient and $\kappa > 0$ is the magnetic diffusion coefficient.

The main difficulty of studying the global well-posedness of (1.1) lies in the absence of the viscosity. Similar to the Navier-Stokes equations, the viscous and resistive (incompressible and compressible) MHD system has a unique global classical solution, at least for the small initial data, see [5,9,18] for instance. On the other hand, it is remarkable that the ideal incompressible homogeneous MHD system in the whole space also admits a unique global classical solution around a nonzero uniform magnetic field [2,3,8,22]. It is then natural to ask whether the MHD systems with only the viscosity or resistivity admit global classical solutions or develop singularities in finite time. The global existence of classical solutions to the viscous and non-resistive MHD systems has been established around a nonzero uniform magnetic field. For the Cauchy problem, we refer to [1,13,15,25,27] for the incompressible homogeneous case and [24] for the 2D compressible isentropic case. For the initial boundary value problem, the global well-posedness has been proved only for the case of a horizontally flat strip domain, see [16] for the 2D incompressible homogeneous system around a uniform horizontal magnetic field and [20] for the 3D (incompressible and compressible) systems around a uniform non-horizontal magnetic field. The inviscid and resistive incompressible homogeneous 2D MHD system has a global weak solution in H^1, but the question whether such weak solutions are unique or can be improved to be global classical solutions remains open [4,10,12]. For a 2D periodic domain, [28] showed the global existence of classical solutions around a nonzero uniform magnetic field when the initial data has certain symmetries, and [23] proved a global well-posedness around the zero magnetic field.

In this paper, we consider the compressible MHD system (1.1) in the strip domain $\Omega = \mathbb{T}^2 \times (0,1)$ for $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, with the following boundary conditions:

$$u \cdot e_3 = 0, \quad \nabla\theta \cdot e_3 = 0, \quad B \cdot e_3 = \bar{B} \cdot e_3, \quad E \times e_3 = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

where $\bar{B}$ is the uniform magnetic field of the background. We will prove the global existence of classical solutions to (1.1)-(1.2) around the steady state $(\bar{\rho}, 0, \bar{\theta}, \bar{B})$ with the constant density $\bar{\rho} > 0$ and temperature $\bar{\theta} > 0$ and $\bar{B}_3 \neq 0$.

Set

$$\varrho = \rho - \bar{\rho}, \quad u = u, \quad \vartheta = \theta - \bar{\theta}, \quad b = B - \bar{B}. \tag{1.3}$$

Then the problem under consideration can be reformulated as

$$\begin{cases} (\partial_t + u \cdot \nabla)\varrho + \rho \operatorname{div} u = 0 & \text{in } \Omega, \\ \rho(\partial_t + u \cdot \nabla)u + \nabla p = \operatorname{curl} b \times (\bar{B} + b) & \text{in } \Omega, \\ c_v \rho(\partial_t + u \cdot \nabla)\vartheta + p \operatorname{div} u - \mu\Delta\vartheta = \kappa|\operatorname{curl} b|^2 & \text{in } \Omega, \\ \partial_t b = \operatorname{curl} E, \quad E = u \times (\bar{B} + b) - \kappa \operatorname{curl} b & \text{in } \Omega, \\ \operatorname{div} b = 0 & \text{in } \Omega, \\ u_3 = 0, \quad \partial_3 \vartheta = 0, \quad b_3 = 0, \quad E \times e_3 = 0 & \text{on } \partial\Omega, \\ (\varrho, u, \vartheta, b)|_{t=0} = (\varrho_0, u_0, \vartheta_0, b_0). \end{cases} \tag{1.4}$$

Recall that the conditions $\operatorname{div} b = 0$ in Ω and $b_3 = 0$ on $\partial\Omega$ in (1.4) should be regarded as the constraints on the initial data

$$\operatorname{div} b_0 = 0 \quad \text{in } \Omega \quad \text{and} \quad b_{0,3} = 0 \quad \text{on } \partial\Omega. \tag{1.5}$$

Indeed, it follows from $\partial_t b = \operatorname{curl} E$ in Ω and $E \times e_3 = 0$ on $\partial\Omega$ in (1.4) that $\partial_t \operatorname{div} b = 0$ in Ω and $\partial_t b_3 = 0$ on $\partial\Omega$.

Note the entropy-dissipation structure of (1.4): for a smooth solution, it holds that

$$\frac{d}{dt} \int_\Omega \left(\frac{1}{2}\rho|u|^2 + \rho c_v \left(\theta - \bar{\theta}\ln\frac{\theta}{\bar{\theta}} - \bar{\theta} \right) + R\bar{\theta} \left(\rho\ln\frac{\rho}{\bar{\rho}} - \rho + \bar{\rho} \right) + \frac{1}{2}|b|^2 \right)$$

$$+ \int_\Omega \left(\frac{\mu\bar{\theta}}{\theta^2}|\nabla\vartheta|^2 + \frac{\kappa\bar{\theta}}{\theta}|\operatorname{curl} b|^2 \right) = 0. \tag{1.6}$$

Hereafter the differential elements dx of the integrals over Ω are omitted. Furthermore, the following conservation laws hold:

$$\frac{d}{dt} \int_\Omega \varrho = 0, \quad \frac{d}{dt} \int_\Omega \left(\frac{1}{2}\rho|u|^2 + c_v\rho\vartheta + \frac{1}{2}|b|^2 \right) = 0, \quad \frac{d}{dt} \int_\Omega b_h = 0, \tag{1.7}$$

where $b_h = (b_1, b_2)$, etc. It follows that if initially

$$\int_\Omega \varrho_0 = 0, \quad \int_\Omega \left(\frac{1}{2}\rho_0|u_0|^2 + c_v\rho_0\vartheta_0 + \frac{1}{2}|b_0|^2 \right) = 0, \quad \int_\Omega b_{0,h} = 0, \tag{1.8}$$

then for any time $t > 0$,

$$\int_\Omega \varrho = 0, \quad \int_\Omega \left(\frac{1}{2}\rho|u|^2 + c_v\rho\vartheta + \frac{1}{2}|b|^2 \right) = 0, \quad \int_\Omega b_h = 0. \tag{1.9}$$

(1.9) allows the use of the Poincaré inequality for ϱ, ϑ, b_h in our analysis of the global well-posedness.

Let $H^k(\Omega)$, $k \geq 0$ and $H^s(\mathbb{T}^2)$, $s \in \mathbb{R}$ be the usual Sobolev spaces with norms denoted by $\|\cdot\|_m$ and $|\cdot|_s$, respectively. For an integer $N \geq 8$, define the high-order energy as

$$\mathcal{E}_{2N} := \sum_{j=0}^{2N} \left\| \partial_t^j(\varrho, u) \right\|_{2N-j}^2 + \sum_{j=0}^{2N-1} \left\| \partial_t^j(\vartheta, b) \right\|_{2N-j+1}^2 + \left\| \partial_t^{2N}(\vartheta, b) \right\|_0^2. \tag{1.10}$$

The key part in proving the global well-posedness of (1.4) is to show that $\mathcal{E}_{2N}(t)$ is bounded for all $t \geq 0$. To this end, we need to derive a sufficiently fast time-decay rate of certain lower-order Sobolev norms of the solution, which will follow from a set of energy-dissipation estimates. For $n = N+4, \ldots, 2N$, define the dissipations as

$$\mathfrak{D}_n := \sum_{j=0}^{n-1} \left\| \partial_t^j(\varrho, u) \right\|_{n-j-1}^2 + \sum_{j=0}^{n-2} \left\| \partial_t^j(\vartheta, b) \right\|_{n-j}^2 + \sum_{j=0}^{n} \left\| \partial_t^j(\vartheta, b) \right\|_{1, n-j}^2 \tag{1.11}$$

and the corresponding energies as

$$\mathfrak{E}_n := \sum_{j=0}^{n} \left\| \partial_t^j \varrho \right\|_{n-j}^2 + \|u\|_{0,n}^2 + \|u\|_{n-1}^2 + \sum_{j=1}^{n} \left\| \partial_t^j u \right\|_{n-j}^2$$

$$+ \|(\vartheta, b)\|_n^2 + \sum_{j=1}^{n-1} \left\| \partial_t^j(\vartheta, b) \right\|_{n-j+1}^2 + \left\| \partial_t^n(\vartheta, b) \right\|_0^2. \tag{1.12}$$

Here the anisotropic Sobolev norm $\|\cdot\|_{m,\ell}$ is defined as

$$\|f\|_{m,\ell} := \sum_{\alpha \in \mathbb{N}^2, |\alpha| \leq \ell} \|\partial^\alpha f\|_m. \tag{1.13}$$

Now the main result of this paper is stated as follows.

Theorem 1.1. *Assume that $\mu > 0$, $\kappa > 0$ and $\bar{\rho} > 0$, $\bar{\theta} > 0$, $\bar{B}_3 \neq 0$. Let $N \geq 8$ be an integer. Assume that $\varrho_0, u_0 \in H^{2N}(\Omega)$ and $\vartheta_0, b_0 \in H^{2N+1}(\Omega)$ are given such that $\mathcal{E}_{2N}(0) < +\infty$ and that the 2N-th order compatibility conditions required for the local well-posedness of (1.4)*

$$\partial_t^j u_3(0) = 0, \quad \partial_3 \partial_t^j \vartheta(0) = 0, \quad \partial_t^j E(0) \times e_3 = 0 \quad \text{on } \partial\Omega \quad \text{for} \quad j = 0, \ldots, 2N-1, \tag{1.14}$$

(1.5) *and* (1.8) *are satisfied. There exists a universal constant* $\varepsilon_0 > 0$ *such that if* $\mathcal{E}_{2N}(0) \leq \varepsilon_0$, *then there exists a unique global solution* $(\varrho, u, \vartheta, b)$ *to* (1.4) *satisfying that for all* $t \geq 0$,

$$\mathcal{E}_{2N}(t) \lesssim \mathcal{E}_{2N}(0) \tag{1.15}$$

and

$$\sum_{j=0}^{N-6} (1+t)^{N-5-j} \mathfrak{E}_{N+4+j}(t) + \sum_{j=0}^{N-6} \int_0^t (1+s)^{N-5-j} \mathfrak{D}_{N+4+j}(s) ds \lesssim \mathcal{E}_{2N}(0). \tag{1.16}$$

Remark 1.1. Note that the global well-posedness of compressible inviscid heat-conductive flows for the small initial data has been established only in 1D (see e.g., [9, 14, 19, 26]), but the one in multi-D is unknown. So our result reveals the strong stabilizing effect of the transversal magnetic field and resistivity. In a forthcoming paper, we will use some ideas developed here to prove the global well-posedness of free interface problems for the compressible MHD system (1.1), which generalizes our previous work [21] of the inviscid and resistive incompressible homogeneous MHD system.

Remark 1.2. Our result here relies crucially on the consideration of the heat-conductive flows, and it does not hold for isentropic flows or the case without heat conduction; for example, taking $B = \bar{B} = e_3$, these flows are reduced to the 1D compressible Euler flows for which in general one may expect the formation of shock in finite time. One of the key points here is that the heat conduction produces the dissipation control of the divergence of the velocity, as explained below.

Note that the local well-posedness of (1.4) (for $\rho, \theta > 0$) for $\mathcal{E}_{2N} < \infty$ with $N \geq 8$ is classical, which can be proved by using an iteration argument based on the solvability of the compressible Euler equations [17] and the parabolic problems for the temperature and magnetic field [6]. Hence, by a standard continuity argument, to prove the global well-posedness in Theorem 1.1 it suffices to derive a priori the estimate (1.15).

The basic strategy in our analysis is to use first the entropy-dissipation structure (1.6) to get the tangential energy evolution estimates of the solution to (1.4) as well as its temporal and horizontal spatial derivatives that preserve the boundary conditions, which will be carried out in Section 2. The next step is to derive the estimates involving normal derivatives by exploiting further the structures of the equations and boundary conditions, and the key here is to observe the following equations for the fluid vorticity curl$u$: applying curl to the second equation in

(1.4), and then using the horizontal components of the fourth equation to replace $\partial_3^2 b_h$ and the first equation to replace $\mathrm{div}\,u$, one finds

$$\rho(\partial_t + u \cdot \nabla)(\mathrm{curl}\,u)_i + \frac{\bar{B}_3^2}{\kappa}(\mathrm{curl}\,u)_i$$

$$= \bar{B}_h \cdot \nabla_h(\mathrm{curl}\,b)_i + (-1)^{i+1}\bar{B}_3 \partial_3 \partial_{3-i} b_3 + (-1)^{i+1}\frac{\bar{B}_3}{\kappa}(\kappa \Delta_h b_{3-i} - \partial_t b_{3-i})$$

$$+ (-1)^{i+1}\frac{\bar{B}_3}{\kappa}\left(\bar{B}_h \cdot \nabla_h u_{3-i} + \bar{B}_3 \partial_{3-i} u_3 + \frac{\bar{B}_{3-i}}{\bar{\rho}}\partial_t \varrho\right) + \cdots, \quad i = 1, 2, \qquad (1.17)$$

where $\nabla_h = (\partial_1, \partial_2)$, $\Delta_h = \partial_1^2 + \partial_2^2$ and "$+\cdots$" means plus some nonlinear terms. As $\kappa > 0$ and $\bar{B}_3 \neq 0$, one thus sees that there is an induced damping effect for $(\mathrm{curl}\,u)_h$ in (1.17). By controlling the linear terms in the last line of (1.17) in terms of the tangential energy at the beginning, in a recursive way in terms of the number of normal derivatives one derives the following full energy estimates:

$$\mathcal{E}_{2N}(t) \lesssim \mathcal{E}_{2N}(0) + \int_0^t \sqrt{\mathfrak{E}_{N+4}}\mathcal{E}_{2N}. \qquad (1.18)$$

Various elliptic estimates are involved in the derivation of (1.18), see Section 3 for the details.

Note that if $\sqrt{\mathfrak{E}_{N+4}}$ is integrable in time, then (1.18) can be closed to be (1.15). This will be achieved by using

$$\frac{d}{dt}\mathfrak{E}_n + \mathfrak{D}_n \leq 0, \quad n = N+4, \dots, 2N-2. \qquad (1.19)$$

(1.19) is derived in an elaborate way by controlling instead the linear terms in the last line of (1.17) in terms of the tangential dissipation. To achieve this, one notes first from the fourth equation in (1.4) that

$$\partial_t b - \kappa \Delta b = \bar{B} \cdot \nabla u - \bar{B}\,\mathrm{div}\,u + \cdots. \qquad (1.20)$$

It is then crucial here to use the third equation in (1.4) to replace $\mathrm{div}\,u$ in (1.20) to find

$$\partial_t b - \kappa \Delta b = \bar{B} \cdot \nabla u + \frac{\bar{B}}{\bar{p}}(c_v \bar{\rho}\partial_t \vartheta - \mu \Delta \vartheta) + \cdots. \qquad (1.21)$$

As mentioned in Remark 1.2, it should be pointed out here that such replacement is effective only for $\mu > 0$ since there is no priori tangential dissipation estimates of $\partial_t \vartheta$ if $\mu = 0$, and the same reason holds also for that one could not use the first equation in (1.4) to replace $\mathrm{div}\,u$ by $\partial_t \varrho / \bar{\rho}$ in (1.20). Now using the vertical

and horizontal components of (1.21), respectively, one then gets the tangential dissipation estimates of the following two quantities:

$$\bar{B}\cdot\nabla\left(u_3-\frac{\mu}{\bar{p}}\partial_3\vartheta\right),\quad \bar{B}\cdot\nabla\left(\kappa\partial_3 b_h+\bar{B}_3 u_h-\frac{\mu\bar{B}_h}{\bar{p}}\partial_3\vartheta\right).\qquad(1.22)$$

Thus one can use the Poincaré-type inequalities related to $\bar{B}\cdot\nabla$ for $\bar{B}_3\neq 0$ together with the boundary conditions on $\partial\Omega$ to derive the tangential dissipation estimates of u. The tangential dissipation estimates of $\partial_t\varrho$ then follow from the interaction between the first and second equations in (1.4).

Now observe that $\mathfrak{E}_\ell\leq\mathfrak{D}_{\ell+1}$. Then by employing a time weighted inductive argument based on (1.19), one concludes (1.16), which implies in particular a decay of $\sqrt{\mathfrak{E}_{N+4}}$ with the rate $(1+t)^{-(N-5)/2}$, see Section 4 for more details. Consequently, this scheme of the a priori estimates is closed by requiring $N\geq 8$.

Notation. Now we list the conventions for notation in this paper. $C>0$ denotes generic constants independent of the data and time, but may depend on the parameters of the problem, $\kappa,\mu,\bar{B},\bar{\rho},\bar{\theta}$ and N, which is referred to as "universal" and allowed to change from line to line. $A_1\lesssim A_2$ means that $A_1\leq CA_2$ for a universal constant $C>0$, $A_1\lesssim A_2+A_3$ means that $A_1\leq A_2+CA_3$ and

$$\partial_t A_1+A_2\lesssim A_3\quad\text{means}\quad\partial_t\widetilde{A}_1+A_2\lesssim A_3\quad\text{for}\quad A_1\lesssim\widetilde{A}_1\lesssim A_1.\qquad(1.23)$$

$\mathbb{N}=\{0,1,...\}$ denotes for the collection of non-negative integers. When using space-time differential multi-indices, we write $\mathbb{N}^{1+d}=\{\alpha=(\alpha_0,\alpha_1,...,\alpha_d)\}$ to emphasize that the 0-index term is related to temporal derivatives. For just spatial derivatives, we write $\mathbb{N}^d$. For $\alpha\in\mathbb{N}^{1+d}$, $\partial^\alpha=\partial_t^{\alpha_0}\partial_1^{\alpha_1}\cdots\partial_d^{\alpha_d}$. We define the standard commutators

$$\left[\partial^\alpha,f\right]g=\partial^\alpha(fg)-f\partial^\alpha g\quad\text{and}\quad\left[\partial^\alpha,f,g\right]=\partial^\alpha(fg)-f\partial^\alpha g-\partial^\alpha fg.\qquad(1.24)$$

2　Tangential energy evolution

In this section, we will derive energy evolution estimates for temporal and horizontal spatial derivatives of the solution to (1.4) by using the entropy-dissipation structure (1.6). It will be assumed throughout the rest of the paper that the solution to (1.4) is given on $[0,T]$ and obeys the a priori assumption

$$\mathcal{E}_{2N}(t)\leq\delta,\quad\forall t\in[0,T]\qquad(2.1)$$

for a sufficiently small constant $\delta > 0$. This implies in particular that

$$\frac{1}{2}\bar{\rho} \leq \rho \leq 2\bar{\rho}, \quad \frac{1}{2}\bar{\theta} \leq \theta \leq 2\bar{\theta}, \quad \forall (t,x) \in [0,T] \times \bar{\Omega}. \tag{2.2}$$

Note that (2.1) and (2.2) will be always used in the following; in particular, the smallness (2.1) is used in many nonlinear estimates so that various polynomials of $\mathcal{E}_{2N}$ are bounded by $C\mathcal{E}_{2N}$.

In order to use the nonlinear structure of (1.4) to derive the tangential energy evolution estimates at the $2N$ level, one applies ∂^α for $\alpha \in \mathbb{N}^{1+2}$ to (1.4) to find

$$\begin{cases} (\partial_t + u \cdot \nabla)\partial^\alpha \varrho + \rho \operatorname{div}\partial^\alpha u = F^{1,\alpha} & \text{in } \Omega, \\ \rho(\partial_t + u \cdot \nabla)\partial^\alpha u + \nabla(R\theta\partial^\alpha \varrho + R\rho\partial^\alpha \vartheta) = \operatorname{curl}\partial^\alpha b \times (\bar{B}+b) + F^{2,\alpha} & \text{in } \Omega, \\ c_v\rho(\partial_t + u \cdot \nabla)\partial^\alpha \vartheta + p \operatorname{div}\partial^\alpha u - \mu\Delta\partial^\alpha \vartheta = F^{3,\alpha} & \text{in } \Omega, \\ \partial_t\partial^\alpha b = \operatorname{curl}\partial^\alpha E, \quad \partial^\alpha E = \partial^\alpha u \times (\bar{B}+b) - \kappa\operatorname{curl}\partial^\alpha b + F^{4,\alpha} & \text{in } \Omega, \\ \operatorname{div}\partial^\alpha b = 0 & \text{in } \Omega, \\ \partial^\alpha u_3 = 0, \quad \partial_3\partial^\alpha \vartheta = 0, \quad \partial^\alpha b_3 = 0, \quad \partial^\alpha E \times e_3 = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

where

$$F^{1,\alpha} = -[\partial^\alpha, u] \cdot \nabla\varrho - [\partial^\alpha, \varrho]\operatorname{div}u, \tag{2.4a}$$

$$F^{2,\alpha} = -[\partial^\alpha, \rho(\partial_t + u \cdot \nabla)]u - \nabla(R[\partial^\alpha, \varrho, \vartheta]) - [\partial^\alpha, b] \times \operatorname{curl}b, \tag{2.4b}$$

$$F^{3,\alpha} = -[\partial^\alpha, c_v\rho(\partial_t + u \cdot \nabla)]\vartheta - [\partial^\alpha, p]\operatorname{div}u + \kappa\partial^\alpha(|\operatorname{curl}b|^2), \tag{2.4c}$$

$$F^{4,\alpha} = -[\partial^\alpha, b] \times u \tag{2.4d}$$

with notation of commutators given in (1.24).

These nonlinear terms $F^{i,\alpha}$ are estimated as follows.

Lemma 2.1. *For $|\alpha| \leq 2N$, it holds that*

$$\left\|(F^{1,\alpha}, F^{2,\alpha}, F^{3,\alpha}, F^{4,\alpha})\right\|_0^2 \lesssim \mathfrak{E}_{N+4}\mathcal{E}_{2N}. \tag{2.5}$$

Proof. Note that all terms in $F^{i,\alpha}$ are at least quadratic and each term can be written in the form XY, where X involves fewer derivatives than Y. One may use the definition of $\mathcal{E}_n$ to estimate $\|Y\|_0^2 \lesssim \mathcal{E}_{2N}$ and the usual Sobolev embeddings along with the definition of $\mathfrak{E}_n$ to estimate $\|X\|_{L^\infty}^2 \lesssim \mathfrak{E}_{N+4}$. Then

$$\|XY\|_0^2 \leq \|X\|_{L^\infty}^2\|Y\|_0^2 \lesssim \mathfrak{E}_{N+4}\mathcal{E}_{2N},$$

and the estimate (2.5) follows by summing terms. $\square$

For $n \geq 3$, define the tangential energy by

$$\bar{\mathcal{E}}_n := \sum_{j=0}^{n} \left\| \partial_t^j (\varrho, u, \vartheta, b) \right\|_{0, n-j}^2 \tag{2.6}$$

and the corresponding dissipation by

$$\bar{\mathcal{D}}_n := \sum_{j=0}^{n} \left\| \partial_t^j (\vartheta, b) \right\|_{1, n-j}^2, \tag{2.7}$$

where anisotropic Sobolev norms (1.13) have been used. Then one has the following tangential energy evolution at the $2N$ level.

Proposition 2.1. *It holds that*

$$\frac{d}{dt} \bar{\mathcal{E}}_{2N} + \bar{\mathcal{D}}_{2N} \lesssim \sqrt{\mathfrak{E}_{N+4}} \mathcal{E}_{2N}. \tag{2.8}$$

Proof. Let $\alpha \in \mathbb{N}^{1+2}$ with $|\alpha| \leq 2N$. First, taking the inner product of the second equation in (2.3) with $\partial^\alpha u$ and then integrating by parts over Ω, by using the first equation in (1.4) and the boundary conditions, one has

$$\frac{1}{2} \frac{d}{dt} \int_\Omega \rho |\partial^\alpha u|^2 - \int_\Omega (R\theta \partial^\alpha \varrho + R\rho \partial^\alpha \vartheta) \operatorname{div} \partial^\alpha u$$
$$= \int_\Omega \operatorname{curl} \partial^\alpha b \times (\bar{B} + b) \cdot \partial^\alpha u + \int_\Omega F^{2,\alpha} \cdot \partial^\alpha u. \tag{2.9}$$

The first equation in (2.3) leads to

$$-\int_\Omega R\theta \partial^\alpha \varrho \operatorname{div} \partial^\alpha u = \int_\Omega \frac{R\theta}{\rho} \left(\partial_t \partial^\alpha \varrho + u \cdot \nabla \partial^\alpha \varrho - F^{1,\alpha} \right) \partial^\alpha \varrho$$
$$= \frac{1}{2} \frac{d}{dt} \int_\Omega \frac{R\theta}{\rho} |\partial^\alpha \varrho|^2 - \frac{1}{2} \int_\Omega R\rho (\partial_t + u \cdot \nabla) \left(\frac{\theta}{\rho^2} \right) |\partial^\alpha \varrho|^2 - \int_\Omega \frac{R\theta}{\rho} F^{1,\alpha} \partial^\alpha \varrho. \tag{2.10}$$

Next, taking the inner product of the third equation with $\partial^\alpha \vartheta / \theta$ and using the relation $p = R\rho\theta$ give

$$\frac{1}{2} \frac{d}{dt} \int_\Omega \frac{c_v \rho}{\theta} |\partial^\alpha \vartheta|^2 + \int_\Omega \frac{\mu}{\theta} |\nabla \partial^\alpha \vartheta|^2 + \int_\Omega R\rho \operatorname{div} \partial^\alpha u \partial^\alpha \vartheta$$
$$= \frac{1}{2} \int_\Omega c_v \rho (\partial_t + u \cdot \nabla) \left(\frac{1}{\theta} \right) |\partial^\alpha \vartheta|^2 - \int_\Omega \mu \partial^\alpha \vartheta \nabla \left(\frac{1}{\theta} \right) \cdot \nabla \partial^\alpha \vartheta + \int_\Omega F^{3,\alpha} \partial^\alpha \vartheta, \tag{2.11}$$

while taking the inner product of the fourth equation with $\partial^\alpha b$ yields

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\partial^\alpha b|^2 + \kappa \int_\Omega |\text{curl}\partial^\alpha b|^2$$
$$= \int_\Omega \partial^\alpha u \times (\bar{B}+b)\cdot\text{curl}\partial^\alpha b + \int_\Omega F^{4,\alpha}\cdot\text{curl}\partial^\alpha b. \tag{2.12}$$

Consequently, combining (2.9)-(2.12) yields that, by (2.5),

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left(\frac{R\theta}{\rho}|\partial^\alpha\varrho|^2 + \rho|\partial^\alpha u|^2 + \frac{c_v\rho}{\theta}|\partial^\alpha\vartheta|^2 + |\partial^\alpha b|^2\right)$$
$$+ \int_\Omega \left(\frac{\mu}{\theta}|\nabla\partial^\alpha\vartheta|^2 + \kappa|\text{curl}\partial^\alpha b|^2\right)$$
$$= \int_\Omega \left(\frac{1}{2}R\rho(\partial_t+u\cdot\nabla)\left(\frac{\theta}{\rho^2}\right)|\partial^\alpha\varrho|^2 + \frac{1}{2}c_v\rho(\partial_t+u\cdot\nabla)\left(\frac{1}{\theta}\right)|\partial^\alpha\vartheta|^2\right.$$
$$\left. -\mu\partial^\alpha\vartheta\nabla\left(\frac{1}{\theta}\right)\cdot\nabla\partial^\alpha\vartheta\right)$$
$$+ \int_\Omega \left(\frac{R\theta}{\rho}F^{1,\alpha}\partial^\alpha\varrho + F^{2,\alpha}\cdot\partial^\alpha u + F^{3,\alpha}\partial^\alpha\vartheta + F^{4,\alpha}\cdot\text{curl}\partial^\alpha b\right)$$
$$\lesssim \sqrt{\mathfrak{E}_{N+4}}\left(\|\partial^\alpha\varrho\|_0^2 + \|\partial^\alpha\vartheta\|_0^2 + \|\partial^\alpha\vartheta\|_0\|\nabla\partial^\alpha\vartheta\|_0\right) + \|F^{1,\alpha}\|_0\|\partial^\alpha\varrho\|_0$$
$$+ \|F^{2,\alpha}\|_0\|\partial^\alpha u\|_0 + \|F^{3,\alpha}\|_0\|\partial^\alpha\vartheta\|_0 + \|F^{4,\alpha}\|_0\|\text{curl}\partial^\alpha b\|_0$$
$$\lesssim \sqrt{\mathfrak{E}_{N+4}}\mathcal{E}_{2N} + \sqrt{\mathfrak{E}_{N+4}}\sqrt{\mathcal{E}_{2N}}\left(\|\nabla\partial^\alpha\vartheta\|_0 + \|\text{curl}\partial^\alpha b\|_0\right). \tag{2.13}$$

By Cauchy's inequality, one deduces from (2.13) that, by the convention notation (1.23),

$$\frac{d}{dt}\|\partial^\alpha(\varrho,u,\vartheta,b)\|_0^2 + \|\nabla\partial^\alpha\vartheta\|_0^2 + \|\text{curl}\partial^\alpha b\|_0^2 \lesssim \sqrt{\mathfrak{E}_{N+4}}\mathcal{E}_{2N}. \tag{2.14}$$

It follows by using the Poincaré inequality and (1.9) that

$$\|\partial^\alpha\vartheta\|_0^2 \lesssim \|\nabla\partial^\alpha\vartheta\|_0^2 + \left|\int_\Omega \partial^\alpha\vartheta\right|^2 \lesssim \|\nabla\partial^\alpha\vartheta\|_0^2 + \mathfrak{E}_{N+4}\mathcal{E}_{2N}. \tag{2.15}$$

Employing the Hodge-type estimates (A.3) of Lemma A.1 with $r=1$ and by (1.9), one finds

$$\|\partial^\alpha b\|_1^2 \lesssim \|\text{curl}\partial^\alpha b\|_0^2. \tag{2.16}$$

Then the estimate (2.8) follows from (2.14) by summing over such α. $\square$

Now we rewrite (1.4) as a perturbation of the linearized equations

$$
\begin{cases}
\partial_t \varrho + \bar{\rho} \operatorname{div} u = G^1 & \text{in } \Omega, \\
\bar{\rho} \partial_t u + \nabla (R \bar{\theta} \varrho + R \bar{\rho} \vartheta) = \operatorname{curl} b \times \bar{B} + G^2 & \text{in } \Omega, \\
c_v \bar{\rho} \partial_t \vartheta + \bar{p} \operatorname{div} u - \mu \Delta \vartheta = G^3 & \text{in } \Omega, \\
\partial_t b - \kappa \Delta b = \bar{B} \cdot \nabla u - \bar{B} \operatorname{div} u + G^4 & \text{in } \Omega, \\
\operatorname{div} b = 0 & \text{in } \Omega, \\
u_3 = 0, \ \partial_3 \vartheta = 0, \ b_3 = 0, \ \kappa \partial_3 b_h = -\bar{B}_3 u_h & \text{on } \partial\Omega,
\end{cases}
\tag{2.17}
$$

where $\bar{p} = R\bar{\rho}\bar{\theta}$ and

$$
G^1 = -\operatorname{div}(\varrho u), \tag{2.18a}
$$

$$
G^2 = -\varrho \partial_t u - \rho u \cdot \nabla u - \nabla(R\varrho\vartheta) + \operatorname{curl} b \times b, \tag{2.18b}
$$

$$
G^3 = -c_v \varrho \partial_t \vartheta - c_v \rho u \cdot \nabla \vartheta - (p - \bar{p}) \operatorname{div} u + \kappa |\operatorname{curl} b|^2, \tag{2.18c}
$$

$$
G^4 = \operatorname{curl}(u \times b). \tag{2.18d}
$$

The nonlinear terms G^i are estimated as follows.

Lemma 2.2. *It holds that*

$$
\sum_{j=0}^{2N-1} \left\| \partial_t^j (G^1, G^2, G^3, G^4) \right\|_{2N-j-1}^2 \lesssim \min\{\mathfrak{E}_{N+4}, \mathfrak{D}_{N+4}\} \mathcal{E}_{2N}. \tag{2.19}
$$

Proof. (2.19) can be proved similarly to Lemma 2.1. □

One has the following tangential energy evolution at the $N+4,\dots,2N-2$ levels.

Proposition 2.2. *For $n = N+4,\dots,2N-2$, it holds that*

$$
\frac{d}{dt} (\bar{\mathcal{E}}_n - \bar{\mathcal{B}}_n) + \bar{\mathcal{D}}_n \lesssim \sqrt{\mathcal{E}_{2N}} \mathfrak{D}_n, \tag{2.20}
$$

where $\bar{\mathcal{B}}_n$ is defined by (2.27), which satisfies

$$
|\bar{\mathcal{B}}_n| \lesssim \sqrt{\mathcal{E}_{2N}} \mathfrak{E}_n. \tag{2.21}
$$

Proof. For $n = N+4,\dots,2N-2$, similarly as the derivation of (2.13), one deduces from (2.17) that for $\alpha \in \mathbb{N}^{1+2}$ with $|\alpha| \le n$,

$$
\frac{1}{2} \frac{d}{dt} \int_\Omega \left(\frac{R\bar{\theta}}{\bar{\rho}} |\partial^\alpha \varrho|^2 + \bar{\rho} |\partial^\alpha u|^2 + \frac{c_v \bar{\rho}}{\bar{\theta}} |\partial^\alpha \vartheta|^2 + |\partial^\alpha b|^2 \right)
$$

$$+\int_{\Omega}\left(\frac{\mu}{\bar{\theta}}|\nabla\partial^{\alpha}\vartheta|^{2}+\kappa|\mathrm{curl}\partial^{\alpha}b|^{2}\right)$$
$$=\int_{\Omega}\left(\frac{R\bar{\theta}}{\bar{\rho}}\partial^{\alpha}G^{1}\partial^{\alpha}\varrho+\partial^{\alpha}G^{2}\cdot\partial^{\alpha}u+\partial^{\alpha}G^{3}\partial^{\alpha}\vartheta+\partial^{\alpha}G^{4}\cdot\mathrm{curl}\partial^{\alpha}b\right). \qquad (2.22)$$

It follows from (2.19) that

$$\int_{\Omega}\left(\partial^{\alpha}G^{3}\partial^{\alpha}\vartheta+\partial^{\alpha}G^{4}\cdot\mathrm{curl}\partial^{\alpha}b\right)$$
$$\lesssim\|\partial^{\alpha}G^{3}\|_{0}\|\partial^{\alpha}\vartheta\|_{0}+\|\partial^{\alpha}G^{4}\|_{0}\|\mathrm{curl}\partial^{\alpha}b\|_{0}\lesssim\sqrt{\mathfrak{D}_{N+4}\mathcal{E}_{2N}}\sqrt{\mathfrak{D}_{n}}. \qquad (2.23)$$

Now we estimate the term G^{1}. If $|\alpha|\leq n-1$, then (2.19) implies

$$\int_{\Omega}\partial^{\alpha}\varrho\partial^{\alpha}G^{1}\lesssim\|\partial^{\alpha}\varrho\|_{0}\|\partial^{\alpha}G^{1}\|_{0}\lesssim\sqrt{\mathfrak{D}_{n}}\sqrt{\mathfrak{D}_{N+4}\mathcal{E}_{2N}}. \qquad (2.24)$$

If $|\alpha|=n$ and $\alpha_{1}+\alpha_{2}\geq 1$, then one writes $\alpha=\alpha'+(\alpha-\alpha')$ for $\alpha'\in\mathbb{N}^{2}$ with $\alpha'\leq\alpha$ and $|\alpha'|=1$ and then integrates by parts over Ω to have, since $n\leq 4N-2$ and by (2.19),

$$\int_{\Omega}\partial^{\alpha}\varrho\partial^{\alpha}G^{1}=\int_{\Omega}\partial^{\alpha-\alpha'}\varrho\partial^{\alpha+\alpha'}G^{1}$$
$$\lesssim\|\partial^{\alpha-\alpha'}\varrho\|_{0}\|\partial^{\alpha+\alpha'}G^{1}\|_{0}\lesssim\sqrt{\mathfrak{D}_{n}}\sqrt{\mathfrak{D}_{N+4}\mathcal{E}_{2N}}. \qquad (2.25)$$

The remaining case, $\alpha_{0}=n$, can be handled by the integration by parts in t and using (2.19) as

$$\int_{\Omega}\partial_{t}^{n}\varrho\partial_{t}^{n}G^{1}=\frac{d}{dt}\int_{\Omega}\partial_{t}^{n-1}\varrho\partial_{t}^{n}G^{1}-\int_{\Omega}\partial_{t}^{n-1}\varrho\partial_{t}^{n+1}G^{1}$$
$$\lesssim\frac{d}{dt}\int_{\Omega}\partial_{t}^{n-1}\varrho\partial_{t}^{n}G^{1}+\|\partial_{t}^{n-1}\varrho\|_{0}\|\partial_{t}^{n+1}G^{1}\|_{0}$$
$$\lesssim\frac{d}{dt}\int_{\Omega}\partial_{t}^{n-1}\varrho\partial_{t}^{n}G^{1}+\sqrt{\mathfrak{D}_{n}}\sqrt{\mathfrak{D}_{N+4}\mathcal{E}_{2N}}. \qquad (2.26)$$

Consequently, combining (2.24)-(2.26) (and doing the similar computations for the G^{2} term) and (2.23), one concludes (2.20) from (2.22) by summing over such α, where

$$\bar{\mathcal{B}}_{n}:=\int_{\Omega}\left(\frac{R\bar{\theta}}{\bar{\rho}}\partial_{t}^{n-1}\varrho\partial_{t}^{n}G^{1}+\partial_{t}^{n-1}u\cdot\partial_{t}^{n}G^{2}\right), \qquad (2.27)$$

which, by (2.19), satisfies (2.21). □

3   Boundedness estimates of $\mathcal{E}_N$

In this section, we will explore further the structures of (1.4) to derive the estimates involving the normal derivatives of the solution, with the tangential energy evolution estimates in hand.

For the estimates of the normal derivatives of the velocity u, as for the Euler equations, a natural way is to estimate first the fluid vorticity, $\mathrm{curl}\,u$, and then to use the Hodge-type estimates. Applying curl to the second equation in (1.4) yields that

$$\rho(\partial_t + u \cdot \nabla)\,\mathrm{curl}\,u = \bar{B} \cdot \nabla\,\mathrm{curl}\,b + \mathrm{curl}(\mathrm{curl}\,b \times b) - \big[\mathrm{curl}, \rho(\partial_t + u \cdot \nabla)\big]u. \qquad (3.1)$$

It follows from the second component of the fourth equation in (2.17) that

$$\bar{B} \cdot \nabla(\mathrm{curl}\,b)_1 \equiv \bar{B}_h \cdot \nabla_h(\mathrm{curl}\,b)_1 + \bar{B}_3 \partial_3 (\mathrm{curl}\,b)_1$$
$$= \bar{B}_h \cdot \nabla_h(\mathrm{curl}\,b)_1 + \bar{B}_3 \partial_3 \partial_2 b_3 + \frac{\bar{B}_3}{\kappa}\big(\kappa \Delta_h b_2 - \partial_t b_2 + \bar{B} \cdot \nabla u_2 - \bar{B}_2 \mathrm{div}\,u + G_2^4\big). \qquad (3.2)$$

On the other hand, one can write, by using the first equation in (2.17), that

$$\bar{B} \cdot \nabla u_2 - \bar{B}_2 \mathrm{div}\,u = \bar{B}_h \cdot \nabla_h u_2 - \bar{B}_3 (\mathrm{curl}\,u)_1 + \bar{B}_3 \partial_2 u_3 + \frac{\bar{B}_2}{\bar{\rho}}(\partial_t \varrho - G^1). \qquad (3.3)$$

Hence, as a consequence of (3.2) and (3.3), the first component of (3.1) can be rewritten as

$$\rho(\partial_t + u \cdot \nabla)(\mathrm{curl}\,u)_1 + \frac{\bar{B}_3^2}{\kappa}(\mathrm{curl}\,u)_1$$
$$= \bar{B}_h \cdot \nabla_h(\mathrm{curl}\,b)_1 + \bar{B}_3 \partial_3 \partial_2 b_3 + \frac{\bar{B}_3}{\kappa}\left(\kappa \Delta_h b_2 - \partial_t b_2 + \bar{B}_h \cdot \nabla_h u_2 + \bar{B}_3 \partial_2 u_3 + \frac{\bar{B}_2}{\bar{\rho}}\partial_t \varrho\right)$$
$$+ \frac{\bar{B}_3}{\kappa}\left(-\frac{\bar{B}_2}{\bar{\rho}}G^1 + G_2^4\right) + \big(\mathrm{curl}(\mathrm{curl}\,b \times b)\big)_1 - \big[\mathrm{curl}, \rho(\partial_t + u \cdot \nabla)\big]u_1. \qquad (3.4)$$

Similarly, one has

$$\rho(\partial_t + u \cdot \nabla)(\mathrm{curl}\,u)_2 + \frac{\bar{B}_3^2}{\kappa}(\mathrm{curl}\,u)_2$$
$$= \bar{B}_h \cdot \nabla_h(\mathrm{curl}\,b)_2 - \bar{B}_3 \partial_3 \partial_1 b_3 - \frac{\bar{B}_3}{\kappa}\left(\kappa \Delta_h b_1 - \partial_t b_1 + \bar{B}_h \cdot \nabla_h u_1 + \bar{B}_3 \partial_1 u_3 + \frac{\bar{B}_1}{\bar{\rho}}\partial_t \varrho\right)$$
$$- \frac{\bar{B}_3}{\kappa}\left(-\frac{\bar{B}_1}{\bar{\rho}}G^1 + G_1^4\right) + \big(\mathrm{curl}(\mathrm{curl}\,b \times b)\big)_2 - \big[\mathrm{curl}, \rho(\partial_t + u \cdot \nabla)\big]u_2. \qquad (3.5)$$

Eqs. (3.4) and (3.5) yield a transport-damping evolution structure for $(\mathrm{curl}\,u)_h$, which depends crucially on the positivity of the magnetic diffusion coefficient $\kappa > 0$ and the non-vanishing of $\bar{B}_3 \neq 0$.

Applying ∂^α for $\alpha \in \mathbb{N}^{1+3}$ with $|\alpha| \geq 1$ to (3.4) and (3.5) gives

$$\rho(\partial_t + u\cdot\nabla)\partial^\alpha(\mathrm{curl}\,u)_h + \frac{\bar{B}_3^2}{\kappa}\partial^\alpha(\mathrm{curl}\,u)_h = \partial^\alpha L_h + \Phi_h^\alpha, \tag{3.6}$$

where for $i=1,2$,

$$L_i = \bar{B}_h\cdot\nabla_h(\mathrm{curl}\,b)_i + (-1)^{i+1}\bar{B}_3\partial_3\partial_{3-i}b_3 \tag{3.7}$$
$$+ (-1)^{i+1}\frac{\bar{B}_3}{\kappa}\left(\kappa\Delta_h b_{3-i} - \partial_t b_{3-i} + \bar{B}_h\cdot\nabla_h u_{3-i} + \bar{B}_3\partial_{3-i}u_3 + \frac{\bar{B}_{3-i}}{\bar{\rho}}\partial_t\varrho\right)$$

and

$$\Phi_h^\alpha = \partial^\alpha\Phi_h - \left[\partial^\alpha, \rho(\partial_t + u\cdot\nabla)\right](\mathrm{curl}\,u)_h \tag{3.8}$$

with that for $i=1,2$,

$$\Phi_i = (-1)^{i+1}\frac{\bar{B}_3}{\kappa}\left(-\frac{\bar{B}_{3-i}}{\bar{\rho}}G^1 + G_{3-i}^4\right) + \left(\mathrm{curl}(\mathrm{curl}\,b\times b)\right)_i \tag{3.9}$$
$$- \left[\mathrm{curl}, \rho(\partial_t + u\cdot\nabla)\right]u_i.$$

The nonlinear terms Φ_h^α can be estimated as follows.

Lemma 3.1. *It holds that for $|\alpha| \leq 2N-1$,*

$$\|\Phi_h^\alpha\|_0^2 \lesssim \min\{\mathfrak{E}_{N+4}, \mathfrak{D}_{N+4}\}\mathcal{E}_{2N}. \tag{3.10}$$

Proof. The proof follows in the same way as for Lemmas 2.1. $\square$

The estimates involving normal derivatives at the $2N$ level are derived as follows.

Proposition 3.1. *It holds that*

$$\frac{d}{dt}\|(\mathrm{curl}\,u)_h\|_{2N-1}^2 + \sum_{j=0}^{2N}\|\partial_t^j(\varrho,u)\|_{2N-j}^2 + \sum_{j=0}^{2N}\|\partial_t^j(\vartheta,b)\|_{2N-j+1}^2$$
$$\lesssim \bar{\mathcal{E}}_{2N} + \bar{\mathcal{D}}_{2N} + \mathfrak{E}_{N+4}\mathcal{E}_{2N} \tag{3.11}$$

and that

$$\mathcal{E}_{2N} \lesssim \bar{\mathcal{E}}_{2N} + \|(\mathrm{curl}\,u)_h\|_{2N-1}^2 + \mathfrak{E}_{N+4}\mathcal{E}_{2N}. \tag{3.12}$$

Proof. Fix $\ell = 0, \ldots, 2N-1$. Let $\alpha \in \mathbb{N}^3$ with $|\alpha| \leq 2N-1$ such that $\alpha_3 \leq 2N-1-\ell$. Taking the inner product of (3.6) with $\partial^\alpha (\mathrm{curl}\, u)_h$ and then integrating by parts over Ω, one obtains

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho |\partial^\alpha (\mathrm{curl}\, u)_h|^2 + \frac{\bar{B}_3^2}{\kappa}\int_\Omega |\partial^\alpha (\mathrm{curl}\, u)_h|^2$$
$$= \int_\Omega \left(\partial^\alpha L_h + \Phi_h^\alpha\right)\cdot\partial^\alpha (\mathrm{curl}\, u)_h. \tag{3.13}$$

By Cauchy's inequality, (3.7) and (3.10), it follows from summing (3.13) over such α that

$$\frac{d}{dt}\|(\mathrm{curl}\, u)_h\|_{2N-1-\ell,\ell}^2 + \|(\mathrm{curl}\, u)_h\|_{2N-1-\ell,\ell}^2 \lesssim \sum_\alpha \|\partial^\alpha L_h\|_0^2 + \|\Phi_h^\alpha\|_0^2$$
$$\lesssim \|u\|_{2N-1-\ell,\ell+1}^2 + \|b\|_{2N-\ell,\ell+1}^2 + \|\partial_t(b,\varrho)\|_{2N-1}^2 + \mathfrak{E}_{N+4}\mathcal{E}_{2N}. \tag{3.14}$$

Now employing the Hodge-type estimates (A.1) of Lemma A.1 with $r = 2N-\ell \geq 1$ and using the first equation in (2.17) and (2.19), one obtains

$$\|u\|_{2N-\ell,\ell}^2 \lesssim \|u\|_{0,\ell+2N-\ell}^2 + \|(\mathrm{curl}\, u)_h\|_{2N-1-\ell,\ell}^2 + \|\mathrm{div}\, u\|_{2N-1-\ell,\ell}^2$$
$$\leq \|u\|_{0,2N}^2 + \|(\mathrm{curl}\, u)_h\|_{2N-1-\ell,\ell}^2 + \|\partial_t\varrho\|_{2N-1-\ell,\ell}^2 + \|G^1\|_{2N-1}^2$$
$$\lesssim \|u\|_{0,2N}^2 + \|(\mathrm{curl}\, u)_h\|_{2N-1-\ell,\ell}^2 + \|\partial_t\varrho\|_{2N-1}^2 + \mathfrak{E}_{N+4}\mathcal{E}_{2N}. \tag{3.15}$$

On the other hand, consider the following elliptic problem:

$$\begin{cases} -\kappa\Delta b = \bar{B}\cdot\nabla u - \bar{B}\,\mathrm{div}\, u - \partial_t b + G^4 & \text{in } \Omega, \\ b_3 = 0, \ \ \kappa\partial_3 b_h = -\bar{B}_3 u_h & \text{on } \partial\Omega. \end{cases} \tag{3.16}$$

By the standard H^r elliptic estimates with $r = 2N-\ell+1 \geq 2$, the trace theory (as $2N-\ell \geq 1$) and (2.19), one has

$$\|b\|_{2N-\ell+1,\ell}^2 \lesssim \|b_h\|_0^2 + \|u\|_{2N-\ell,\ell}^2 + \|\partial_t b\|_{2N-\ell-1,\ell}^2 + \|G^4\|_{2N-\ell-1,\ell}^2 + |u_h|_{2N-\frac{1}{2}}^2$$
$$\lesssim \|b_h\|_0^2 + \|u\|_{2N-\ell,\ell}^2 + \|\partial_t b\|_{2N-1}^2 + \mathfrak{E}_{N+4}\mathcal{E}_{2N}. \tag{3.17}$$

Then it follows from (3.14), (3.15) and (3.17) that

$$\frac{d}{dt}\|(\mathrm{curl}\, u)_h\|_{2N-1-\ell,\ell}^2 + \|u\|_{2N-\ell,\ell}^2 + \|b\|_{2N-\ell+1,\ell}^2$$
$$\lesssim \|u\|_{2N-1-\ell,\ell+1}^2 + \|b\|_{2N-\ell,\ell+1}^2 + \|\partial_t(b,\varrho)\|_{2N-1}^2 + \mathfrak{E}_{N+4}\mathcal{E}_{2N}. \tag{3.18}$$

A suitable linear combination of (3.18) for $\ell = 0, \ldots, 2N-1$ yields that

$$\frac{d}{dt} \|(\operatorname{curl} u)_h\|_{2N-1}^2 + \|u\|_{2N}^2 + \|b\|_{2N+1}^2$$
$$\lesssim \|u\|_{0,2N}^2 + \|b\|_{1,2N}^2 + \|\partial_t(b,\varrho)\|_{2N-1}^2 + \mathfrak{E}_{N+4}\mathcal{E}_{2N}. \tag{3.19}$$

Next, applying curl to the second equation in (2.17) yields

$$\partial_t \operatorname{curl} u = \bar{B} \cdot \nabla \operatorname{curl} b + \operatorname{curl} G^2. \tag{3.20}$$

For $j = 1, \ldots, 2N-1$, employing the Hodge-type estimates (A.2) of Lemma A.1 with $r = 2N - j \geq 1$, by (3.20), the first equation in (2.17) and (2.19), one obtains

$$\|\partial_t^j u\|_{2N-j}^2 \lesssim \|\partial_t^j u_h\|_0^2 + \|\partial_t^j \operatorname{curl} u\|_{2N-j-1}^2 + \|\partial_t^j \operatorname{div} u\|_{2N-j-1}^2$$
$$\lesssim \|\partial_t^j u_h\|_0^2 + \|\partial_t^{j-1} b\|_{2N-j+1}^2 + \|\partial_t^{j-1} G^2\|_{2N-j}^2$$
$$\quad + \|\partial_t^{j+1} \varrho\|_{2N-j-1}^2 + \|\partial_t^j G^1\|_{2N-j-1}^2$$
$$\lesssim \|\partial_t^j u_h\|_0^2 + \|\partial_t^{j-1} b\|_{2N-(j-1)}^2 + \|\partial_t^{j+1} \varrho\|_{2N-(j+1)}^2 + \mathfrak{E}_{N+4}\mathcal{E}_{2N}. \tag{3.21}$$

On the other hand, for $j = 0, \ldots, 2N-1$, by the Poincaré inequality and (1.9) and using the second equation in (2.17) and (2.19), one has

$$\|\partial_t^j \varrho\|_{2N-j}^2 \lesssim \|\partial_t^j \nabla \varrho\|_{2N-j-1}^2$$
$$\lesssim \|\partial_t^{j+1} u\|_{2N-j-1}^2 + \|\partial_t^j(\vartheta,b)\|_{2N-j}^2 + \|\partial_t^j G^2\|_{2N-j-1}^2$$
$$\lesssim \|\partial_t^{j+1} u\|_{2N-(j+1)}^2 + \|\partial_t^j(\vartheta,b)\|_{2N-j}^2 + \mathfrak{E}_{N+4}\mathcal{E}_{2N}. \tag{3.22}$$

We shall use the estimates (3.21) and (3.22) repeatedly. First, letting $j = 2N-1$ in (3.21) yields

$$\|\partial_t^{2N-1} u\|_1^2 \lesssim \|\partial_t^{2N-1} u\|_0^2 + \|\partial_t^{2N-2} b\|_2^2 + \|\partial_t^{2N} \varrho\|_0^2 + \mathfrak{E}_{N+4}\mathcal{E}_{2N}. \tag{3.23}$$

Next, for $j = 1, \ldots, 2N-2$, (3.21) together with (3.22) (with j replaced by $j+1$) implies

$$\|\partial_t^j u\|_{2N-j}^2 \lesssim \|\partial_t^j u\|_0^2 + \|\partial_t^{j-1} b\|_{2N-(j-1)}^2 + \|\partial_t^{j+2} u\|_{2N-(j+2)}^2$$
$$\quad + \|\partial_t^{j+1}(\vartheta,b)\|_{2N-(j+1)}^2 + \mathfrak{E}_{N+4}\mathcal{E}_{2N}. \tag{3.24}$$

Using a simple induction based on the estimate (3.24), one obtains

$$\sum_{j=1}^{2N-2}\big\|\partial_t^j u\big\|_{2N-j}^2 \lesssim \sum_{j=1}^{2N-2}\big\|\partial_t^j u\big\|_0^2 + \sum_{j=0}^{2N-3}\big\|\partial_t^j b\big\|_{2N-j}^2 + \big\|\partial_t^{2N-1}u\big\|_1^2 + \big\|\partial_t^{2N}u\big\|_0^2$$

$$+ \sum_{j=2}^{2N-1}\big\|\partial_t^j(\vartheta,b)\big\|_{2N-j}^2 + \mathfrak{E}_{N+4}\mathcal{E}_{2N}. \tag{3.25}$$

This together with the estimates (3.23) and (3.22) for $j=0,\dots,2N-1$ implies

$$\sum_{j=0}^{2N}\big\|\partial_t^j \varrho\big\|_{2N-j}^2 + \sum_{j=1}^{2N}\big\|\partial_t^j u\big\|_{2N-j}^2$$

$$\lesssim \sum_{j=1}^{2N}\big\|\partial_t^j u\big\|_0^2 + \big\|\partial_t^{2N}\varrho\big\|_0^2 + \sum_{j=0}^{2N-1}\big\|\partial_t^j(\vartheta,b)\big\|_{2N-j}^2 + \mathfrak{E}_{N+4}\mathcal{E}_{2N}. \tag{3.26}$$

Now, applying ∂_t^j, $j=1,\dots,2N-1$, to the problem (3.16) and the standard H^r elliptic estimates with $r=2N-j+1\geq 2$, the trace theory (as $2N-j\geq 1$) and (2.19) show that

$$\big\|\partial_t^j b\big\|_{2N-j+1}^2 \lesssim \big\|\partial_t^j b_h\big\|_0^2 + \big\|\partial_t^j u\big\|_{2N-j}^2 + \big\|\partial_t^{j+1}b\big\|_{2N-j-1}^2$$

$$+ \big\|\partial_t^j G^4\big\|_{2N-j-1}^2 + \big|\partial_t^j u_h\big|_{2N-j-\frac12}^2$$

$$\lesssim \big\|\partial_t^j b_h\big\|_0^2 + \big\|\partial_t^j u\big\|_{2N-j}^2 + \big\|\partial_t^{j+1}b\big\|_{2N-(j+1)}^2 + \mathfrak{E}_{N+4}\mathcal{E}_{2N}. \tag{3.27}$$

On the other hand, consider the following elliptic problem:

$$\begin{cases} -\mu\Delta\vartheta = -\bar{p}\,\mathrm{div}\,u - c_v\bar{\rho}\partial_t\theta + G^3 & \text{in } \Omega, \\ \partial_3\vartheta = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.28}$$

Applying ∂_t^j, $j=0,\dots,2N-1$, to the problem (3.28) and the standard H^r elliptic estimates with $r=2N-j+1\geq 2$ and (2.19) show that

$$\big\|\partial_t^j\vartheta\big\|_{2N-j+1}^2 \lesssim \big\|\partial_t^j\vartheta\big\|_0^2 + \big\|\partial_t^j u\big\|_{2N-j}^2 + \big\|\partial_t^{j+1}\vartheta\big\|_{2N-j-1}^2 + \big\|\partial_t^j G^3\big\|_{2N-j-1}^2$$

$$\lesssim \big\|\partial_t^j\vartheta\big\|_0^2 + \big\|\partial_t^j u\big\|_{2N-j}^2 + \big\|\partial_t^{j+1}\vartheta\big\|_{2N-(j+1)}^2 + \mathfrak{E}_{N+4}\mathcal{E}_{2N}. \tag{3.29}$$

Collecting (3.26), (3.27) with summing over $j=1,\dots,2N-1$ and (3.29) with summing over $j=0,\dots,2N-1$ yields that

$$\sum_{j=0}^{2N}\big\|\partial_t^j\varrho\big\|_{2N-j}^2 + \sum_{j=1}^{2N}\big\|\partial_t^j u\big\|_{2N-j}^2 + \sum_{j=0}^{2N-1}\big\|\partial_t^j\vartheta\big\|_{2N-j+1}^2 + \sum_{j=1}^{2N-1}\big\|\partial_t^j b\big\|_{2N-j+1}^2$$

$$\lesssim \|u\|_{2N}^2 + \sum_{j=1}^{2N}\|\partial_t^j u\|_0^2 + \|\partial_t^{2N}\varrho\|_0^2 + \sum_{j=0}^{2N-1}\|\partial_t^j(\vartheta,b)\|_{2N-j}^2 + \mathfrak{E}_{N+4}\mathcal{E}_{2N}. \tag{3.30}$$

Now combining (3.19) and (3.30) leads to

$$\frac{d}{dt}\|(\mathrm{curl}\,u)_h\|_{2N-1}^2 + \sum_{j=0}^{2N}\|\partial_t^j(\varrho,u)\|_{2N-j}^2 + \sum_{j=0}^{2N-1}\|\partial_t^j(\vartheta,b)\|_{2N-j+1}^2 \tag{3.31}$$

$$\lesssim \|u\|_{0,2N}^2 + \sum_{j=1}^{2N}\|\partial_t^j u\|_0^2 + \|\partial_t^{2N}\varrho\|_0^2 + \sum_{j=0}^{2N-1}\|\partial_t^j(\vartheta,b)\|_{2N-j}^2 + \|b\|_{1,2N}^2 + \mathfrak{E}_{N+4}\mathcal{E}_{2N}.$$

This together with the Sobolev interpolation implies that

$$\frac{d}{dt}\|(\mathrm{curl}\,u)_h\|_{2N-1}^2 + \sum_{j=0}^{2N}\|\partial_t^j(\varrho,u)\|_{2N-j}^2 + \sum_{j=0}^{2N-1}\|\partial_t^j(\vartheta,b)\|_{2N-j+1}^2 \tag{3.32}$$

$$\lesssim \|u\|_{0,2N}^2 + \sum_{j=1}^{2N}\|\partial_t^j u\|_0^2 + \|\partial_t^{2N}\varrho\|_0^2 + \sum_{j=0}^{2N-1}\|\partial_t^j(\vartheta,b)\|_0^2 + \|b\|_{1,2N}^2 + \mathfrak{E}_{N+4}\mathcal{E}_{2N},$$

which yields (3.11) by controlling the first fourth terms in the right hand side by $\bar{\mathcal{E}}_{2N}$ and the fifth term by $\bar{\mathcal{D}}_{2N}$.

We now prove (3.12). Recalling (3.15) with $\ell = 0$, (3.26), (3.27) and (3.29) for $j = 0,\dots,2N-1$, and then summing them up, similarly as the derivation of (3.32), one deduces

$$\mathcal{E}_{2N} \lesssim \|u\|_{0,2N}^2 + \sum_{j=1}^{2N}\|\partial_t^j u\|_0^2 + \|\partial_t^{2N}\varrho\|_0^2 + \sum_{j=0}^{2N-1}\|\partial_t^j(\vartheta,b)\|_0^2$$

$$+ \|(\mathrm{curl}\,u)_h\|_{2N-1}^2 + \mathfrak{E}_{N+4}\mathcal{E}_{2N}, \tag{3.33}$$

which yields (3.12) since the first four terms in the right hand side can be controlled by $\bar{\mathcal{E}}_{2N}$. $\qquad\square$

We now deduce the boundedness estimates of $\mathcal{E}_{2N}$. Set

$$\mathfrak{E}_{N+4}^w(t) := (1+t)^{N-5}\mathfrak{E}_{N+4}(t). \tag{3.34}$$

Theorem 3.1. *Let $N \ge 8$. It holds that*

$$\mathcal{E}_{2N}(t) + \int_0^t \bar{\mathcal{D}}_{2N}(s)\,ds \lesssim \mathcal{E}_{2N}(0) + \sup_{0\le s\le t}\mathcal{E}_{2N}(s)\sqrt{\mathfrak{E}_{N+4}^w(s)}, \quad \forall t\in[0,T]. \tag{3.35}$$

Proof. First, integrating (2.8) in time implies that for $N \geq 8$,

$$\bar{\mathcal{E}}_{2N}(t) + \int_0^t \bar{\mathcal{D}}_{2N}(s)\,ds$$

$$\lesssim \bar{\mathcal{E}}_{2N}(0) + \sup_{0 \leq s \leq t} \mathcal{E}_{2N}(s)\sqrt{\mathfrak{E}_{N+4}^w(s)} \int_0^t (1+s)^{-(N-5)/2}\,ds$$

$$\lesssim \bar{\mathcal{E}}_{2N}(0) + \sup_{0 \leq s \leq t} \mathcal{E}_{2N}(s)\sqrt{\mathfrak{E}_{N+4}^w(s)}. \tag{3.36}$$

Next, a Gronwall type argument for (3.11) yields

$$\|(\mathrm{curl}\,u)_h(t)\|_{2N-1}^2 \lesssim \mathcal{E}_{2N}(0) + \sup_{0 \leq s \leq t} \bar{\mathcal{E}}_{2N}(s) + \int_0^t \bar{\mathcal{D}}_{2N}(s)\,ds$$

$$+ \sup_{0 \leq s \leq t} \left(\mathcal{E}_{2N}(s)\mathcal{E}_{N+4}(s)\right)^2. \tag{3.37}$$

Hence, one concludes (3.35) from (3.36), (3.37) and (3.12). $\qquad\square$

4 Decay estimates of $\mathfrak{E}_{N+4}$

In this section, to close the full energy estimates (3.35), we will derive the decay estimates of $\mathfrak{E}_{N+4}$. This will follow from a set of energy-dissipation estimates related to $\mathfrak{E}_n$ and $\mathfrak{D}_n$.

Note that the tangential dissipation estimates of $\bar{\mathcal{D}}_n$ control only the temperature ϑ and magnetic field b. The tangential dissipation estimates for the velocity u rely on the coupling between the fluid and the magnetic field and the presence of the heat conductivity, and one has the following.

Proposition 4.1. *For $n = N+4,\ldots,2N$, it holds that*

$$\sum_{j=0}^{n-1} \|\partial_t^j u\|_{0,n-j-1}^2 \lesssim \bar{\mathcal{D}}_n + \mathfrak{D}_{N+4}\mathcal{E}_{2N}. \tag{4.1}$$

Proof. Let $n = N+4,\ldots,2N$. It follows from the fourth and third equations in (2.17) that

$$\partial_t b - \kappa \Delta b = \bar{B} \cdot \nabla u + \frac{\bar{B}}{\bar{p}}\left(c_v \bar{\rho} \partial_t \vartheta - \mu \Delta \vartheta - G^3\right) + G^4. \tag{4.2}$$

By the vertical component of (4.2) and using the fifth equation in (2.17), one obtains

$$\bar{B} \cdot \nabla \left(u_3 - \frac{\mu}{\bar{p}}\partial_3 \vartheta\right) \equiv \bar{B}_h \cdot \nabla_h \left(-\frac{\mu}{\bar{p}}\partial_3 \vartheta\right) + \bar{B} \cdot \nabla u_3 - \frac{\mu \bar{B}_3}{\bar{p}}\partial_3^2 \vartheta \tag{4.3}$$

$$= \bar{B}_h \cdot \nabla_h \left(-\frac{\mu}{\bar{p}} \partial_3 \vartheta \right) + \partial_t b_3 - \kappa \Delta_h b_3 + \kappa \partial_3 \mathrm{div}_h b_h - \frac{\bar{B}_3}{\bar{p}} \left(c_v \bar{\rho} \partial_t \vartheta - \mu \Delta_h \vartheta - G^3 \right) - G_3^4.$$

As $\bar{B}_3 \neq 0$ and since $u_3 = 0$ and $\partial_3 \vartheta = 0$ on $\partial\Omega$, by the Poincaré-type inequality, (4.3) and (2.19), one deduces that for $j = 0, \dots, n-1$,

$$\left\| \partial_t^j \left(u_3 - \frac{\mu}{\bar{p}} \partial_3 \vartheta \right) \right\|_{0,n-j-1}^2 \lesssim \left\| \bar{B} \cdot \nabla \partial_t^j \left(u_3 - \frac{\mu}{\bar{p}} \partial_3 \vartheta \right) \right\|_{0,n-j-1}^2$$

$$\lesssim \left\| \partial_t^j (\vartheta, b) \right\|_{1,n-j}^2 + \left\| \partial_t^{j+1} (\vartheta, b_3) \right\|_{0,n-j-1}^2 + \left\| \partial_t^j G^3 \right\|_{n-j-1}^2 + \left\| \partial_t^j G_3^4 \right\|_{n-j-1}^2$$

$$\lesssim \bar{\mathcal{D}}_n + \mathfrak{D}_{N+4} \mathcal{E}_{2N}. \tag{4.4}$$

This implies

$$\left\| \partial_t^j u_3 \right\|_{0,n-j-1}^2 \lesssim \left\| \partial_t^j \left(u_3 - \frac{\mu}{\bar{p}} \partial_3 \vartheta \right) \right\|_{0,n-j-1}^2 + \left\| \partial_t^j \vartheta \right\|_{1,n-j-1}^2$$

$$\lesssim \bar{\mathcal{D}}_n + \mathfrak{D}_{N+4} \mathcal{E}_{2N}. \tag{4.5}$$

Now by the horizontal components of (4.2), one finds

$$\bar{B} \cdot \nabla \left(\kappa \partial_3 b_h + \bar{B}_3 u_h - \frac{\mu \bar{B}_h}{\bar{p}} \partial_3 \vartheta \right) \tag{4.6}$$

$$\equiv \bar{B}_h \cdot \nabla_h \left(\kappa \partial_3 b_h - \frac{\mu \bar{B}_h}{\bar{p}} \partial_3 \vartheta \right) + \bar{B}_3 \left(\kappa \partial_3^2 b_h + \bar{B} \cdot \nabla u_h - \frac{\mu \bar{B}_h}{\bar{p}} \partial_3^2 \vartheta \right)$$

$$= \bar{B}_h \cdot \nabla_h \left(\kappa \partial_3 b_h - \frac{\mu \bar{B}_h}{\bar{p}} \partial_3 \vartheta \right) + \bar{B}_3 \left(\partial_t b_h - \kappa \Delta_h b_h - \frac{\bar{B}_h}{\bar{p}} \left(c_v \bar{\rho} \partial_t \vartheta - \mu \Delta_h \vartheta - G^3 \right) - G_h^4 \right).$$

Since $\partial_3 \vartheta = 0$ and $\kappa \partial_3 b_h + \bar{B}_3 u_h = 0$ on $\partial\Omega$, by the Poincaré-type inequality again, (4.6) and (2.19), one deduces that for $j = 0, \dots, n-1$,

$$\left\| \partial_t^j \left(\kappa \partial_3 b_h + \bar{B}_3 u_h - \frac{\mu \bar{B}_h}{\bar{p}} \partial_3 \vartheta \right) \right\|_{0,n-j-1}^2$$

$$\lesssim \left\| \bar{B} \cdot \nabla \partial_t^j \left(\kappa \partial_3 b_h + \bar{B}_3 u_h - \frac{\mu \bar{B}_h}{\bar{p}} \partial_3 \vartheta \right) \right\|_{0,n-j-1}^2$$

$$\lesssim \left\| \partial_t^j (\vartheta, b_h) \right\|_{1,n-j}^2 + \left\| \partial_t^{j+1} (\vartheta, b_h) \right\|_{0,n-j-1}^2 + \left\| \partial_t^j G^3 \right\|_{n-j-1}^2 + \left\| \partial_t^j G_h^4 \right\|_{n-j-1}^2$$

$$\lesssim \bar{\mathcal{D}}_n + \mathfrak{D}_{N+4} \mathcal{E}_{2N}. \tag{4.7}$$

This implies, using $\bar{B}_3 \neq 0$ again, that

$$\left\|\partial_t^j u_h\right\|_{0,n-j-1}^2 \lesssim \left\|\partial_t^j\left(\kappa\partial_3 b_h + \bar{B}_3 u_h - \frac{\mu\bar{B}_h}{\bar{p}}\partial_3\vartheta\right)\right\|_{0,n-j-1}^2 + \left\|\partial_t^j(\vartheta,b_h)\right\|_{1,n-j-1}^2$$

$$\lesssim \bar{\mathcal{D}}_n + \mathfrak{D}_{N+4}\mathcal{E}_{2N}. \tag{4.8}$$

This together with (4.5) gives (4.1). $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Next, we derive the dissipation estimates of $\partial_t^{n-1}\varrho$.

Proposition 4.2. *For $n = N+4,\dots,2N$, it holds that*

$$\frac{d}{dt}\int_\Omega \bar{\rho}\partial_t^{n-1}u\cdot\partial_t^{n-2}u + \left\|\partial_t^{n-1}\varrho\right\|_0^2 \lesssim \bar{\mathcal{D}}_n + \sqrt{\mathcal{E}_{2N}}\mathfrak{D}_n. \tag{4.9}$$

Proof. Let $n = N+4,\dots,2N$. One applies ∂_t^{n-2} to the first equation in (2.17) and then takes the inner product with $\frac{R\bar{\theta}}{\bar{\rho}}\partial_t^{n-1}\varrho$ to have

$$\int_\Omega \frac{R\bar{\theta}}{\bar{\rho}}\left|\partial_t^{n-1}\varrho\right|^2 + \int_\Omega R\bar{\theta}\partial_t^{n-1}\varrho\operatorname{div}\partial_t^{n-2}u = \int_\Omega \frac{R\bar{\theta}}{\bar{\rho}}\partial_t^{n-1}\varrho\partial_t^{n-2}G^1. \tag{4.10}$$

By (2.19), one obtains

$$\int_\Omega \frac{R\bar{\theta}}{\bar{\rho}}\partial_t^{n-1}\varrho\partial_t^{n-2}G^1 \lesssim \left\|\partial_t^{n-1}\varrho\right\|_0\left\|\partial_t^{n-2}G^1\right\|_0 \lesssim \sqrt{\mathfrak{D}_n}\sqrt{\mathfrak{D}_{N+4}\mathcal{E}_{2N}}. \tag{4.11}$$

Integrating by parts over Ω and using the second equation in (2.17), one has

$$\int_\Omega R\bar{\theta}\partial_t^{n-1}\varrho\operatorname{div}\partial_t^{n-2}u = -\int_\Omega \nabla\left(R\bar{\theta}\partial_t^{n-1}\varrho\right)\cdot\partial_t^{n-2}u$$

$$= \int_\Omega \left(\bar{\rho}\partial_t^n u + \nabla\left(R\bar{\rho}\partial_t^{n-1}\vartheta\right) - \operatorname{curl}\partial_t^{n-1}b\times\bar{B} - \partial_t^{n-1}G^2\right)\cdot\partial_t^{n-2}u. \tag{4.12}$$

Integrating by parts in time yields

$$\int_\Omega \bar{\rho}\partial_t^n u\cdot\partial_t^{n-2}u = \frac{d}{dt}\int_\Omega \bar{\rho}\partial_t^{n-1}u\cdot\partial_t^{n-2}u - \int_\Omega \bar{\rho}\left|\partial_t^{n-1}u\right|^2. \tag{4.13}$$

One obtains directly

$$-\int_\Omega \left(\nabla\left(R\bar{\rho}\partial_t^{n-1}\vartheta\right) - \operatorname{curl}\partial_t^{n-1}b\times\bar{B}\right)\cdot\partial_t^{n-2}u \lesssim \left\|\partial_t^{n-1}(\vartheta,b)\right\|_1\left\|\partial_t^{n-2}u\right\|_0 \tag{4.14}$$

and by (2.19),

$$\int_\Omega \partial_t^{n-1} G^2 \cdot \bar\rho \partial_t^{n-2} u \lesssim \left\| \partial_t^{n-1} G^2 \right\|_0 \left\| \partial_t^{n-2} u \right\|_0 \lesssim \sqrt{\mathfrak{D}_{N+4}\mathcal{E}_{2N}} \sqrt{\mathfrak{D}_n}. \tag{4.15}$$

Consequently, combining (4.10)-(4.15) yields, by Cauchy's inequality,

$$\frac{d}{dt} \int_\Omega \bar\rho \partial_t^{n-1} u \cdot \partial_t^{n-2} u + \left\| \partial_t^{n-1} \varrho \right\|_0^2$$
$$\lesssim \left\| (\partial_t^{n-2} u, \partial_t^{n-3} u) \right\|_0^2 + \left\| \partial_t^{n-1}(\vartheta, b) \right\|_1^2 + \sqrt{\mathcal{E}_{2N}}\mathfrak{D}_n. \tag{4.16}$$

One thus concludes (4.9) by controlling the first term in the right hand side by using (4.1) and the second term by $\bar{\mathcal{D}}_{n-1}$, since $n \geq N+4$. □

Now we derive the full energy-dissipation estimates.

Proposition 4.3. *For* $n = N+4, \ldots, 2N$, *it holds that*

$$\frac{d}{dt} \left(\left\| (\mathrm{curl}\, u)_h \right\|_{n-2}^2 + \int_\Omega \bar\rho \partial_t^{n-1} u \cdot \partial_t^{n-2} u \right) + \mathfrak{D}_n \lesssim \bar{\mathcal{D}}_n + \sqrt{\mathcal{E}_{2N}}\mathfrak{D}_n \tag{4.17}$$

and that

$$\mathfrak{E}_n \lesssim \bar{\mathcal{E}}_n + \left\| (\mathrm{curl}\, u)_h \right\|_{n-2}^2 + \mathfrak{E}_{N+4}\mathcal{E}_{2N}. \tag{4.18}$$

Proof. Let $n = N+4, \ldots, 2N$. It follows similarly as the derivation of (3.32), with $2N$ replaced by $n-1$, that

$$\frac{d}{dt} \left\| (\mathrm{curl}\, u)_h \right\|_{n-2}^2 + \sum_{j=0}^{n-1} \left\| \partial_t^j(\varrho, u) \right\|_{n-j-1}^2 + \sum_{j=0}^{n-2} \left\| \partial_t^j(\vartheta, b) \right\|_{n-j}^2 \tag{4.19}$$
$$\lesssim \left\| u \right\|_{0,n-1}^2 + \sum_{j=1}^{n-1} \left\| \partial_t^j u \right\|_0^2 + \left\| \partial_t^{n-1} \varrho \right\|_0^2 + \sum_{j=0}^{n-2} \left\| \partial_t^j(\vartheta, b) \right\|_0^2 + \left\| b \right\|_{1,n-1}^2 + \mathfrak{D}_{N+4}\mathcal{E}_{2N}.$$

One may appeal to (4.1) to control the first two terms in the right hand side, and the fourth and fifth terms are bounded by $\bar{\mathcal{D}}_{n-2}$. One then combines the resulting inequality and (4.9) to conclude (4.17).

We now prove (4.18). Taking $\ell = 0$ and replacing $2N$ by $n-1$ in (3.15), taking $j = 0$ and replacing $2N$ by $n-1$ in (3.27) and (3.29), and then combining them yield

$$\left\| u \right\|_{n-1}^2 + \left\| (\vartheta, b) \right\|_n^2 \lesssim \left\| u \right\|_{0,n-1}^2 + \left\| (\mathrm{curl}\, u)_h \right\|_{n-2}^2 + \left\| (\vartheta, b) \right\|_0^2$$
$$+ \left\| \partial_t(\varrho, \vartheta, b) \right\|_{n-2}^2 + \mathfrak{E}_{N+4}\mathcal{E}_{2N}. \tag{4.20}$$

Replacing $2N$ by $n-1$ in (3.30) (but without estimating $\|\vartheta\|_n^2$) leads to

$$\sum_{j=0}^{n}\left\|\partial_t^j\varrho\right\|_{n-j}^2+\sum_{j=1}^{n}\left\|\partial_t^j u\right\|_{n-j}^2+\sum_{j=1}^{n-1}\left\|\partial_t^j(\vartheta,b)\right\|_{n-j+1}^2$$

$$\lesssim\sum_{j=1}^{n}\left\|\partial_t^j u\right\|_0^2+\left\|\partial_t^n\varrho\right\|_0^2+\sum_{j=0}^{n-1}\left\|\partial_t^j(\vartheta,b)\right\|_{2N-j}^2+\mathfrak{E}_{N+4}\mathcal{E}_{2N}. \tag{4.21}$$

Hence, combining (4.20)-(4.21) gives

$$\mathfrak{E}_n\lesssim\bar{\mathcal{E}}_n+\|(\mathrm{curl}u)_h\|_{n-2}^2+\|\partial_t\varrho\|_{n-2}^2+\sum_{j=0}^{n-1}\left\|\partial_t^j(\vartheta,b)\right\|_{n-j}^2+\mathfrak{E}_{N+4}\mathcal{E}_{2N}. \tag{4.22}$$

This together with the Sobolev interpolation implies that

$$\mathfrak{E}_n\lesssim\bar{\mathcal{E}}_n+\|(\mathrm{curl}u)_h\|_{n-2}^2+\|\partial_t\varrho\|_0^2+\sum_{j=0}^{n-1}\left\|\partial_t^j(\vartheta,b)\right\|_0^2+\mathfrak{E}_{N+4}\mathcal{E}_{2N}$$

$$\lesssim\bar{\mathcal{E}}_n+\|(\mathrm{curl}u)_h\|_{n-2}^2+\mathfrak{E}_{N+4}\mathcal{E}_{2N}. \tag{4.23}$$

This is (4.18). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

With the previous estimates in hand, we now derive the decay estimates.

Theorem 4.1. *It holds that*

$$\int_0^t\mathfrak{D}_{2N}(s)ds+\sum_{j=0}^{N-6}(1+t)^{N-5-j}\mathfrak{E}_{N+4+j}(t)+\sum_{j=0}^{N-6}\int_0^t(1+s)^{N-5-j}\mathfrak{D}_{N+4+j}(s)ds$$

$$\lesssim\mathcal{E}_{2N}(0)+\sup_{0\le s\le t}\mathcal{E}_{2N}(s)\sqrt{\mathfrak{E}_{N+4}^w(s)}. \tag{4.24}$$

Proof. First, taking $n=2N$ in (4.17), one deduces that for $\mathcal{E}_{2N}\le\delta$ small,

$$\frac{d}{dt}\left(\|(\mathrm{curl}u)_h\|_{2N-2}^2+\int_\Omega\bar{\rho}\partial_t^{2N-1}u\cdot\partial_t^{2N-2}u\right)+\mathfrak{D}_{2N}\lesssim\bar{\mathcal{D}}_{2N}. \tag{4.25}$$

Integrating (4.25) in time and by (3.35) gives in particular that

$$\int_0^t\mathfrak{D}_{2N}(s)ds\lesssim\mathcal{E}_{2N}(0)+\bar{\mathcal{E}}_{2N}(t)+\int_0^t\bar{\mathcal{D}}_{2N}(s)ds$$

$$\lesssim\mathcal{E}_{2N}(0)+\sup_{0\le s\le t}\mathcal{E}_{2N}(s)\sqrt{\mathfrak{E}_{N+4}^w(s)}. \tag{4.26}$$

Next, it follows from (4.17) that for $n = N+4,\dots,2N-2$,

$$\frac{d}{dt}\left(\|(\mathrm{curl}\,u)_h\|_{n-2}^2 + \int_\Omega \bar\rho \partial_t^{n-1} u \cdot \partial_t^{n-2} u\right) + \mathfrak{D}_n \lesssim \bar{\mathcal{D}}_n. \qquad (4.27)$$

On the other hand, it follows from (4.18) that

$$\mathfrak{E}_n \lesssim \bar{\mathcal{E}}_n + \|(\mathrm{curl}\,u)_h\|_{n-2}^2. \qquad (4.28)$$

One can deduce from (4.27), (4.28), (2.20) and (2.21) that

$$\frac{d}{dt}\mathfrak{E}_n + \mathfrak{D}_n \le 0. \qquad (4.29)$$

Observe that $\mathfrak{E}_\ell \le \mathfrak{D}_{\ell+1}$. Then we will employ a time weighted inductive argument here. To begin with, it follows from (4.29) that for $j = 0,\dots,N-4$,

$$\frac{d}{dt}\mathfrak{E}_{N+4+j} + \mathfrak{D}_{N+4+j} \le 0. \qquad (4.30)$$

Multiplying (4.30) by $(1+t)^{N-5-j}$ and using $\mathfrak{E}_{N+4+j} \le \mathfrak{D}_{N+5+j}$, one gets

$$\frac{d}{dt}\left((1+t)^{N-5-j}\mathfrak{E}_{N+4+j}\right) + (1+t)^{N-5-j}\mathfrak{D}_{N+4+j}$$
$$\le (N-5-j)(1+t)^{N-6-j}\mathfrak{E}_{N+4+j} \lesssim (1+t)^{N-5-(j+1)}\mathfrak{D}_{N+4+(j+1)}. \qquad (4.31)$$

Integrating (4.31) in time directly and using a suitable linear combination of the resulting inequalities, one obtains

$$\sum_{j=0}^{N-6}(1+t)^{N-5-j}\mathfrak{E}_{N+4+j}(t) + \sum_{j=0}^{N-6}\int_0^t (1+s)^{N-5-j}\mathfrak{D}_{N+4+j}(s)\,ds$$
$$\lesssim \mathcal{E}_{2N}(0) + \int_0^t \mathfrak{D}_{2N-1}(s)\,ds. \qquad (4.32)$$

This together with (4.26) implies (4.24). $\qquad\qquad\square$

5   Global well-posedness

We are now ready to present the proof of Theorem 1.1.

Proof. The estimate (4.24) implies in particular (1.16), for $\mathcal{E}_{2N} \le \delta$ small. Then combining the estimates (3.35) and (1.16) yields (1.15). This closes the a priori assumption (2.1) if $\mathcal{E}_{2N}(0) \le \varepsilon_0$ for $\varepsilon_0 > 0$ sufficiently small. Therefore, the global well-posedness follows by using a standard continuity argument. $\qquad\square$

Appendix A . Hodge-type estimates

Lemma A.1. *Let $r \geq 1$ be an integer. Then it holds that*

$$\|v\|_r \lesssim \|v\|_{0,r} + \|(\mathrm{curl}\,v)_h\|_{r-1} + \|\mathrm{div}\,v\|_{r-1}. \tag{A.1}$$

If $v_3 = 0$ on $\partial\Omega$, then

$$\|v\|_r \lesssim \|v_h\|_0 + \|\mathrm{curl}\,v\|_{r-1} + \|\mathrm{div}\,v\|_{r-1}. \tag{A.2}$$

If $v_3 = 0$ on $\partial\Omega$ and $\int_\Omega v_h = 0$, then

$$\|v\|_r \lesssim \|\mathrm{curl}\,v\|_{r-1} + \|\mathrm{div}\,v\|_{r-1}. \tag{A.3}$$

Proof. One may refer to [21, Lemma A.9] for the proof of (A.1). Now for $v_3 = 0$ on $\partial\Omega$, by the standard elliptic estimates on $-\Delta v_3 = (\mathrm{curl}\,\mathrm{curl}\,v)_3 - \partial_3 \mathrm{div}\,v$, one obtains

$$\|v_3\|_r \lesssim \|(\mathrm{curl}\,v)_h\|_{r-1} + \|\mathrm{div}\,v\|_{r-1}. \tag{A.4}$$

On the other hand, one has

$$\|\nabla v_h\|_{r-1} \lesssim \|\nabla v_3\|_{r-1} + \|\mathrm{curl}\,v\|_{r-1} + \|\mathrm{div}\,v\|_{r-1}. \tag{A.5}$$

Then (A.2) follows from (A.4)-(A.5), and (A.3) follows by using further the Poincaré inequality. $\square$

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