

Global Solutions of 3-D Inhomogeneous Navier-Stokes System with Large Viscosity in One Variable

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Abstract. We consider the global well-posedness of three dimensional incompressible inhomogeneous Navier-Stokes equation with different viscous coefficients in the vertical and horizontal variables. In particular, when one of these viscous coefficients is large enough compared with the initial data and the initial density is close enough to a positive constant, we prove the global well-posedness of this system. This result extends the previous results in [9, 11] for the classical Navier-Stokes system.

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1 Introduction

We consider the global existence of strong solution to the following 3-D inhomogeneous incompressible anisotropic Navier-Stokes equations with initial density being sufficiently close to a positive constant in the critical space, $\dot{B}_{p,1}^{\frac{3}{p}}(\mathbb{R}^3)$ for some $p \in]3, 4[$, and with a large viscous coefficient in one direction:

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$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \rho(\partial_t u + u \cdot \nabla u) - \nu_h \Delta_h u - \nu_v \partial_3^2 u + \nabla P = 0, \\ \operatorname{div} u = 0, \\ (\rho, u)|_{t=0} = (\rho_0, u_0), \end{cases} \quad (1.1)$$

where ρ, u stand for the density and velocity of the fluid respectively, ∇P is a scalar pressure function, which guarantees the divergence free condition of the velocity field, ν_h and ν_v are viscous coefficients so that ν_v is much larger than $\nu_h > 0$, $\Delta_h \stackrel{\text{def}}{=} \partial_{x_1}^2 + \partial_{x_2}^2$ designates the horizontal Laplacian. One may check [8] for more background of this system. The system (1.1) has three major basic features. First, the incompressibility expressed by fact that the vector field u is divergence free gives

$$\|\rho(t)\|_{L^\infty} = \|\rho_0\|_{L^\infty}.$$

Second, this system has the following energy law

$$\frac{1}{2} \|\sqrt{\rho}u(t)\|_{L^2}^2 + \nu_h \|\nabla_h u\|_{L_t^2(L^2)}^2 + \nu_v \|\partial_3 u\|_{L_t^2(L^2)}^2 = \frac{1}{2} \|\sqrt{\rho_0}u_0\|_{L^2}^2. \quad (1.2)$$

The third basic feature is the scaling invariance property: If (ρ, u, P) is a solution of (1.1) on $[0, T] \times \mathbb{R}^3$, then the rescaled triplet $(\rho, u, P)_\lambda$ defined by

$$(\rho, u, P)_\lambda(t, x) \stackrel{\text{def}}{=} \left(\rho(\lambda^2 t, \lambda x), \lambda u(\lambda^2 t, \lambda x), \lambda^2 P(\lambda^2 t, \lambda x) \right) \quad \text{with } \lambda \in \mathbb{R}^+ \quad (1.3)$$

is also a solution of (1.1) on $[0, T/\lambda^2] \times \mathbb{R}^3$. Motivated by (1.3), Danchin [7] established the well-posedness of (1.1) in the so-called critical functional framework for small perturbations of some positive constant density.

When the density ρ is away from zero, we set $a \stackrel{\text{def}}{=} \frac{1}{\rho} - 1$. Then (1.1) can be equivalently formulated as

$$\begin{cases} \partial_t a + u \cdot \nabla a = 0, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \partial_t u + u \cdot \nabla u - \nu_h(1+a)\Delta_h u - \nu_v(1+a)\partial_3^2 u + (1+a)\nabla P = 0, \\ \operatorname{div} u = 0, \\ (a, u)|_{t=0} = (a_0, u_0). \end{cases} \quad (1.4)$$

One may check [5] and the references therein concerning the well-posedness theory of the system (1.1) or (1.4). In particular, when $\rho_0 \in L^\infty$ with a positive lower bound and initial velocity being sufficiently small in the critical Besov space,

$B_{2,1}^{\frac{1}{2}}$, Zhang [13] proved the global existence of weak solutions to the system (1.1) and also provided a lower bound for the lifespan of smooth enough solutions of (1.1).

Motivated by [9, 11] concerning the global wellposedness of 3-D anisotropic incompressible Navier-Stokes system with different viscous coefficients in the vertical and horizontal variables, we are going to study the inhomogeneous Navier-Stokes system, which also has different viscous coefficients. Our result below (see Theorem 1.1) ensures that given initial data u_0 , for small enough a_0 , if ν_v is sufficiently large compared with $\nu_h > 0$, then (1.4) has a unique global solution.

Notation. In what follows, we shall denote

$$B_p^s \stackrel{\text{def}}{=} B_{p,1}^s, \quad B_p^{s_1, s_2} \stackrel{\text{def}}{=} B_{p,1}^{s_1, s_2}.$$

The definitions of the above spaces will be recalled in Section 2.

The main result states as follows:

Theorem 1.1. *Let $p \in]3, 4[$ and $\theta \in]3/4, 1[$. Let $a_0 \in B_p^{\frac{3}{p}}(\mathbb{R}^3)$ and $u_0 \in B_2^{0, \frac{1}{2}}(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$ be a solenoidal vector field. Then there exist two positive constants c_0 and C such that if*

$$\left(\nu_h \|a_0\|_{B_p^{\frac{3}{p}}} + \|a_0\|_{B_p^{\frac{3}{p}}} \|u_0\|_{B_2^{0, \frac{1}{2}}} + \frac{\nu_h^{1-\theta}}{\nu_v^{1-\theta}} \|a_0\|_{B_p^{\frac{3}{p}}} \|u_0\|_{B_2^{\frac{1}{2}}} \right. \tag{1.5}$$

$$\left. + \frac{1}{\nu_h^\theta \nu_v^{1-\theta}} \|u_0\|_{B_2^{0, \frac{1}{2}}} \left(\|u_0\|_{B_2^{0, \frac{1}{2}}} + \|u_0\|_{B_2^{\frac{1}{2}}} \right) \right) \exp \left(\frac{C \left(\|u_0\|_{B_2^{0, \frac{1}{2}}} + \|u_0\|_{B_2^{\frac{1}{2}}} \right)}{\nu_h^\theta \nu_v^{1-\theta}} \right) \leq c_0 \nu_h,$$

the system (1.4) has a unique global solution

$$a \in C\left([0, \infty); B_p^{\frac{3}{p}}(\mathbb{R}^3)\right), \quad u \in C\left([0, \infty); H^1(\mathbb{R}^3)\right) \cap L^2\left(\mathbb{R}^+; \dot{H}^2(\mathbb{R}^3)\right).$$

This paper is organized as follows. In Section 2, we shall recall some basic facts on Littlewood-Paley theory and some direct consequences. In Section 3, we shall prove a regularity criteria, namely (3.2), for the local smooth solution constructed in [12]. In Section 4, we shall prove that under the assumption of (1.5), the regularity criteria (3.2) holds for any $T < T^*$, where T^* denotes the lifespan of the local smooth solution of (1.1). In order to do so, we denote $u_L \stackrel{\text{def}}{=} e^{t(\nu_h \Delta_h + \nu_v \partial_3^2)} u_0$. Then u_L solves

$$\begin{cases} \partial_t u_L - \nu_h \Delta_h u_L - \nu_v \partial_3^2 u_L = 0, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ u_L|_{t=0} = u_0. \end{cases} \tag{1.6}$$

Motivated by [11], we write

$$u = u_L + R,$$

then it follows from (1.4) and (1.6), that

$$\begin{cases} \partial_t R + R \cdot \nabla R - \nu_h \Delta_h R - \nu_v \partial_3^2 R + \nabla P = F & \text{with} \\ F = a(\nu_h \Delta_h R + \nu_v \partial_3^2 R + \nu_h \Delta_h u_L + \nu_v \partial_3^2 u_L) \\ \quad - u_L \cdot \nabla u_L - R \cdot \nabla u_L - u_L \cdot \nabla R - a \nabla P, \\ \operatorname{div} R = 0, \\ R|_{t=0} = 0. \end{cases} \quad (1.7)$$

Indeed we shall prove that the regularity criteria (3.2) can not happen for the remaining term R through anisotropic Littlewood-Paley analysis.

2 Preliminaries

Before we present the function spaces we are going to work with in this context, let us briefly recall some basic facts on Littlewood-Paley theory (see, e.g., [1]).

Let φ and χ be smooth functions supported in $\mathcal{C} \stackrel{\text{def}}{=} \{\tau \in \mathbb{R}^+, \frac{3}{4} \leq \tau \leq \frac{8}{3}\}$ and $\mathfrak{B} \stackrel{\text{def}}{=} \{\tau \in \mathbb{R}^+, \tau \leq \frac{4}{3}\}$ respectively such that

$$\begin{aligned} \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\tau) &= 1 & \text{for } \tau > 0, \\ \chi(\tau) + \sum_{j \geq 0} \varphi(2^{-j}\tau) &= 1 & \text{for } \tau \geq 0. \end{aligned}$$

For $a \in \mathcal{S}'(\mathbb{R}^3)$, we set

$$\begin{aligned} \Delta_k^h a &\stackrel{\text{def}}{=} \mathcal{F}^{-1}(\varphi(2^{-k}|\xi_h|)\hat{a}), & S_k^h a &\stackrel{\text{def}}{=} \mathcal{F}^{-1}(\chi(2^{-k}|\xi_h|)\hat{a}), \\ \Delta_\ell^v a &\stackrel{\text{def}}{=} \mathcal{F}^{-1}(\varphi(2^{-\ell}|\xi_3|)\hat{a}), & S_\ell^v a &\stackrel{\text{def}}{=} \mathcal{F}^{-1}(\chi(2^{-\ell}|\xi_3|)\hat{a}), \\ \Delta_j a &\stackrel{\text{def}}{=} \mathcal{F}^{-1}(\varphi(2^{-j}|\xi|)\hat{a}), & S_j a &\stackrel{\text{def}}{=} \mathcal{F}^{-1}(\chi(2^{-j}|\xi|)\hat{a}), \end{aligned} \quad (2.1)$$

where $\xi_h = (\xi_1, \xi_2)$, $\xi = (\xi_h, \xi_3)$, $\mathcal{F}a$ and \hat{a} denote the Fourier transform of the distribution a . The dyadic operators satisfy the property of almost orthogonality

$$\begin{aligned} \Delta_k \Delta_j a &\equiv 0, & \text{if } |k-j| \geq 2, \\ \Delta_k (S_{j-1} a \Delta_j b) &\equiv 0, & \text{if } |k-j| \geq 5. \end{aligned} \quad (2.2)$$

Similar properties hold for Δ_k^h and Δ_ℓ^v .

Let us recall the anisotropic Bernstein lemma from [3, 10].

Lemma 2.1. *Let \mathcal{B}_h (resp. \mathcal{B}_v) be a ball of \mathbb{R}_h^2 (resp. \mathbb{R}_v), and \mathcal{C}_h (resp. \mathcal{C}_v) be a ring of \mathbb{R}_h^2 (resp. \mathbb{R}_v); let $1 \leq p_2 \leq p_1 \leq \infty$ and $1 \leq q_2 \leq q_1 \leq \infty$. Then there hold:*

- If $\text{Supp } \hat{a} \subset 2^k \mathcal{B}_h$, then $\|\partial_{x_h}^\alpha a\|_{L_h^{p_1}(L_v^{q_1})} \lesssim 2^{k(|\alpha|+2(1/p_2-1/p_1))} \|a\|_{L_h^{p_2}(L_v^{q_1})}$.
- If $\text{Supp } \hat{a} \subset 2^\ell \mathcal{B}_v$, then $\|\partial_{x_3}^\beta a\|_{L_h^{p_1}(L_v^{q_1})} \lesssim 2^{\ell(\beta+(1/q_2-1/q_1))} \|a\|_{L_h^{p_1}(L_v^{q_2})}$.
- If $\text{Supp } \hat{a} \subset 2^k \mathcal{C}_h$, then $\|a\|_{L_h^{p_1}(L_v^{q_1})} \lesssim 2^{-kN} \sup_{|\alpha|=N} \|\partial_{x_h}^\alpha a\|_{L_h^{p_1}(L_v^{q_1})}$.
- If $\text{Supp } \hat{a} \subset 2^\ell \mathcal{C}_v$, then $\|a\|_{L_h^{p_1}(L_v^{q_1})} \lesssim 2^{-\ell N} \|\partial_{x_3}^N a\|_{L_h^{p_1}(L_v^{q_1})}$.

Due to the anisotropic spectral properties of the linear equation (1.6), we need the following anisotropic Besov norm:

Definition 2.1. *Let $s, s_1, s_2 \in \mathbb{R}$, $1 \leq p \leq \infty$, $1 \leq q \leq \infty$ and $a \in \mathcal{S}'_h(\mathbb{R}^3)$, we define the norm*

$$\begin{aligned} \|a\|_{B_{p,1}^s} &\stackrel{\text{def}}{=} \left(2^{js} \|\Delta_j a\|_{L^p} \right)_{\ell^1}, \\ \|a\|_{B_{p,1}^{s_1,s_2}} &\stackrel{\text{def}}{=} \left(2^{\ell s_2} 2^{ks_1} \|\Delta_k^h \Delta_\ell^v a\|_{L^p} \right)_{\ell^1}, \\ \|a\|_{\tilde{L}_t^q(B_{p,1}^s)} &\stackrel{\text{def}}{=} \left(2^{js} \|\Delta_j a\|_{L^q([0,t];L^p)} \right)_{\ell^1}, \\ \|a\|_{\tilde{L}_t^q(B_{p,1}^{s_1,s_2})} &\stackrel{\text{def}}{=} \left(2^{\ell s_2} 2^{ks_1} \|\Delta_k^h \Delta_\ell^v a\|_{L^q([0,t];L^p)} \right)_{\ell^1}. \end{aligned}$$

We recall the classical homogeneous anisotropic Sobolev norm as follows:

$$\|a\|_{\dot{H}^{s_1,s_2}} \stackrel{\text{def}}{=} \left(\sum_{(k,\ell) \in \mathbb{Z}^2} 2^{2ks_1} 2^{2\ell s_2} \|\Delta_k^h \Delta_\ell^v a\|_{L^2}^2 \right)^{\frac{1}{2}}. \quad (2.3)$$

For the convenience of readers, we also recall the following product laws in the anisotropic spaces.

Lemma 2.2 ([2]). *Let $s_1 \leq \frac{2}{p}, s_2 \leq \frac{2}{p}$, with $s_1 + s_2 > 0$. Let $\sigma_1 \leq \frac{1}{p}, \sigma_2 \leq \frac{1}{p}$, with $\sigma_1 + \sigma_2 > 0$.*

If $a \in B_p^{s_1,\sigma_1}(\mathbb{R}^3)$, $b \in B_p^{s_2,\sigma_2}(\mathbb{R}^3)$, then $ab \in B_p^{s_1+s_2-\frac{2}{p},\sigma_1+\sigma_2-\frac{1}{p}}(\mathbb{R}^3)$, and

$$\|ab\|_{B_p^{s_1+s_2-\frac{2}{p},\sigma_1+\sigma_2-\frac{1}{p}}} \lesssim \|a\|_{B_p^{s_1,\sigma_1}} \|b\|_{B_p^{s_2,\sigma_2}}.$$

As an application of the above facts, we present the following lemma concerning the solution of the linear equation (1.6), which tells us the small quantities that will be used in what follows.

Lemma 2.3. *Let $p \in]3, 4[$, $\theta \in]3/4, 1[$, and $u_0 \in B_2^{0, \frac{1}{2}} \cap B_2^{\frac{1}{2}}$. Let $u_L \stackrel{def}{=} e^{t(\nu_h \Delta_h + \nu_v \partial_3^2)} u_0$. Then we have*

$$\|u_L\|_{\tilde{L}_t^\infty(B_p^{-1+\frac{2}{p}, \frac{1}{p}})} \lesssim \|u_0\|_{B_2^{0, \frac{1}{2}}}, \tag{2.4a}$$

$$\|\nabla_h u_L\|_{L_t^1(B_p^{\frac{2}{p}, \frac{1}{p}})} \lesssim \frac{1}{\nu_h^\theta \nu_v^{1-\theta}} \|u_0\|_{B_2^{\frac{1}{2}}}, \tag{2.4b}$$

$$\|\partial_3 u_L\|_{L_t^1(B_p^{\frac{2}{p}, \frac{1}{p}})} \lesssim \frac{1}{\sqrt{\nu_h \nu_v}} \|u_0\|_{B_2^{0, \frac{1}{2}}}, \tag{2.4c}$$

$$\|\partial_3^2 u_L\|_{L_t^1(B_p^{-1+\frac{2}{p}, \frac{1}{p}})} \lesssim \frac{1}{\nu_v} \|u_0\|_{B_2^{0, \frac{1}{2}}}. \tag{2.4d}$$

Proof. By the definition of u_L , we have

$$\|\Delta_k^h \Delta_\ell^v u_L\|_{L_t^\infty(L^p)} \lesssim \|\Delta_k^h \Delta_\ell^v u_0\|_{L^p} \lesssim 2^{k(1-\frac{2}{p})} 2^{\ell(\frac{1}{2}-\frac{1}{p})} \|\Delta_k^h \Delta_\ell^v u_0\|_{L^2},$$

which implies (2.4a).

We also observe that

$$\begin{aligned} \|\Delta_k^h \Delta_\ell^v u_L\|_{L_t^1(L^p)} &\lesssim \|\Delta_k^h \Delta_\ell^v u_0\|_{L^p} \int_0^t e^{-ct'(\nu_h 2^{2k} + \nu_v 2^{2\ell})} dt' \\ &\lesssim \frac{1}{\nu_h 2^{2k} + \nu_v 2^{2\ell}} \|\Delta_k^h \Delta_\ell^v u_0\|_{L^p}, \end{aligned}$$

which together with Lemma 2.1 ensures that

$$\|\Delta_k^h \Delta_\ell^v u_L\|_{L_t^1(L^p)} \lesssim \frac{2^{k(1-\frac{2}{p})} 2^{\ell(\frac{1}{2}-\frac{1}{p})}}{\nu_h 2^{2k} + \nu_v 2^{2\ell}} \|\Delta_k^h \Delta_\ell^v u_0\|_{L^2}. \tag{2.5}$$

Thanks to (2.5), for any $\theta \in [0, 1]$, we infer

$$\begin{aligned} \|\nabla_h \Delta_k^h \Delta_\ell^v u_L\|_{L_t^1(L^p)} &\lesssim \frac{2^{k(2-\frac{2}{p})} 2^{\ell(\frac{1}{2}-\frac{1}{p})}}{\nu_h^\theta \nu_v^{1-\theta} 2^{2k\theta} 2^{2\ell(1-\theta)}} \|\Delta_k^h \Delta_\ell^v u_0\|_{L^2} \\ &\lesssim \frac{2^{k(2-\frac{2}{p}-2\theta)} 2^{\ell(2\theta-\frac{3}{2}-\frac{1}{p})}}{\nu_h^\theta \nu_v^{1-\theta}} \|\Delta_k^h \Delta_\ell^v u_0\|_{L^2}, \end{aligned}$$

which together with [4, Lemma 4.3] implies that for $\theta \in]3/4, 1[$

$$\|\nabla_h u_L\|_{L_t^1(B_p^{\frac{2}{p}, \frac{1}{p}})} \lesssim \frac{1}{\nu_h^\theta \nu_v^{1-\theta}} \|u_0\|_{B^{2-2\theta, 2\theta-\frac{3}{2}}} \lesssim \frac{1}{\nu_h^\theta \nu_v^{1-\theta}} \|u_0\|_{B_2^{\frac{1}{2}}}.$$

This proves (2.4b).

Along the same line, we have

$$\begin{aligned} \|\partial_3 \Delta_k^h \Delta_\ell^v u_L\|_{L_t^1(L^p)} &\lesssim \frac{2^{k(1-\frac{2}{p})} 2^{\ell(\frac{3}{2}-\frac{1}{p})}}{\nu_h 2^{2k} + \nu_v 2^{2\ell}} \|\Delta_k^h \Delta_\ell^v u_0\|_{L^2} \\ &\lesssim \frac{2^{k(1-\frac{2}{p})} 2^{\ell(\frac{3}{2}-\frac{1}{p})}}{\sqrt{\nu_h \nu_v} 2^k 2^\ell} \|\Delta_k^h \Delta_\ell^v u_0\|_{L^2} \lesssim \frac{2^{-\frac{2}{p}k} 2^{\ell(\frac{1}{2}-\frac{1}{p})}}{\sqrt{\nu_h \nu_v}} \|\Delta_k^h \Delta_\ell^v u_0\|_{L^2}, \end{aligned}$$

which leads to (2.4c).

Similarly, we observe that

$$\begin{aligned} \|\partial_3^2 \Delta_k^h \Delta_\ell^v u_L\|_{L_t^1(L^p)} &\lesssim \frac{2^{k(1-\frac{2}{p})} 2^{\ell(\frac{5}{2}-\frac{1}{p})}}{\nu_h 2^{2k} + \nu_v 2^{2\ell}} \|\Delta_k^h \Delta_\ell^v u_0\|_{L^2} \\ &\lesssim \frac{2^{k(1-\frac{2}{p})} 2^{\ell(\frac{1}{2}-\frac{1}{p})}}{\nu_v} \|\Delta_k^h \Delta_\ell^v u_0\|_{L^2}, \end{aligned}$$

which ensures (2.4d). This completes the proof of Lemma 2.3. □

Remark 2.1. We deduce from the law of product Lemmas 2.2 and 2.3 that for any $\theta \in]3/4, 1[$

$$\begin{aligned} &\|u_L \cdot \nabla u_L\|_{L_t^1(B_p^{-1+\frac{2}{p}, \frac{1}{p}})} \\ &\lesssim \|u_L\|_{\tilde{L}_t^\infty(B_p^{-1+\frac{2}{p}, \frac{1}{p}})} \left(\|\nabla_h u_L\|_{L_t^1(B_p^{\frac{2}{p}, \frac{1}{p}})} + \|\partial_3 u_L\|_{L_t^1(B_p^{\frac{2}{p}, \frac{1}{p}})} \right) \\ &\lesssim \frac{\|u_0\|_{B_2^{0, \frac{1}{2}}}^2}{\sqrt{\nu_h \nu_v}} + \frac{\|u_0\|_{B_2^{0, \frac{1}{2}}} \|u_0\|_{B_2^{\frac{1}{2}}}}{\nu_h^\theta \nu_v^{1-\theta}} \lesssim \frac{\|u_0\|_{B_2^{0, \frac{1}{2}}} \left(\|u_0\|_{B_2^{0, \frac{1}{2}}} + \|u_0\|_{B_2^{\frac{1}{2}}} \right)}{\nu_h^\theta \nu_v^{1-\theta}}. \end{aligned} \tag{2.6}$$

Finally, we also recall a lemma concerning the propagation of regularities for the transport equation.

Lemma 2.4 ([1]). *Let $p \in]3, 4[$. Then given a_0 in $B_p^{\frac{3}{p}}(\mathbb{R}^3)$ and a vector field u in $L^1([0, T]; Lip(\mathbb{R}^3))$ with $\text{div } u = 0$, the following transport equation*

$$\partial_t a + u \cdot \nabla a = 0, \quad a|_{t=0} = a_0 \tag{2.7}$$

has a unique solution a satisfying

$$\|a\|_{\dot{L}_t^\infty(B_p^{\frac{3}{p}})} \leq \|a_0\|_{B_p^{\frac{3}{p}}} \exp\left(C \int_0^t \|\nabla u(t')\|_{L^\infty} dt'\right).$$

3 The regularity criteria

In this section, we investigate the regularity criteria for the system (1.1).

Proposition 3.1. *Let (ρ_0, u_0) satisfy $0 < m \leq \rho_0 \leq M$ and $u_0 \in H^1(\mathbb{R}^3)$. Let (ρ, u) be the unique solution of the system (1.1) on $[0, T[$ which satisfies*

$$m \leq \rho \leq M \quad \text{and} \quad u \in C([0, T[; H^1(\mathbb{R}^3)) \cap L^2([0, T[; \dot{H}^2(\mathbb{R}^3))). \quad (3.1)$$

If

$$\int_0^T \left(\|u(t)\|_{L^\infty}^2 + \|\nabla u(t)\|_{L^\infty} \right) dt < +\infty, \quad (3.2)$$

then (ρ, u) may be continued beyond T .

Proof. We remark that the unique solution (ρ, u) of the system (1.1) which satisfies (3.1) has been constructed in [12]. In what follows, we just present the a priori estimates. Without loss of generality, we may assume that $\nu_v > \nu_h$. As a convention in the rest of this section, we shall always denote by C a constant depending on m and M in (3.1), which may be different from line to line. We first get, by taking L^2 inner product of momentum equation of (1.1) with u , that

$$\frac{1}{2} \frac{d}{dt} \|\sqrt{\rho}u(t)\|_{L^2}^2 + \nu_h \|\nabla_h u\|_{L^2}^2 + \nu_v \|\partial_3 u\|_{L^2}^2 = 0. \quad (3.3)$$

Integrating the above equality over $[0, t]$ yields (1.2).

We split the rest of the proof into the following two steps.

Step 1. \dot{H}^1 energy estimate of u .

By taking L^2 inner product of (1.1) with $\partial_t u$, we obtain

$$\|\sqrt{\rho}u_t\|_{L^2}^2 + \frac{1}{2} \frac{d}{dt} \left(\nu_h \|\nabla_h u\|_{L^2}^2 + \nu_v \|\partial_3 u\|_{L^2}^2 \right) \leq |(\rho u \cdot \nabla u | u_t)|. \quad (3.4)$$

Observing that

$$\begin{aligned} |(\rho u \cdot \nabla u | u_t)| &\leq C \|\sqrt{\rho}u_t\|_{L^2} \|u\|_{L^\infty} \|\nabla u\|_{L^2} \\ &\leq \frac{1}{2} \|\sqrt{\rho}u_t\|_{L^2}^2 + C \|u\|_{L^\infty}^2 \|\nabla u\|_{L^2}^2 \\ &\leq \frac{1}{2} \|\sqrt{\rho}u_t\|_{L^2}^2 + \frac{C}{\nu_h} \|u\|_{L^\infty}^2 \left(\nu_h \|\nabla_h u\|_{L^2}^2 + \nu_v \|\partial_3 u\|_{L^2}^2 \right). \end{aligned}$$

By inserting the above estimate into (3.4) and then applying Gronwall inequality to the resulting inequality, we find

$$\begin{aligned} & \nu_h \|\nabla_h u\|_{L_t^\infty(L^2)}^2 + \nu_v \|\partial_3 u\|_{L_t^\infty(L^2)}^2 + \|\sqrt{\rho} u_t\|_{L_t^2(L^2)}^2 \\ & \leq \left(\nu_h \|\nabla_h u_0\|_{L^2}^2 + \nu_v \|\partial_3 u_0\|_{L^2}^2 \right) \exp \left(\frac{C}{\nu_h} \int_0^t \|u\|_{L^\infty}^2 dt' \right). \end{aligned} \quad (3.5)$$

On the other hand, in view of (1.1), we write

$$-\nu_h \Delta_h u - \nu_v \partial_3^2 u + \nabla P = -\rho(u_t + u \cdot \nabla u),$$

from which and the classical estimates on the Stokes system, we infer

$$\begin{aligned} & \nu_h \|\nabla_h^2 u\|_{L^2} + \nu_v \|\partial_3^2 u\|_{L^2} + \|\nabla P\|_{L^2} \\ & \leq C(\|\rho u_t\|_{L^2} + \|\rho u \cdot \nabla u\|_{L^2}) \leq C(\|\rho u_t\|_{L^2} + \|u\|_{L^\infty} \|\nabla u\|_{L^2}), \end{aligned} \quad (3.6)$$

which together with (3.5) ensures that

$$\begin{aligned} & \nu_h \|\nabla_h^2 u\|_{L_t^2(L^2)}^2 + \nu_v \|\partial_3^2 u\|_{L_t^2(L^2)}^2 + \|\nabla P\|_{L_t^2(L^2)}^2 \\ & \leq \frac{1}{\nu_h} \left(\nu_h \|\nabla_h u_0\|_{L^2}^2 + \nu_v \|\partial_3 u_0\|_{L^2}^2 \right) \exp \left(\frac{C}{\nu_h} \int_0^t \|u\|_{L^\infty}^2 dt' \right). \end{aligned} \quad (3.7)$$

Step 2. \dot{H}^2 energy estimate of u .

We first get, by taking the time derivative to the momentum equation (1.1), that

$$\rho(u_{tt} + u \cdot \nabla u_t) - \nu_h \Delta_h u_t - \nu_v \partial_3^2 u_t + \nabla P_t = -\rho_t(u_t + u \cdot \nabla u) - \rho u_t \cdot \nabla u.$$

Taking the L^2 inner product of the above equation with u_t , we write

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\sqrt{\rho} u_t\|_{L^2}^2 + \nu_h \|\nabla_h u_t\|_{L^2}^2 + \nu_v \|\partial_3 u_t\|_{L^2}^2 \\ & = -(\rho u_t \cdot \nabla u | u_t) - (\rho_t u_t | u_t) - (\rho_t u \cdot \nabla u | u_t) \\ & \stackrel{\text{def}}{=} E + F + G. \end{aligned} \quad (3.8)$$

Obviously, we have

$$|E| \lesssim \|\nabla u\|_{L^\infty} \|\sqrt{\rho} u_t\|_{L^2}^2.$$

While noticing that $\rho_t = -u \cdot \nabla \rho$, by using integration by parts, we get

$$\begin{aligned} F &= -2 \int_{\mathbb{R}^3} (\rho u \cdot \nabla u_t | u_t) dx \leq C \|u\|_{L^\infty} \|\nabla u_t\|_{L^2} \|\sqrt{\rho} u_t\|_{L^2} \\ &\leq \frac{\nu_h}{4} \|\nabla u_t\|_{L^2}^2 + \frac{C}{\nu_h} \|u\|_{L^\infty}^2 \|\sqrt{\rho} u_t\|_{L^2}^2. \end{aligned}$$

To deal with G , by using integrating by parts, we have

$$\begin{aligned} G &= \int_{\mathbb{R}^3} ((\rho u \cdot \nabla u) \cdot \nabla u | u_t) dx + \int_{\mathbb{R}^3} ((\rho u \otimes u) : \nabla^2 u | u_t) dx \\ &\quad + \int_{\mathbb{R}^3} ((\rho u \cdot \nabla u) | u \cdot \nabla u_t) dx \\ &\stackrel{\text{def}}{=} G_1 + G_2 + G_3. \end{aligned}$$

Applying Sobolev inequality gives

$$G_1 \leq C \|u\|_{L^6} \|\nabla u\|_{L^6}^2 \|\sqrt{\rho} u_t\|_{L^2} \leq C \|\nabla u\|_{L^2} \|\nabla^2 u\|_{L^2}^2 \|\sqrt{\rho} u_t\|_{L^2},$$

which together with (3.6) ensures that

$$\begin{aligned} G_1 &\leq \frac{C}{\nu_h} \|\nabla u\|_{L^2} \|\nabla^2 u\|_{L^2} (\|\rho u_t\|_{L^2} + \|u\|_{L^\infty} \|\nabla u\|_{L^2}) \|\sqrt{\rho} u_t\|_{L^2} \\ &\leq \frac{C}{\nu_h} \left(\|u\|_{L^\infty}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 \right) \|\sqrt{\rho} u_t\|_{L^2}^2 + \frac{C}{\nu_h} \|\nabla u\|_{L^2}^4 \|\nabla^2 u\|_{L^2}^2. \end{aligned}$$

While along the same line to the proof of (3.6), one has

$$\begin{aligned} &\nu_h \|\nabla_h^2 u\|_{L^2} + \nu_v \|\partial_3^2 u\|_{L^2} + \|\nabla P\|_{L^2} \\ &\leq C (\|\sqrt{\rho} u_t\|_{L^2} + \|u \cdot \nabla u\|_{L^2}) \leq \frac{C}{\nu_h} \left(\|\sqrt{\rho} u_t\|_{L^2} + \|\nabla u\|_{L^2}^3 \right) + \frac{\nu_h}{2} \|\nabla^2 u\|_{L^2}, \end{aligned}$$

which implies

$$\nu_h \|\nabla_h^2 u\|_{L^2} + \nu_v \|\partial_3^2 u\|_{L^2} + \|\nabla P\|_{L^2} \leq \frac{C}{\nu_h} \left(\|\sqrt{\rho} u_t\|_{L^2} + \|\nabla u\|_{L^2}^3 \right). \tag{3.9}$$

Hence we obtain

$$\begin{aligned} G_2 &\leq C \|u\|_{L^\infty}^2 \|\nabla^2 u\|_{L^2} \|\sqrt{\rho} u_t\|_{L^2} \\ &\leq \frac{C}{\nu_h^2} \|u\|_{L^\infty}^2 \left(\|\sqrt{\rho} u_t\|_{L^2} + \|\nabla u\|_{L^2}^3 \right) \|\sqrt{\rho} u_t\|_{L^2} \\ &\leq \frac{C}{\nu_h^2} \|u\|_{L^\infty}^2 \left(\|\sqrt{\rho} u_t\|_{L^2}^2 + \|\nabla u\|_{L^2}^6 \right). \end{aligned}$$

Finally we observe that

$$\begin{aligned} G_3 &\leq C \|u\|_{L^6}^2 \|\nabla u\|_{L^6} \|\nabla u_t\|_{L^2} \leq C \|\nabla u\|_{L^2}^2 \|\nabla^2 u\|_{L^2} \|\nabla u_t\|_{L^2} \\ &\leq \frac{\nu_h}{4} \|\nabla u_t\|_{L^2}^2 + \frac{C}{\nu_h} \|\nabla u\|_{L^2}^4 \|\nabla^2 u\|_{L^2}^2. \end{aligned}$$

By inserting the above estimates into (3.8), we conclude that

$$\begin{aligned} &\frac{d}{dt} \|\sqrt{\rho} u_t\|_{L^2}^2 + \nu_h \|\nabla_h u_t\|_{L^2}^2 + \nu_v \|\partial_3 u_t\|_{L^2}^2 \\ &\lesssim \left(\|\nabla u\|_{L^\infty} + \frac{1}{\nu_h^2} \|u\|_{L^\infty}^2 + \frac{1}{\nu_h} \|\nabla u\|_{L^2}^2 + \frac{1}{\nu_h} \|\nabla^2 u\|_{L^2}^2 \right) \|\sqrt{\rho} u_t\|_{L^2}^2 \\ &\quad + \frac{1}{\nu_h} \|\nabla u\|_{L^2}^4 \|\nabla^2 u\|_{L^2}^2 + \frac{1}{\nu_h^2} \|u\|_{L^\infty}^2 \|\nabla u\|_{L^2}^6. \end{aligned} \quad (3.10)$$

Due to the initial data $u_0 \in H^1$ instead of H^2 , as in [12], we introduce $\sigma(t) \stackrel{\text{def}}{=} \min(1, t)$ multiply the above inequality by $\sigma(t)$ to get

$$\begin{aligned} &\frac{d}{dt} \left(\sigma(t) \|\sqrt{\rho} u_t\|_{L^2}^2 \right) + \sigma(t) \left(\nu_h \|\nabla_h u_t\|_{L^2}^2 + \nu_v \|\partial_3 u_t\|_{L^2}^2 \right) \\ &\lesssim \|\sqrt{\rho} u_t\|_{L^2}^2 + \sigma(t) \left(\|\nabla u\|_{L^\infty} + \frac{1}{\nu_h^2} \|u\|_{L^\infty}^2 + \frac{1}{\nu_h} \|\nabla u\|_{L^2}^2 + \frac{1}{\nu_h} \|\nabla^2 u\|_{L^2}^2 \right) \|\sqrt{\rho} u_t\|_{L^2}^2 \\ &\quad + \frac{\sigma(t)}{\nu_h} \|\nabla u\|_{L^2}^4 \|\nabla^2 u\|_{L^2}^2 + \frac{\sigma(t)}{\nu_h^2} \|u\|_{L^\infty}^2 \|\nabla u\|_{L^2}^6. \end{aligned}$$

Applying Gronwall inequality to above estimate and using the fact $\sigma(t) \leq 1$, we obtain

$$\begin{aligned} &\sigma(t) \|\sqrt{\rho} u_t\|_{L^2}^2 + \int_0^t \sigma(t') \left(\nu_h \|\nabla_h u_t\|_{L^2}^2 + \nu_v \|\partial_3 u_t\|_{L^2}^2 \right) dt' \\ &\lesssim \left(\|\sqrt{\rho} u_t\|_{L_t^2(L^2)}^2 + \int_0^t \left(\frac{1}{\nu_h} \|\nabla u\|_{L^2}^4 \|\nabla^2 u\|_{L^2}^2 + \frac{1}{\nu_h^2} \|u\|_{L^\infty}^2 \|\nabla u\|_{L^2}^6 \right) dt' \right) \\ &\quad \times \exp \int_0^t \left(\|\nabla u\|_{L^\infty} + \frac{1}{\nu_h^2} \|u\|_{L^\infty}^2 + \frac{1}{\nu_h} \|\nabla^2 u\|_{L^2}^2 + \frac{1}{\nu_h} \|\nabla u\|_{L^2}^2 \right) dt'. \end{aligned} \quad (3.11)$$

By using (1.2), (3.5) and (3.7), we can get

$$\int_0^t \frac{1}{\nu_h} \|\nabla u\|_{L^2}^4 \|\nabla^2 u\|_{L^2}^2 dt' \leq \frac{1}{\nu_h} \|\nabla u\|_{L_t^\infty(L^2)}^4 \|\nabla^2 u\|_{L_t^2(L^2)}^2$$

$$\begin{aligned} &\leq \frac{1}{\nu_h^5} \left(\nu_h^3 \|\nabla_h u_0\|_{L^2}^6 + \nu_v^3 \|\partial_3 u_0\|_{L^2}^6 \right) \exp \left(\frac{C}{\nu_h} \int_0^t \|u\|_{L^\infty}^2 dt' \right), \\ &\int_0^t \frac{1}{\nu_h^2} \|u\|_{L^\infty}^2 \|\nabla u\|_{L^2}^6 dt' \leq \frac{1}{\nu_h^2} \|\nabla u\|_{L_t^\infty(L^2)}^6 \int_0^t \|u\|_{L^\infty}^2 dt' \\ &\leq \frac{1}{\nu_h^5} \left(\nu_h^3 \|\nabla_h u_0\|_{L^2}^6 + \nu_v^3 \|\partial_3 u_0\|_{L^2}^6 \right) \exp \left(\frac{C}{\nu_h} \int_0^t \|u\|_{L^\infty}^2 dt' \right). \end{aligned}$$

By substituting (3.5) and the above estimates into (3.11), we achieve

$$\begin{aligned} &\|\sigma(t)\sqrt{\rho}u_t\|_{L_t^\infty(L^2)}^2 + \int_0^t \sigma(t') \left(\nu_h \|\nabla_h u_t\|_{L^2}^2 + \nu_v \|\partial_3 u_t\|_{L^2}^2 \right) dt' \\ &\leq \frac{1}{\nu_h^5} \left(\nu_h^3 \|\nabla_h u_0\|_{L^2}^6 + \nu_v^3 \|\partial_3 u_0\|_{L^2}^6 \right) \exp \left(\frac{C}{\nu_h} \int_0^t \left(\|u\|_{L^\infty}^2 + \|\nabla u\|_{L^\infty} \right) dt' \right) \\ &\quad \times \exp \left(\frac{1}{\nu_h^2} \|\sqrt{\rho}u_0\|_{L^2}^2 + \frac{1}{\nu_h^3} \left(\nu_h \|\nabla_h u_0\|_{L^2}^2 + \nu_v \|\partial_3 u_0\|_{L^2}^2 \right) \exp \left(\int_0^t \frac{1}{\nu_h} \|u\|_{L^\infty}^2 dt' \right) \right), \end{aligned}$$

which together with (3.9) ensures that

$$\begin{aligned} &\nu_h \|\sigma(t)\nabla_h^2 u\|_{L_t^\infty(L^2)}^2 + \nu_v \|\sigma(t)\partial_3^2 u\|_{L_t^\infty(L^2)}^2 + \|\sigma(t)\nabla P\|_{L_t^\infty(L^2)}^2 \tag{3.12} \\ &\leq \frac{1}{\nu_h^8} \left(\nu_h^3 \|\nabla_h u_0\|_{L^2}^6 + \nu_v^3 \|\partial_3 u_0\|_{L^2}^6 \right) \exp \left(\frac{C}{\nu_h} \int_0^t \left(\|u\|_{L^\infty}^2 + \|\nabla u\|_{L^\infty} \right) dt' \right) \\ &\quad \times \exp \left(\frac{1}{\nu_h^2} \|\sqrt{\rho}u_0\|_{L^2}^2 + \frac{1}{\nu_h^3} \left(\nu_h \|\nabla_h u_0\|_{L^2}^2 + \nu_v \|\partial_3 u_0\|_{L^2}^2 \right) \exp \left(\int_0^t \frac{1}{\nu_h} \|u\|_{L^\infty}^2 dt' \right) \right). \end{aligned}$$

With the estimates (1.2), (3.5), (3.7) and (3.12), we can follow the same line to the proof of [12, Theorem 1.2] that under the assumption (3.2), we can prove the solution (ρ, u) can be extended beyond T . Furthermore, by using Lagrangian approach, such a solution is unique. This completes the proof of Proposition 3.1. \square

4 The proof of the Theorem 1.1

The goal of this section is to present the proof of Theorem 1.1. Indeed according to Proposition 3.1, it remains to show that there holds (3.2) for any $T < T^*$, with T^* being the lifespan of the solution (a, u) . In order to do this, we first present the estimate of the pressure function appearing in (1.7).

Lemma 4.1. *Let $p \in]3, 4[$. Then the pressure function ∇P determined by (1.7) satisfies*

$$\begin{aligned} \|\nabla P\|_{B_p^{-1+\frac{2}{p}, \frac{1}{p}}} &\lesssim \|R \cdot \nabla R\|_{B_p^{-1+\frac{2}{p}, \frac{1}{p}}} + \nu_h \|a\|_{B_p^{\frac{2}{p}, \frac{1}{p}}} \left(\|\Delta_h R\|_{B_p^{-1+\frac{2}{p}, \frac{1}{p}}} + \|\Delta_h u_L\|_{B_p^{-1+\frac{2}{p}, \frac{1}{p}}} \right) \\ &\quad + \nu_v \|a\|_{B_p^{\frac{2}{p}, \frac{1}{p}}} \left(\|\partial_3^2 R\|_{B_p^{-1+\frac{2}{p}, \frac{1}{p}}} + \|\partial_3^2 u_L\|_{B_p^{-1+\frac{2}{p}, \frac{1}{p}}} \right) \\ &\quad + \|a\|_{B_p^{\frac{2}{p}, \frac{1}{p}}} \|\nabla P\|_{B_p^{-1+\frac{2}{p}, \frac{1}{p}}} + \|u_L \cdot \nabla u_L\|_{B_p^{-1+\frac{2}{p}, \frac{1}{p}}} \\ &\quad + \|R \cdot \nabla u_L\|_{B_p^{-1+\frac{2}{p}, \frac{1}{p}}} + \|u_L \cdot \nabla R\|_{B_p^{-1+\frac{2}{p}, \frac{1}{p}}}. \end{aligned} \quad (4.1)$$

Proof. Applying the operator div to the R equation of (1.7) yields

$$\begin{aligned} \Delta P = \operatorname{div} &(-R \cdot \nabla R + \nu_h a \Delta_h R + \nu_v a \partial_3^2 R + \nu_h a \Delta_h u_L \\ &+ \nu_v a \partial_3^2 u_L - a \nabla P - u_L \cdot \nabla u_L - R \cdot \nabla u_L - u_L \cdot \nabla R), \end{aligned}$$

we thus write

$$\begin{aligned} \nabla P = \nabla(-\Delta)^{-1} \operatorname{div} &(R \cdot \nabla R - \nu_h a \Delta_h R - \nu_v a \partial_3^2 R - \nu_h a \Delta_h u_L \\ &- \nu_v a \partial_3^2 u_L + a \nabla P + u_L \cdot \nabla u_L + R \cdot \nabla u_L + u_L \cdot \nabla R). \end{aligned}$$

As $\nabla(-\Delta)^{-1} \operatorname{div}$ is a homogeneous Fourier multiplier of degree 0, then by using the product laws, Lemma 2.2, we conclude the proof of (4.1). \square

Let us now present the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $p \in]3, 4[$. By applying $\Delta_k^h \Delta_\ell^v$ to the (1.7) and then taking L^2 inner product of the resulting momentum equation with $|\Delta_k^h \Delta_\ell^v R^i|^{p-2} \Delta_k^h \Delta_\ell^v R^i$, we get

$$\begin{aligned} &\frac{1}{p} \frac{d}{dt} \|\Delta_k^h \Delta_\ell^v R^i\|_{L^p}^p - \int_{\mathbb{R}^3} \Delta_k^h \Delta_\ell^v (\nu_h \Delta_h R^i + \nu_v \partial_3^2 R^i) |\Delta_k^h \Delta_\ell^v R^i|^{p-2} \Delta_k^h \Delta_\ell^v R^i dx \\ &= - \int_{\mathbb{R}^3} \Delta_k^h \Delta_\ell^v (R \cdot \nabla R^i) |\Delta_k^h \Delta_\ell^v R^i|^{p-2} \Delta_k^h \Delta_\ell^v R^i dx \\ &\quad + \nu_h \int_{\mathbb{R}^3} \Delta_k^h \Delta_\ell^v (a \Delta_h R^i + a \Delta_h u_L^i) |\Delta_k^h \Delta_\ell^v R^i|^{p-2} \Delta_k^h \Delta_\ell^v R^i dx \\ &\quad + \nu_v \int_{\mathbb{R}^3} \Delta_k^h \Delta_\ell^v (a \partial_3^2 R^i + a \partial_3^2 u_L^i) |\Delta_k^h \Delta_\ell^v R^i|^{p-2} \Delta_k^h \Delta_\ell^v R^i dx \\ &\quad - \int_{\mathbb{R}^3} \Delta_k^h \Delta_\ell^v (u_L \cdot \nabla u_L^i + R \cdot \nabla u_L^i + u_L \cdot \nabla R^i) |\Delta_k^h \Delta_\ell^v R^i|^{p-2} \Delta_k^h \Delta_\ell^v R^i dx \\ &\quad - \int_{\mathbb{R}^3} \Delta_k^h \Delta_\ell^v ((a+1) \partial_i P) |\Delta_k^h \Delta_\ell^v R^i|^{p-2} \Delta_k^h \Delta_\ell^v R^i dx. \end{aligned} \quad (4.2)$$

It follows from [6, Lemma A.5] that there exists a positive \bar{c} so that

$$\begin{aligned} & - \int_{\mathbb{R}^3} \Delta_h \Delta_k^h \Delta_\ell^v R^{(i)} |\Delta_k^h \Delta_\ell^v R^{(i)}|^{p-2} \Delta_k^h \Delta_\ell^v R^{(i)} dx \geq \bar{c} 2^{2k} \|\Delta_k^h \Delta_\ell^v R^{(i)}\|_{L^p}^p, \\ & - \int_{\mathbb{R}^3} \partial_3^2 \Delta_k^h \Delta_\ell^v R^{(i)} |\Delta_k^h \Delta_\ell^v R^{(i)}|^{p-2} \Delta_k^h \Delta_\ell^v R^{(i)} dx \geq \bar{c} 2^{2\ell} \|\Delta_k^h \Delta_\ell^v R^{(i)}\|_{L^p}^p. \end{aligned}$$

By inserting the above inequalities into (4.2), we get, by a standard derivation, that

$$\begin{aligned} & \|\Delta_k^h \Delta_\ell^v R\|_{L_t^\infty(L^p)} + \nu_h 2^{2k} \|\Delta_k^h \Delta_\ell^v R\|_{L_t^1(L^p)} + \nu_v 2^{2\ell} \|\Delta_k^h \Delta_\ell^v R\|_{L_t^1(L^p)} \\ & \lesssim \|\Delta_k^h \Delta_\ell^v (R \cdot \nabla R)\|_{L_t^1(L^p)} + \nu_h \|\Delta_k^h \Delta_\ell^v (a \Delta_h R)\|_{L_t^1(L^p)} + \nu_v \|\Delta_k^h \Delta_\ell^v (a \partial_3^2 R)\|_{L_t^1(L^p)} \\ & \quad + \nu_h \|\Delta_k^h \Delta_\ell^v (a \Delta_h u_L)\|_{L_t^1(L^p)} + \nu_v \|\Delta_k^h \Delta_\ell^v (a \partial_3^2 u_L)\|_{L_t^1(L^p)} + \|\Delta_k^h \Delta_\ell^v (u_L \cdot \nabla u_L)\|_{L_t^1(L^p)} \\ & \quad + \|\Delta_k^h \Delta_\ell^v (R \cdot \nabla u_L)\|_{L_t^1(L^p)} + \|\Delta_k^h \Delta_\ell^v (u_L \cdot \nabla R)\|_{L_t^1(L^p)} + \|\Delta_k^h \Delta_\ell^v ((a+1) \nabla P)\|_{L_t^1(L^p)}. \end{aligned}$$

By multiplying the above inequality by $2^{k(-1+\frac{2}{p})} 2^{\frac{\ell}{p}}$ and then summing up the resulting inequalities for $k, \ell \in \mathbb{Z}$, we obtain

$$\begin{aligned} & \|R\|_{\tilde{L}_t^\infty(B_p^{-1+\frac{2}{p}, \frac{1}{p}})} + \nu_h \|R\|_{L_t^1(B_p^{1+\frac{2}{p}, \frac{1}{p}})} + \nu_v \|R\|_{L_t^1(B_p^{-1+\frac{2}{p}, 2+\frac{1}{p}})} \tag{4.3} \\ & \lesssim \|R \cdot \nabla R\|_{L_t^1(B_p^{-1+\frac{2}{p}, \frac{1}{p}})} + \nu_h \|a\|_{\tilde{L}_t^\infty(B_p^{\frac{2}{p}, \frac{1}{p}})} \|\Delta_h R\|_{L_t^1(B_p^{-1+\frac{2}{p}, \frac{1}{p}})} \\ & \quad + \nu_v \|a\|_{\tilde{L}_t^\infty(B_p^{\frac{2}{p}, \frac{1}{p}})} \|\partial_3^2 R\|_{L_t^1(B_p^{-1+\frac{2}{p}, \frac{1}{p}})} + \nu_h \|a\|_{\tilde{L}_t^\infty(B_p^{\frac{2}{p}, \frac{1}{p}})} \|\Delta_h u_L\|_{L_t^1(B_p^{-1+\frac{2}{p}, \frac{1}{p}})} \\ & \quad + \nu_v \|a\|_{\tilde{L}_t^\infty(B_p^{\frac{2}{p}, \frac{1}{p}})} \|\partial_3^2 u_L\|_{L_t^1(B_p^{-1+\frac{2}{p}, \frac{1}{p}})} + \|u_L \cdot \nabla u_L\|_{L_t^1(B_p^{-1+\frac{2}{p}, \frac{1}{p}})} \\ & \quad + \|R \cdot \nabla u_L\|_{L_t^1(B_p^{-1+\frac{2}{p}, \frac{1}{p}})} + \|u_L \cdot \nabla R\|_{L_t^1(B_p^{-1+\frac{2}{p}, \frac{1}{p}})} + \|(1+a) \nabla P\|_{L_t^1(B_p^{-1+\frac{2}{p}, \frac{1}{p}})}. \end{aligned}$$

It follows from the law of product, Lemma 2.2, that

$$\|R \cdot \nabla R\|_{L_t^1(B_p^{-1+\frac{2}{p}, \frac{1}{p}})} \lesssim \|R\|_{\tilde{L}_t^\infty(B_p^{-1+\frac{2}{p}, \frac{1}{p}})} \left(\|\nabla_h R\|_{L_t^1(B_p^{\frac{2}{p}, \frac{1}{p}})} + \|\partial_3 R\|_{L_t^1(B_p^{\frac{2}{p}, \frac{1}{p}})} \right).$$

Yet we observe from interpolation inequality that

$$\|R\|_{L_t^1(B_p^{\frac{2}{p}, 1+\frac{1}{p}})} \lesssim \int_0^t \|R(t')\|_{B_p^{1+\frac{2}{p}, \frac{1}{p}}}^{\frac{1}{2}} \|R(t')\|_{B_p^{-1+\frac{2}{p}, 2+\frac{1}{p}}}^{\frac{1}{2}} dt'. \tag{4.4}$$

As a result, it comes out

$$\|R \cdot \nabla R\|_{L_t^1(B_p^{-1+\frac{2}{p}, \frac{1}{p}})} \lesssim \|R\|_{\tilde{L}_t^\infty(B_p^{-1+\frac{2}{p}, \frac{1}{p}})} \left(\|R\|_{L_t^1(B_p^{1+\frac{2}{p}, \frac{1}{p}})} + \|R\|_{L_t^1(B_p^{-1+\frac{2}{p}, 2+\frac{1}{p}})} \right).$$

Along the same line, we have

$$\begin{aligned} \|u_L \cdot \nabla R\|_{B_p^{-1+\frac{2}{p}, \frac{1}{p}}} &\lesssim \|u_L\|_{B_p^{\frac{2}{p}, \frac{1}{p}}} \|\nabla R\|_{B_p^{-1+\frac{2}{p}, \frac{1}{p}}} \\ &\lesssim \|u_L\|_{B_p^{-1+\frac{2}{p}, \frac{1}{p}}}^{\frac{1}{2}} \|\nabla_h u_L\|_{B_p^{\frac{2}{p}, \frac{1}{p}}}^{\frac{1}{2}} \left(\|\nabla_h R\|_{B_p^{-1+\frac{2}{p}, \frac{1}{p}}} + \|\partial_3 R\|_{B_p^{-1+\frac{2}{p}, \frac{1}{p}}} \right) \\ &\lesssim \|u_L\|_{B_p^{-1+\frac{2}{p}, \frac{1}{p}}} \|\nabla_h u_L\|_{B_p^{\frac{2}{p}, \frac{1}{p}}} + \|\nabla_h R\|_{B_p^{-1+\frac{2}{p}, \frac{1}{p}}}^2 + \|\partial_3 R\|_{B_p^{-1+\frac{2}{p}, \frac{1}{p}}}^2. \end{aligned}$$

Notice that

$$\begin{aligned} \|\nabla_h R\|_{B_p^{-1+\frac{2}{p}, \frac{1}{p}}}^2 &\lesssim \|R\|_{B_p^{\frac{2}{p}, \frac{1}{p}}}^2 \lesssim \|R\|_{B_p^{-1+\frac{2}{p}, \frac{1}{p}}} \|R\|_{B_p^{1+\frac{2}{p}, \frac{1}{p}}}, \\ \|\partial_3 R\|_{B_p^{-1+\frac{2}{p}, \frac{1}{p}}}^2 &\lesssim \|R\|_{B_p^{-1+\frac{2}{p}, 1+\frac{1}{p}}}^2 \lesssim \|R\|_{B_p^{-1+\frac{2}{p}, \frac{1}{p}}} \|R\|_{B_p^{-1+\frac{2}{p}, 2+\frac{1}{p}}}, \end{aligned}$$

so that there holds

$$\begin{aligned} \|u_L \cdot \nabla R\|_{L_t^1(B_p^{-1+\frac{2}{p}, \frac{1}{p}})} &\lesssim \|u_L\|_{\tilde{L}_t^\infty(B_p^{-1+\frac{2}{p}, \frac{1}{p}})} \|\nabla_h u_L\|_{L_t^1(B_p^{\frac{2}{p}, \frac{1}{p}})} \\ &\quad + \|R\|_{\tilde{L}_t^\infty(B_p^{-1+\frac{2}{p}, \frac{1}{p}})} \left(\|R\|_{L_t^1(B_p^{1+\frac{2}{p}, \frac{1}{p}})} + \|R\|_{L_t^1(B_p^{-1+\frac{2}{p}, 2+\frac{1}{p}})} \right). \end{aligned}$$

Obviously, we have

$$\|R \cdot \nabla u_L\|_{L_t^1(B_p^{-1+\frac{2}{p}, \frac{1}{p}})} \lesssim \int_0^t \|R(t')\|_{B_p^{-1+\frac{2}{p}, \frac{1}{p}}} \left(\|\nabla_h u_L(t')\|_{B_p^{\frac{2}{p}, \frac{1}{p}}} + \|\partial_3 u_L(t')\|_{B_p^{\frac{2}{p}, \frac{1}{p}}} \right) dt'.$$

By substituting the above estimates and (4.1) into (4.3), and using [4, Lemma 4.3], we achieve

$$\begin{aligned} &\|R\|_{\tilde{L}_t^\infty(B_p^{-1+\frac{2}{p}, \frac{1}{p}})} + \nu_h \|R\|_{L_t^1(B_p^{1+\frac{2}{p}, \frac{1}{p}})} + \nu_v \|R\|_{L_t^1(B_p^{-1+\frac{2}{p}, 2+\frac{1}{p}})} \\ &\lesssim \|R\|_{\tilde{L}_t^\infty(B_p^{-1+\frac{2}{p}, \frac{1}{p}})} \left(\|R\|_{L_t^1(B_p^{1+\frac{2}{p}, \frac{1}{p}})} + \|R\|_{L_t^1(B_p^{-1+\frac{2}{p}, 2+\frac{1}{p}})} \right) \\ &\quad + \|a\|_{\tilde{L}_t^\infty(B_p^{\frac{3}{p}})} \left(\nu_h \|R\|_{L_t^1(B_p^{1+\frac{2}{p}, \frac{1}{p}})} + \nu_v \|R\|_{L_t^1(B_p^{-1+\frac{2}{p}, 2+\frac{1}{p}})} \right) \end{aligned}$$

$$\begin{aligned}
 & + \|a\|_{\tilde{L}_t^\infty(B_p^{\frac{3}{p}})} \left(\nu_h \|\Delta_h u_L\|_{L_t^1(B_p^{-1+\frac{2}{p}, \frac{1}{p}})} + \nu_v \|\partial_3^2 u_L\|_{L_t^1(B_p^{-1+\frac{2}{p}, \frac{1}{p}})} \right) \\
 & + \|u_L\|_{\tilde{L}_t^\infty(B_p^{-1+\frac{2}{p}, \frac{1}{p}})} \left(\|\nabla_h u_L\|_{L_t^1(B_p^{\frac{2}{p}, \frac{1}{p}})} + \|\partial_3 u_L\|_{L_t^1(B_p^{\frac{2}{p}, \frac{1}{p}})} \right) \\
 & + \int_0^t \|R(t')\|_{B_p^{-1+\frac{2}{p}, \frac{1}{p}}} \left(\|\nabla_h u_L(t')\|_{B_p^{\frac{2}{p}, \frac{1}{p}}} + \|\partial_3 u_L(t')\|_{B_p^{\frac{2}{p}, \frac{1}{p}}} \right) dt'. \tag{4.5}
 \end{aligned}$$

Let us denote

$$\begin{aligned}
 T^* \stackrel{\text{def}}{=} \sup \left\{ t < T^* : \nu_h \|a\|_{\tilde{L}_t^\infty(B_p^{\frac{3}{p}})} + \|R\|_{\tilde{L}_t^\infty(B_p^{-1+\frac{2}{p}, \frac{1}{p}})} \right. \\
 \left. + \nu_h \|R\|_{L_t^1(B_p^{1+\frac{2}{p}, \frac{1}{p}})} + \nu_v \|R\|_{L_t^1(B_p^{-1+\frac{2}{p}, 2+\frac{1}{p}})} \leq 2c_0 \nu_h \right\}, \tag{4.6}
 \end{aligned}$$

where c_0 is a small enough positive constant, which will be fixed later on. Then for $t \leq T^*$, one has

$$\|R\|_{\tilde{L}_t^\infty(B_p^{-1+\frac{2}{p}, \frac{1}{p}})} \left(\|R\|_{L_t^1(B_p^{1+\frac{2}{p}, \frac{1}{p}})} + \|R\|_{L_t^1(B_p^{-1+\frac{2}{p}, 2+\frac{1}{p}})} \right) \leq 2c_0 \|R\|_{\tilde{L}_t^\infty(B_p^{-1+\frac{2}{p}, \frac{1}{p}})}, \tag{4.7}$$

$$\|a\|_{\tilde{L}_t^\infty(B_p^{\frac{3}{p}})} \left(\nu_h \|R\|_{L_t^1(B_p^{1+\frac{2}{p}, \frac{1}{p}})} + \nu_v \|R\|_{L_t^1(B_p^{-1+\frac{2}{p}, 2+\frac{1}{p}})} \right) \leq 2c_0 \nu_h \|a\|_{\tilde{L}_t^\infty(B_p^{\frac{3}{p}})}. \tag{4.8}$$

By substituting (4.7) and (4.8) into (4.5), and using Lemma 2.3, we achieve

$$\begin{aligned}
 & \|R\|_{\tilde{L}_t^\infty(B_p^{-1+\frac{2}{p}, \frac{1}{p}})} + \nu_h \|R\|_{L_t^1(B_p^{1+\frac{2}{p}, \frac{1}{p}})} + \nu_v \|R\|_{L_t^1(B_p^{-1+\frac{2}{p}, 2+\frac{1}{p}})} \\
 & \leq C \left(\nu_h \|a\|_{\tilde{L}_t^\infty(B_p^{\frac{3}{p}})} + \frac{\nu_h^{1-\theta}}{\nu_v^{1-\theta}} \|a\|_{\tilde{L}_t^\infty(B_p^{\frac{3}{p}})} \|u_0\|_{B_2^{\frac{1}{2}}} \right. \\
 & \quad + \|a\|_{\tilde{L}_t^\infty(B_p^{\frac{3}{p}})} \|u_0\|_{B_2^{0, \frac{1}{2}}} + \frac{1}{\nu_h^\theta \nu_v^{1-\theta}} \|u_0\|_{B_2^{0, \frac{1}{2}}} \left(\|u_0\|_{B_2^{0, \frac{1}{2}}} + \|u_0\|_{B_2^{\frac{1}{2}}} \right) \\
 & \quad \left. + \int_0^t \|R(t')\|_{B_p^{-1+\frac{2}{p}, \frac{1}{p}}} \left(\|\nabla_h u_L(t')\|_{B_p^{\frac{2}{p}, \frac{1}{p}}} + \|\partial_3 u_L(t')\|_{B_p^{\frac{2}{p}, \frac{1}{p}}} \right) dt' \right). \tag{4.9}
 \end{aligned}$$

Applying Gronwall inequality to (4.9) yields

$$\begin{aligned}
 & \|R\|_{\tilde{L}_t^\infty(B_p^{-1+\frac{2}{p}, \frac{1}{p}})} + \nu_h \|R\|_{L_t^1(B_p^{1+\frac{2}{p}, \frac{1}{p}})} + \nu_v \|R\|_{L_t^1(B_p^{-1+\frac{2}{p}, 2+\frac{1}{p}})} \tag{4.10} \\
 & \leq \left(\nu_h \|a\|_{\tilde{L}_t^\infty(B_p^{\frac{3}{p}})} + \frac{\nu_h^{1-\theta}}{\nu_v^{1-\theta}} \|a\|_{\tilde{L}_t^\infty(B_p^{\frac{3}{p}})} \|u_0\|_{B_2^{\frac{1}{2}}} + \|a\|_{\tilde{L}_t^\infty(B_p^{\frac{3}{p}})} \|u_0\|_{B_2^{0, \frac{1}{2}}} \right)
 \end{aligned}$$

$$+ \frac{1}{\nu_h^\theta \nu_v^{1-\theta}} \|u_0\|_{B_2^{0,\frac{1}{2}}} \left(\|u_0\|_{B_2^{0,\frac{1}{2}}} + \|u_0\|_{B_2^{\frac{1}{2}}} \right) \exp \left(\frac{C(\|u_0\|_{B_2^{0,\frac{1}{2}}} + \|u_0\|_{B_2^{\frac{1}{2}}})}{\nu_h^\theta \nu_v^{1-\theta}} \right).$$

On the other hand, by using interpolation inequality, we have

$$\begin{aligned} \|\nabla R\|_{L_t^1(L^\infty)} &\lesssim \|\nabla R\|_{L_t^1(B_p^{\frac{2}{p},\frac{1}{p}})} \lesssim \|\nabla_h R\|_{L_t^1(B_p^{\frac{2}{p},\frac{1}{p}})} + \|\partial_3 R\|_{L_t^1(B_p^{\frac{2}{p},\frac{1}{p}})} \\ &\lesssim \|R\|_{L_t^1(B_p^{1+\frac{2}{p},\frac{1}{p}})} + \|R\|_{L_t^1(B_p^{\frac{2}{p},1+\frac{1}{p}})} \\ &\lesssim \|R\|_{L_t^1(B_p^{1+\frac{2}{p},\frac{1}{p}})} + \int_0^t \|R(t')\|_{B_p^{1+\frac{2}{p},\frac{1}{p}}}^{\frac{1}{2}} \|R(t')\|_{B_p^{-1+\frac{2}{p},2+\frac{1}{p}}}^{\frac{1}{2}} dt'. \end{aligned}$$

Then applying Young inequality, we get

$$\|\nabla R\|_{L_t^1(L^\infty)} \lesssim \|R\|_{L_t^1(B_p^{1+\frac{2}{p},\frac{1}{p}})} + \|R\|_{L_t^1(B_p^{-1+\frac{2}{p},2+\frac{1}{p}})}. \quad (4.11)$$

So it follows from Lemma 2.4 that

$$\begin{aligned} \|a\|_{\tilde{L}_t^\infty(B_p^{\frac{3}{p}})} &\leq \|a_0\|_{B_p^{\frac{3}{p}}} \exp \left(C' \int_0^t \|\nabla u(t')\|_{L^\infty} dt' \right) \\ &\leq \|a_0\|_{B_p^{\frac{3}{p}}} \exp \left(C' \int_0^t \left(\|\nabla_h u_L(t')\|_{B_p^{\frac{2}{p},\frac{1}{p}}} + \|\partial_3 u_L(t')\|_{B_p^{\frac{2}{p},\frac{1}{p}}} \right. \right. \\ &\quad \left. \left. + \|\nabla_h R(t')\|_{B_p^{\frac{2}{p},\frac{1}{p}}} + \|\partial_3 R(t')\|_{B_p^{\frac{2}{p},\frac{1}{p}}} \right) dt' \right). \end{aligned}$$

Then for $t \leq T^*$, we deduce from (2.4b), (2.4c), (4.11) and (4.6) that

$$\|a\|_{\tilde{L}_t^\infty(B_p^{\frac{3}{p}})} \leq \|a_0\|_{B_p^{\frac{3}{p}}} \exp \left(\frac{C(\|u_0\|_{B_2^{0,\frac{1}{2}}} + \|u_0\|_{B_2^{\frac{1}{2}}})}{\nu_h^\theta \nu_v^{1-\theta}} \right). \quad (4.12)$$

Then we deduce from (4.10) that, under the small condition (1.5), for all $t \leq T^*$ we obtain

$$\begin{aligned} &\|R\|_{L_t^\infty(B_p^{-1+\frac{2}{p},\frac{1}{p}})} + \nu_h \|R\|_{L_t^1(B_p^{1+\frac{2}{p},\frac{1}{p}})} + \nu_v \|R\|_{L_t^1(B_p^{-1+\frac{2}{p},2+\frac{1}{p}})} \\ &\leq \left(\nu_h \|a_0\|_{B_p^{\frac{3}{p}}} + \frac{\nu_h^{1-\theta}}{\nu_v^{1-\theta}} \|a_0\|_{B_p^{\frac{3}{p}}} \|u_0\|_{B_2^{\frac{1}{2}}} + \|a_0\|_{B_p^{\frac{3}{p}}} \|u_0\|_{B_2^{0,\frac{1}{2}}} \right) \end{aligned} \quad (4.13)$$

$$+ \frac{1}{\nu_h^\theta \nu_v^{1-\theta}} \|u_0\|_{B_2^{0, \frac{1}{2}}} \left(\|u_0\|_{B_2^{0, \frac{1}{2}}} + \|u_0\|_{B_2^{\frac{1}{2}}} \right) \exp \left(\frac{C(\|u_0\|_{B_2^{0, \frac{1}{2}}} + \|u_0\|_{B_2^{\frac{1}{2}}})}{\nu_h^\theta \nu_v^{1-\theta}} \right) \leq c_0 \nu_h,$$

which contradicts the definition of T^* . This in turn shows that $T^* = T^*$. More precisely, thanks to (4.13), we infer

$$\|\partial_3 R\|_{L_t^1(B_p^{\frac{2}{p}, \frac{1}{p}})} \lesssim \|R\|_{L_t^1(B_p^{1+\frac{2}{p}, \frac{1}{p}})}^{\frac{1}{2}} \|R\|_{L_t^1(B_p^{-1+\frac{2}{p}, 2+\frac{1}{p}})}^{\frac{1}{2}} \lesssim c_0 \nu_h^{\frac{1}{2}} \nu_v^{-\frac{1}{2}},$$

so that we deduce from (4.6) that

$$\int_0^{T^*} \left(\nu_h \|\nabla_h R(t)\|_{L^\infty} + \nu_v \|\partial_3 R(t)\|_{L^\infty} \right) dt \leq C c_0 \nu_h^{\frac{1}{2}} \nu_v^{\frac{1}{2}}.$$

And we also have

$$\begin{aligned} \|R\|_{L_t^2(L^\infty)}^2 &\lesssim \|R\|_{L_t^2(B_p^{\frac{2}{p}, \frac{1}{p}})}^2 \lesssim \int_0^t \|R(t')\|_{B_p^{-1+\frac{2}{p}, \frac{1}{p}}} \|R(t')\|_{B_p^{1+\frac{2}{p}, \frac{1}{p}}} dt' \\ &\lesssim \|R\|_{L_t^\infty(B_p^{-1+\frac{2}{p}, \frac{1}{p}})} \|R\|_{L_t^1(B_p^{1+\frac{2}{p}, \frac{1}{p}})} \\ &\lesssim \|R\|_{L_t^\infty(B_p^{-1+\frac{2}{p}, \frac{1}{p}})} \|R\|_{L_t^1(B_p^{1+\frac{2}{p}, \frac{1}{p}})} \leq C c_0^2 \nu_h. \end{aligned}$$

Similarly, by Lemma 2.3, we have

$$\begin{aligned} \|u_L\|_{L_t^2(L^\infty)}^2 &\lesssim \|u_L\|_{L_t^\infty(B_p^{-1+\frac{2}{p}, \frac{1}{p}})} \|u_L\|_{L_t^1(B_p^{1+\frac{2}{p}, \frac{1}{p}})} \lesssim \frac{\|u_0\|_{B_2^{0, \frac{1}{2}}} \|u_0\|_{B_2^{\frac{1}{2}}}}{\nu_h^\theta \nu_v^{1-\theta}}, \\ \|\nabla u_L\|_{L_t^1(B_p^{\frac{2}{p}, \frac{1}{p}})} &\lesssim \|\nabla_h u_L\|_{L_t^1(B_p^{\frac{2}{p}, \frac{1}{p}})} + \|\partial_3 u_L\|_{L_t^1(B_p^{\frac{2}{p}, \frac{1}{p}})} \lesssim \frac{\|u_0\|_{B_2^{0, \frac{1}{2}}} + \|u_0\|_{B_2^{\frac{1}{2}}}}{\nu_h^\theta \nu_v^{1-\theta}}. \end{aligned}$$

Consequently, there holds

$$\int_0^{T^*} \left(\|\nabla u(t)\|_{L^\infty} + \|u(t)\|_{L^\infty}^2 \right) dt < +\infty.$$

Then by Proposition 3.1, we get $T^* = +\infty$. This completes the proof of Theorem 1.1. □

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