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Omni-Representations of Leibniz Algebras

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Abstract. In this paper, first we introduce the notion of an omni-representation of a Leibniz algebra $\mathfrak{g}$ on a vector space V as a Leibniz algebra homomorphism from $\mathfrak{g}$ to the omni-Lie algebra $\mathfrak{gl}(V)\oplus V$. Then we introduce the omni-cohomology theory associated to omni-representations and establish the relation between omni-cohomology groups and Loday-Pirashvili cohomology groups.

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1 Introduction

Leibniz algebras were first discovered by Bloh [5] who called them D-algebras. Then Loday [18] rediscovered this algebraic structure and called them Leibniz algebras. A Leibniz algebra is a vector space $\mathfrak{g}$, endowed with a linear map $[\cdot,\cdot]_{\mathfrak{g}}$: $\mathfrak{g}\otimes\mathfrak{g}\longrightarrow\mathfrak{g}$ satisfying

$$[x,[y,z]_{\mathfrak{g}}]_{\mathfrak{g}}=[[x,y]_{\mathfrak{g}},z]_{\mathfrak{g}}+[y,[x,z]_{\mathfrak{g}}]_{\mathfrak{g}},\quad \forall x,y,z\in\mathfrak{g}. \tag{1.1}$$

In particular, if the bracket $[\cdot,\cdot]_{\mathfrak{g}}$ is skew-symmetric, then it is a Lie algebra. Leibniz algebras have important applications in both mathematics and mathematical physics, e.g. the section space of a Courant algebroid is a Leibniz algebra [19]

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and the underlying algebraic structure of an embedding tensor is also a Leibniz algebra which further leads to applications in higher gauge theories [15].

The theory of representations of Leibniz algebras was introduced and studied in [18].

Definition 1.1. *A representation of a Leibniz algebra* $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}})$ *is a triple* (V,l,r)*, where* V *is a vector space equipped with two linear maps* $l:\mathfrak{g}\longrightarrow\mathfrak{gl}(V)$ *and* $r:\mathfrak{g}\longrightarrow\mathfrak{gl}(V)$ *such that the following equalities hold:*

$$l_{[x,y]_{\mathfrak{g}}}=[l_x,l_y],\quad r_{[x,y]_{\mathfrak{g}}}=[l_x,r_y],\quad r_y\circ l_x=-r_y\circ r_x,\quad \forall x,y\in\mathfrak{g}. \tag{1.2}$$

Especially, faithful representations of Leibniz algebras were studied by Barnes [3], conformal representations of Leibniz algebras were studied by Kolesnikov [13], dual representations of Leibniz algebras were given in [21] in their study of Leibniz bialgebras. Representations of symmetric Leibniz algebras were studied by Benayadi [4].

Note that a representation of a Lie algebra $\mathfrak{g}$ on a vector space V is a Lie algebra homomorphism from $\mathfrak{g}$ to the Lie algebra $\mathfrak{gl}(V)$, which realizes an abstract Lie algebra as a subalgebra of the general linear Lie algebra. While the above representation of a Leibniz algebra does not have this advantage. The purpose of this paper is to introduce a new representation theory so that it can realize an abstract Leibniz algebra as a subalgebra of a concrete Leibniz algebra. The omni-Lie algebra $\mathfrak{gl}(V)\oplus V$ introduced by Weinstein [22] is naturally a Leibniz algebra and the main ingredient in our study. We introduce the notion of an omni-representation of a Leibniz algebra $\mathfrak{g}$ on a vector space V which is a homomorphism from $\mathfrak{g}$ to the Leibniz algebra $\mathfrak{gl}(V)\oplus V$. We show that a usual representation (V,l,r) gives rise to an omni-representation $\rho=(l^*\otimes 1+1\otimes l)+r$ of $\mathfrak{g}$ on $V^*\otimes V$.

The cohomology theory of Leibniz algebras was also developed by Loday and Pirashvili [18]. See [1, 6–9, 11] for more applications of Loday-Pirashvili cohomologies. We also develop the omni-cohomology theory for omni-representations introduced above, and give the relation between omni-cohomology groups and Loday-Pirashvili cohomology groups.

The paper is organized as follows. In Section 2, we restudy representation of Leibniz algebras and the corresponding semidirect products. In Section 3, we introduce the notion of omni-representations of Leibniz algebras and study the relation between omni-representations and the usual representations. In Section 4, we introduce omni-cohomology groups for Leibniz algebras with coefficients in omni-representations, and establish the relation between omni-cohomology groups and Loday-Pirashvili cohomology groups.

2 Representations of Leibniz algebras

Let (V,l,r) be a representation of a Leibniz algebra $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}})$.

Definition 2.1. *The Loday-Pirashvili cohomology of* $\mathfrak{g}$ *with coefficients in* V *is the cohomology of the cochain complex* $C^k(\mathfrak{g},V)=\mathrm{Hom}(\otimes^k\mathfrak{g},V),(k\geq 0)$ *with the coboundary operator*

$$\partial: C^k(\mathfrak{g},V) \longrightarrow C^{k+1}(\mathfrak{g},V)$$

defined by

$$\begin{aligned}\partial c^k(x_1,\dots,x_{k+1})=&\sum_{i=1}^{k}(-1)^{i+1}l_{x_i}\big(c^k(x_1,\dots,\widehat{x_i},\dots,x_{k+1})\big)+(-1)^{k+1}r_{x_{k+1}}\big(c^k(x_1,\dots,x_k)\big)\\&+\sum_{1\leq i<j\leq k+1}(-1)^i c^k(x_1,\dots,\widehat{x_i},\dots,x_{j-1},[x_i,x_j]_{\mathfrak{g}},x_{j+1},\dots,x_{k+1}).\end{aligned} \tag{2.1}$$

The resulting cohomology is denoted by $H^\bullet(\mathfrak{g};l,r)$*.*

Obviously, $(\mathbb{R},0,0)$ is a representation of a Leibniz algebra $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}})$, which is called the trivial representation. Denote the corresponding cohomology by $H^\bullet(\mathfrak{g})$. Another important representation is the adjoint representation $(\mathfrak{g},\mathrm{ad}_L,\mathrm{ad}_R)$, where $\mathrm{ad}_L:\mathfrak{g}\longrightarrow\mathfrak{gl}(\mathfrak{g})$ and $\mathrm{ad}_R:\mathfrak{g}\longrightarrow\mathfrak{gl}(\mathfrak{g})$ are defined as follows:

$$\mathrm{ad}_L(x)(y)=[x,y]_{\mathfrak{g}},\quad \mathrm{ad}_R(x)(y)=[y,x]_{\mathfrak{g}},\quad \forall x,y\in\mathfrak{g}. \tag{2.2}$$

The corresponding cohomology is denoted by $H^\bullet(\mathfrak{g};\mathrm{ad}_L,\mathrm{ad}_R)$.

A permutation $\sigma\in\mathbb{S}_n$ is called an $(i,n-i)$-shuffle if $\sigma(1)<\cdots<\sigma(i)$ and $\sigma(i+1)<\cdots<\sigma(n)$. If $i=0$ or n we assume $\sigma=\mathrm{id}$. The set of all $(i,n-i)$-shuffles will be denoted by $\mathbb{S}_{(i,n-i)}$. The notion of an $(i_1,\cdots,i_k)$-shuffle and the set $\mathbb{S}_{(i_1,\cdots,i_k)}$ are defined analogously.

Let $\mathfrak{g}$ be a vector space. We consider the graded vector space

$$C^*(\mathfrak{g},\mathfrak{g})=\oplus_{n\geq 1}C^n(\mathfrak{g},\mathfrak{g})=\oplus_{n\geq 1}\mathrm{Hom}(\otimes^n\mathfrak{g},\mathfrak{g}).$$

The Balavoine bracket on the graded vector space $C^*(\mathfrak{g},\mathfrak{g})$ is given by

$$[P,Q]_{\mathsf{B}}=P\bar{\circ}Q-(-1)^{pq}Q\bar{\circ}P \tag{2.3}$$

for all $P\in C^{p+1}(\mathfrak{g},\mathfrak{g}),Q\in C^{q+1}(\mathfrak{g},\mathfrak{g})$, where $P\bar{\circ}Q\in C^{p+q+1}(\mathfrak{g},\mathfrak{g})$ is defined by

$$P\bar{\circ}Q=\sum_{k=1}^{p+1}P\circ_k Q, \tag{2.4}$$

and $\circ_k$ is defined by

$$\begin{aligned}&(P\circ_k Q)(x_1,\cdots,x_{p+q+1})\\=&\sum_{\sigma\in\mathbb{S}_{(k-1,q)}}(-1)^{\sigma}(-1)^{(k-1)q}\\&\times P\big(x_{\sigma(1)},\cdots,x_{\sigma(k-1)},Q(x_{\sigma(k)},\cdots,x_{\sigma(k+q-1)},x_{k+q}),\\&\qquad x_{k+q+1},\cdots,x_{p+q+1}\big).\end{aligned}\tag{2.5}$$

It is well known that

Theorem 2.1 ([2,10])**.** *With the above notations, $(C^*(\mathfrak{g},\mathfrak{g}),[\cdot,\cdot]_{\mathsf{B}})$ is a graded Lie algebra.*

In particular, for $\alpha\in C^2(\mathfrak{g},\mathfrak{g})$, we have

$$\begin{aligned}[\alpha,\alpha]_{\mathsf{B}}(x,y,z)&=2\alpha\circ\alpha(x,y,z)\\&=2\big(\alpha\big(\alpha(x,y),z\big)-\alpha\big(x,\alpha(y,z)\big)+\alpha\big(y,\alpha(x,z)\big)\big).\end{aligned}\tag{2.6}$$

Thus, α defines a Leibniz algebra structure if and only if $[\alpha,\alpha]_{\mathsf{B}}=0$.

For a representation (V,l,r) of $\mathfrak{g}$, it is obvious that $(V,l,0)$ is also a representation of $\mathfrak{g}$. Thus, we have two semidirect product Leibniz algebras $\mathfrak{g}\ltimes_{(l,r)}V$ and $\mathfrak{g}\ltimes_{(l,0)}V$ with the brackets $[\cdot,\cdot]_{(l,r)}$ and $[\cdot,\cdot]_{(l,0)}$ given respectively by

$$\begin{aligned}&[x+u,y+v]_{(l,r)}=[x,y]_{\mathfrak{g}}+l_xv+r_yu,\\&[x+u,y+v]_{(l,0)}=[x,y]_{\mathfrak{g}}+l_xv.\end{aligned}$$

The right action r induces a linear map $\overline{r}:(\mathfrak{g}\oplus V)\otimes(\mathfrak{g}\oplus V)\longrightarrow\mathfrak{g}\oplus V$ as follows:

$$\overline{r}(x+u,y+v)=r_yu,\quad\forall x,y\in\mathfrak{g},\quad u,v\in V.$$

The adjoint representations ad_L and ad_R of the Leibniz algebra $\mathfrak{g}\ltimes_{(l,0)}V$ are given by

$$\begin{aligned}&\mathrm{ad}_L(x+u)(y+v)=[x,y]_{\mathfrak{g}}+l_xv,\\&\mathrm{ad}_R(x+u)(y+v)=[y,x]_{\mathfrak{g}}+l_yu.\end{aligned}$$

Theorem 2.2. *With the above notations, $\overline{r}$ satisfies the following Maurer-Cartan equation on the Leibniz algebra $\mathfrak{g}\ltimes_{(l,0)}V$:*

$$\partial\overline{r}-\frac{1}{2}[\overline{r},\overline{r}]_{\mathsf{B}}=0,$$

where ∂ is the coboundary operator for the Leibniz algebra $\mathfrak{g}\ltimes_{(l,0)}V$ with coefficients in the adjoint representation. Consequently, the Leibniz algebra $\mathfrak{g}\ltimes_{(l,r)}V$ is a deformation of the Leibniz algebra $\mathfrak{g}\ltimes_{(l,0)}V$ via the Maurer-Cartan element $\overline{r}$.

Proof. By direct computation, we have

$$
\begin{aligned}
&\partial\bar{r}(x+u,y+v,z+w)\\
=&\mathrm{ad}_L(x+u)\bar{r}(y+v,z+w)-\mathrm{ad}_L(y+v)\bar{r}(x+u,z+w)\\
&-\mathrm{ad}_R(z+w)\bar{r}(x+u,y+v)-\bar{r}\big([x+u,y+v]_{(l,0)},z+w\big)\\
&+\bar{r}\big(x+u,[y+v,z+w]_{(l,0)}\big)-\bar{r}\big(y+v,[x+u,z+w]_{(l,0)}\big)\\
=&l_xr_zv-l_yr_zu-r_zl_xv+r_{[y,z]_\mathfrak{g}}u-r_{[x,z]_\mathfrak{g}}v.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
&[\bar{r},\bar{r}]_\mathsf{B}(x+u,y+v,z+w)\\
=&2\big(\bar{r}(\bar{r}(x+u,y+v),z+w)-\bar{r}(x+u,\bar{r}(y+v,z+w))\\
&+\bar{r}(y+v,\bar{r}(x+u,z+w))\big)=2r_zr_yu.
\end{aligned}
$$

Thus, by (1.2), we have

$$
\begin{aligned}
&\left(\partial\bar{r}-\frac{1}{2}[\bar{r},\bar{r}]_\mathsf{B}\right)(x+u,y+v,z+w)\\
=&l_xr_zv-l_yr_zu-r_zl_xv+r_{[y,z]_\mathfrak{g}}u-r_{[x,z]_\mathfrak{g}}v-r_zr_yu\\
=&l_xr_zv-l_yr_zu-r_zl_xv+r_{[y,z]_\mathfrak{g}}u-r_{[x,z]_\mathfrak{g}}v+r_zl_yu=0.
\end{aligned}
$$

The proof is complete. □

Define $l^*:\mathfrak{g}\longrightarrow\mathfrak{gl}(V^*)$ by

$$\langle l_x^*(\xi),u\rangle=-\langle\xi,l_xu\rangle,\quad\forall x\in\mathfrak{g},\quad\xi\in V^*,\quad u\in V.$$

It is straightforward to see that $(V^*\otimes V,l^*\otimes 1+1\otimes l,0)$ is also a representation of $\mathfrak{g}$, where $l^*\otimes 1+1\otimes l:\mathfrak{g}\longrightarrow\mathfrak{gl}(V^*\otimes V)$ is given by

$$(l^*\otimes 1+1\otimes l)_x(\xi\otimes u)=(l_x^*\xi)\otimes u+\xi\otimes l_xu,\quad\forall\xi\in V^*,\quad u\in V.$$

Since $V^*\otimes V\cong\mathfrak{gl}(V)$, an element $\xi\otimes u$ in $V^*\otimes V$ can be identified with a linear map $A\in\mathfrak{gl}(V)$ via $A(v)=\langle\xi,v\rangle u$.

Proposition 2.1. *With the above notations, for all $A\in V^*\otimes V\cong\mathfrak{gl}(V)$, we have*

$$(l^*\otimes 1+1\otimes l)_xA=[l_x,A]=l_x\circ A-A\circ l_x.$$

Moreover, the right action $r:\mathfrak{g}\longrightarrow\mathfrak{gl}(V)$ is a 1-cocycle on $\mathfrak{g}$ with coefficients in the representation $(V^*\otimes V,l^*\otimes 1+1\otimes l,0)$.

Proof. Write $A=\xi\otimes u$, then we have

$$\begin{aligned}(l^*\otimes 1+1\otimes l)_x A(v) &= (l^*\otimes 1+1\otimes l)_x(\xi\otimes u)(v)\\ &= \big((l_x^*\xi)\otimes u+\xi\otimes l_x u\big)(v)\\ &= \langle l_x^*\xi,v\rangle u+\langle\xi,v\rangle l_x u\\ &= -\langle\xi,l_x v\rangle u+\langle\xi,v\rangle l_x u\\ &= [l_x,A](v).\end{aligned}$$

Therefore, by (1.2), we have

$$\partial r(x,y)=(l^*\otimes 1+1\otimes l)_x r(y)-r([x,y]_{\mathfrak{g}})=[l_x,r_y]-r_{[x,y]_{\mathfrak{g}}}=0,$$

which implies that r is a 1-cocycle. □

3 Omni-representations of Leibniz algebras

It is known that the aim of a representation is to realize an abstract algebraic structure as a class of linear transformations on a vector space. Such as a Lie algebra representation is a homomorphism from $\mathfrak{g}$ to the general linear Lie algebra $\mathfrak{gl}(V)$. Unfortunately, the representation of a Leibniz algebra discussed above does not realize a Leibniz algebra as a subalgebra of certain explicit Leibniz algebra. Therefore, it is reasonable for us to provide an alternative definition for the representation of a Leibniz algebra. It is lucky that there is a god-given Leibniz algebra worked as a general linear algebra defined as follows:

Given a vector space V, then $(V,l=\mathrm{id},r=0)$ is a natural representation of $\mathfrak{gl}(V)$, which is viewed as a Leibniz algebra. The corresponding semidirect product Leibniz algebra structure on $\mathfrak{gl}(V)\oplus V$ is given by

$$\{A+u,B+v\}=[A,B]+Av,\quad \forall A,B\in\mathfrak{gl}(V),\quad u,v\in V.$$

This Leibniz algebra is called an omni-Lie algebra and denoted by $\mathfrak{ol}(V)$. The notion of an omni-Lie algebra was introduced by Weinstein [22] as the linearization of a Courant algebroid. The notion of a Courant algebroid was introduced in [17], which has been widely applied in many fields both for mathematics and physics (see [14] for more details). Its Leibniz algebra structure also played an important role when studying the integrability of Courant brackets [12,16].

Notice that the skew-symmetric bracket,

$$[\![A+u,B+v]\!]=[A,B]+\frac{1}{2}(Av-Bu), \tag{3.1}$$

which is obtained via the skew-symmetrization of $\{\cdot,\cdot\}$, is used in Weinstein's original definition. Even though an omni-Lie algebra is not a Lie algebra, all Lie algebra structures on V can be characterized by the Dirac structures in $\mathfrak{ol}(V)$. In fact, the next proposition will show that every Leibniz algebra structure on V can be realized as a Leibniz subalgebra of $\mathfrak{ol}(V)$. For any $\varphi: V \longrightarrow \mathfrak{gl}(V)$, consider its graph

$$\mathcal{G}_\varphi = \{\varphi(u)+u \in \mathfrak{gl}(V)\oplus V \mid \forall u\in V\}.$$

Proposition 3.1. *With the above notations, $\mathcal{G}_\varphi$ is a Leibniz subalgebra of $\mathfrak{ol}(V)$ if and only if*

$$[\varphi(u),\varphi(v)] = \varphi(\varphi(u)v), \quad \forall u,v\in V. \tag{3.2}$$

Furthermore, under this condition, $(V,[\cdot,\cdot]_\varphi)$ is a Leibniz algebra, where the linear map $[\cdot,\cdot]_\varphi : V\otimes V \longrightarrow V$ is given by

$$[u,v]_\varphi = \varphi(u)v, \quad \forall u,v\in V. \tag{3.3}$$

Proof. Since $\mathfrak{ol}(V)$ is a Leibniz algebra, we only need to show that $\mathcal{G}_\varphi$ is closed if and only if (3.2) holds. The conclusion follows from

$$\{\varphi(u)+u,\varphi(v)+v\} = [\varphi(u),\varphi(v)]+\varphi(u)v.$$

The other conclusion is straightforward. The proof is complete. $\square$

Remark 3.1. The condition (3.2) actually means that φ is an embedding tensor. See [20] for more details about embedding tensors.

Recall that a representation of a Lie algebra $\mathfrak{g}$ on a vector space V is a Lie algebra homomorphism from $\mathfrak{g}$ to the Lie algebra $\mathfrak{gl}(V)$, which realizes an abstract Lie algebra as a subalgebra of the general linear Lie algebra. Similarly, for a Leibniz algebra, we suggest the following definition.

Definition 3.1. *An omni-representation of a Leibniz algebra $(\mathfrak{g},[\cdot,\cdot]_\mathfrak{g})$ on a vector space V is a Leibniz algebra homomorphism $\rho : \mathfrak{g} \longrightarrow \mathfrak{ol}(V)$.*

According to the two components of $\mathfrak{gl}(V)\oplus V$, every linear map $\rho:\mathfrak{g}\longrightarrow \mathfrak{ol}(V)$ splits to two linear maps: $\phi:\mathfrak{g}\longrightarrow\mathfrak{gl}(V)$ and $\theta:\mathfrak{g}\longrightarrow V$. Then, we have

Proposition 3.2. *A linear map $\rho=\phi+\theta:\mathfrak{g}\longrightarrow\mathfrak{gl}(V)\oplus V$ is an omni-representation of a Leibniz algebra $(\mathfrak{g},[\cdot,\cdot]_\mathfrak{g})$ if and only if $(V,\phi,0)$ is a representation of the Leibniz algebra $(\mathfrak{g},[\cdot,\cdot]_\mathfrak{g})$ and $\theta:\mathfrak{g}\longrightarrow V$ is a 1-cocycle on the Leibniz algebra $(\mathfrak{g},[\cdot,\cdot]_\mathfrak{g})$ with coefficients in the representation $(V,\phi,0)$.*

Proof. On one hand, we have

$$\rho([x,y]_\mathfrak{g}) = \phi([x,y]_\mathfrak{g}) + \theta([x,y]_\mathfrak{g}).$$

On the other hand, we have

$$\begin{aligned}\{\rho(x),\rho(y)\} &= \{\phi(x)+\theta(x),\phi(y)+\theta(y)\} \\ &= [\phi(x),\phi(y)] + \phi(x)\theta(y).\end{aligned}$$

Thus, ρ is a homomorphism if and only if

$$\phi([x,y]_\mathfrak{g}) = [\phi(x),\phi(y)], \tag{3.4}$$

$$\theta([x,y]_\mathfrak{g}) = \phi(x)\theta(y). \tag{3.5}$$

Eq. (3.4) implies that $(V,\phi,0)$ is a representation of the Leibniz algebra $(\mathfrak{g},[\cdot,\cdot]_\mathfrak{g})$, and Eq. (3.5) implies that $\theta:\mathfrak{g}\longrightarrow V$ is a 1-cocycle on the Leibniz algebra $(\mathfrak{g},[\cdot,\cdot]_\mathfrak{g})$ with coefficients in the representation $(V,\phi,0)$. $\square$

A usual representation in the sense of Definition 1.1 gives rise to an omni-representation naturally.

Theorem 3.1. *Let (V,l,r) be a representation of a Leibniz algebra $(\mathfrak{g},[\cdot,\cdot]_\mathfrak{g})$. Then*

$$\rho = (l^*\otimes 1 + 1\otimes l) + r : \mathfrak{g} \longrightarrow \mathfrak{ol}(V^*\otimes V)$$

is an omni-representation of $\mathfrak{g}$ on $V^*\otimes V$.

Proof. By Proposition 2.1, $r:\mathfrak{g}\longrightarrow\mathfrak{gl}(V)$ is a 1-cocycle on $\mathfrak{g}$ with coefficients in the representation $(V^*\otimes V, l^*\otimes 1+1\otimes l, 0)$. By Proposition 3.2, $\rho=(l^*\otimes 1+1\otimes l)+r$ is a homomorphism from $\mathfrak{g}$ to $\mathfrak{ol}(V^*\otimes V)$, which implies that ρ is an omni-representation of $\mathfrak{g}$ on $V^*\otimes V$. $\square$

A trivial omni-representation of a Leibniz algebra $(\mathfrak{g},[\cdot,\cdot]_\mathfrak{g})$ on $\mathbb{R}$ is defined to be a homomorphism

$$\rho_T = \phi+\theta : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathbb{R})\oplus\mathbb{R}$$

such that $\phi=0$. By Proposition 3.2, we have

Proposition 3.3. *Trivial omni-representations of a Leibniz algebra $(\mathfrak{g},[\cdot,\cdot]_\mathfrak{g})$ are in one-to-one correspondence to $\xi\in\mathfrak{g}^*$ such that $\xi|_{[\mathfrak{g},\mathfrak{g}]_\mathfrak{g}}=0$.*

The adjoint omni-representation $\mathfrak{ad}$ of a Leibniz algebra $(\mathfrak{g},[\cdot,\cdot]_\mathfrak{g})$ on $\mathfrak{g}$ is defined to be the homomorphism

$$\mathfrak{ad} = \mathrm{ad}_L + \mathrm{id} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})\oplus\mathfrak{g}.$$

4 Omni-cohomologies of Leibniz algebras

In this section, we introduce omni-cohomologies of Leibniz algebras associated to omni-representations, and show that omni-cohomology groups and Loday-Pirashvili cohomology groups are isomorphic for the trivial representations and adjoint representations.

Let $\rho:\mathfrak{g}\longrightarrow\mathfrak{ol}(V)$ be an omni-representation of a Leibniz algebra $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}})$. It is obvious that $\mathrm{im}(\rho)\subset\mathfrak{ol}(V)$ is a Leibniz subalgebra so that one can define the set of k-cochains by

$$C^k(\mathfrak{g};\rho)=\{f:\otimes^k\mathfrak{g}\longrightarrow \mathrm{im}(\rho)\}$$

and an operator $\delta:C^k(\mathfrak{g};\rho)\longrightarrow C^{k+1}(\mathfrak{g};\rho)$ by

$$\begin{aligned}\delta c^k(x_1,\dots,x_{k+1})=&\sum_{i=1}^{k}(-1)^{i+1}\{\rho(x_i),c^k(x_1,\dots,\widehat{x_i},\dots,x_{k+1})\}\\&+(-1)^{k+1}\{c^k(x_1,\dots,x_k),\rho(x_{k+1})\}\\&+\sum_{1\le i<j\le k+1}(-1)^i c^k\big(x_1,\dots,\widehat{x_i},\dots,x_{j-1},[x_i,x_j]_{\mathfrak{g}},x_{j+1},\dots,x_{k+1}\big).\end{aligned}\tag{4.1}$$

Lemma 4.1. *With the above notations, we have $\delta^2=0$.*

Proof. For all $x\in\mathfrak{g}$ and $u\in\mathrm{im}(\rho)$, define

$$l_x(u)=\{\rho(x),u\},\quad r_x(u)=\{u,\rho(x)\}.$$

By the fact that $\mathfrak{ol}(V)$ is a Leibniz algebra, we can deduce that $(\mathrm{im}(\rho);l,r)$ is a representation of $\mathfrak{g}$ on $\mathrm{im}(\rho)$ in the sense of Definition 1.1, and δ is just the usual coboundary operator for this representation so that $\delta^2=0$. □

Thus, we have a well-defined cochain complex $(C^\bullet(\mathfrak{g};\rho),\delta)$. The corresponding cohomology is called the omni-cohomology of the Leibniz algebra $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}})$ with coefficients in the omni-representation ρ, and denoted by $H^\bullet_{omni}(\mathfrak{g};\rho)$. In particular, $H^\bullet_{omni}(\mathfrak{g})$ and $H^\bullet_{omni}(\mathfrak{g};\mathfrak{ad})$ denote the omni-cohomologies with coefficients in trivial omni-representation and adjoint omni-representation of $\mathfrak{g}$ respectively.

Theorem 4.1. *With the above notations, we have $H^\bullet_{omni}(\mathfrak{g})=H^\bullet(\mathfrak{g})$.*

Proof. If $[\mathfrak{g},\mathfrak{g}]_{\mathfrak{g}}=\mathfrak{g}$, there is only one trivial omni-representation $\rho=0$ by Proposition 3.3. In this case, all the cochains are also 0. Thus, $H^\bullet_{omni}(\mathfrak{g})=0$. On the other hand, under the condition $[\mathfrak{g},\mathfrak{g}]_{\mathfrak{g}}=\mathfrak{g}$, it is straightforward to deduce that for any $\xi\in C^k(\mathfrak{g})$, $\partial\xi=0$ if and only if $\xi=0$. Thus, $H^\bullet(\mathfrak{g})=0$.

If $[\mathfrak{g},\mathfrak{g}]_{\mathfrak{g}} \neq \mathfrak{g}$, any $0 \neq \xi \in \mathfrak{g}^*$ such that $\xi|_{[\mathfrak{g},\mathfrak{g}]_{\mathfrak{g}}} = 0$ gives rise to a trivial omni-representation ρ_T. Furthermore, we have $\mathrm{im}(\rho_T) = \mathbb{R}$ and $C^k(\mathfrak{g}) = \otimes^k \mathfrak{g}^*$. Thus, the sets of cochains are the same associated to two kinds of representations. Since V is an abelian subalgebra in $\mathfrak{ol}(V)$, for any $\xi \in C^k(\mathfrak{g})$, we have

$$\begin{aligned}
\delta\xi(x_1,\dots,x_{k+1}) &= \sum_{i=1}^{k}(-1)^{i+1}\{\rho_T(x_i),c^k(x_1,\dots,\widehat{x_i},\dots,x_{k+1})\} \\
&\quad +(-1)^{k+1}\{c^k(x_1,\dots,x_k),\rho_T(x_{k+1})\} \\
&\quad + \sum_{1\le i<j\le k+1}(-1)^i c^k\big(x_1,\dots,\widehat{x_i},\dots,x_{j-1},[x_i,x_j]_{\mathfrak{g}},x_{j+1},\dots,x_{k+1}\big) \\
&= \sum_{1\le i<j\le k+1}(-1)^i c^k\big(x_1,\dots,\widehat{x_i},\dots,x_{j-1},[x_i,x_j]_{\mathfrak{g}},x_{j+1},\dots,x_{k+1}\big) \\
&= \partial\xi(x_1,\dots,x_{k+1}).
\end{aligned}$$

Thus, we have $H^\bullet_{omni}(\mathfrak{g}) = H^\bullet(\mathfrak{g})$. $\square$

For the adjoint omni-representation $\mathfrak{ad} = \mathrm{ad}_L + \mathrm{id} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g}) \oplus \mathfrak{g}$, any k-cochain f is uniquely determined by a linear map $\mathfrak{f} : \otimes^k \mathfrak{g} \longrightarrow \mathfrak{g}$ such that

$$f = (\mathrm{ad}_L \circ \mathfrak{f}, \mathfrak{f}) : \otimes^k \mathfrak{g} \;\longrightarrow\; \mathrm{im}(\mathfrak{ad}). \tag{4.2}$$

Theorem 4.2. *With the above notations, we have* $H^\bullet_{omni}(\mathfrak{g};\mathfrak{ad}) = H^\bullet(\mathfrak{g};\mathrm{ad}_L,\mathrm{ad}_R)$.

Proof. Since any k-cochain $f : \otimes^k \mathfrak{g} \longrightarrow \mathrm{im}(\mathfrak{ad})$ is uniquely determined by a linear map $\mathfrak{f} : \otimes^k \mathfrak{g} \longrightarrow \mathfrak{g}$ via (4.2). Thus, there is a one-to-one correspondence between the sets of cochains associated to the two kinds representations via $f \leftrightsquigarrow \mathfrak{f}$. Furthermore, we have

$$\begin{aligned}
&\delta f(x_1,\dots,x_{k+1}) \\
&= \sum_{i=1}^{k}(-1)^{i+1}\{\mathfrak{ad}(x_i),f(x_1,\dots,\widehat{x_i},\dots,x_{k+1})\} \\
&\quad +(-1)^{k+1}\{f(x_1,\dots,x_k),\mathfrak{ad}(x_{k+1})\} \\
&\quad + \sum_{1\le i<j\le k+1}(-1)^i f\big(x_1,\dots,\widehat{x_i},\dots,x_{j-1},[x_i,x_j]_{\mathfrak{g}},x_{j+1},\dots,x_{k+1}\big) \\
&= \sum_{i=1}^{k}(-1)^{i+1}\{\mathrm{ad}_L(x_i)+x_i,\mathrm{ad}_L\mathfrak{f}(x_1,\dots,\widehat{x_i},\dots,x_{k+1})+\mathfrak{f}(x_1,\dots,\widehat{x_i},\dots,x_{k+1})\} \\
&\quad +(-1)^{k+1}\{\mathrm{ad}_L\mathfrak{f}(x_1,\dots,x_k)+\mathfrak{f}(x_1,\dots,x_k),\mathrm{ad}_L(x_{k+1})+x_{k+1}\}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{1\le i<j\le k+1}(-1)^i\big(\mathrm{ad}_L\mathfrak{f}(x_1,\dots,\widehat{x_i},\dots,x_{j-1},[x_i,x_j]_{\mathfrak{g}},x_{j+1},\dots,x_{k+1})\\
&\qquad\qquad +\mathfrak{f}(x_1,\dots,\widehat{x_i},\dots,x_{j-1},[x_i,x_j]_{\mathfrak{g}},x_{j+1},\dots,x_{k+1})\big)\\
=&\sum_{i=1}^{k}(-1)^{i+1}\big([\mathrm{ad}_L(x_i),\mathrm{ad}_L\mathfrak{f}(x_1,\dots,\widehat{x_i},\dots,x_{k+1})]+\mathrm{ad}_L(x_i)\mathfrak{f}(x_1,\dots,\widehat{x_i},\dots,x_{k+1})\big)\\
&+(-1)^{k+1}\big([\mathrm{ad}_L\mathfrak{f}(x_1,\dots,x_k),\mathrm{ad}_L(x_{k+1})]+\mathrm{ad}_L\mathfrak{f}(x_1,\dots,x_k)x_{k+1}\big)\\
&+\sum_{1\le i<j\le k+1}(-1)^i\big(\mathrm{ad}_L\mathfrak{f}(x_1,\dots,\widehat{x_i},\dots,x_{j-1},[x_i,x_j]_{\mathfrak{g}},x_{j+1},\dots,x_{k+1})\\
&\qquad\qquad +\mathfrak{f}(x_1,\dots,\widehat{x_i},\dots,x_{j-1},[x_i,x_j]_{\mathfrak{g}},x_{j+1},\dots,x_{k+1})\big)\\
=&\sum_{i=1}^{k}(-1)^{i+1}\big(\mathrm{ad}_L[x_i,\mathfrak{f}(x_1,\dots,\widehat{x_i},\dots,x_{k+1})]_{\mathfrak{g}}+[x_i,\mathfrak{f}(x_1,\dots,\widehat{x_i},\dots,x_{k+1})]_{\mathfrak{g}}\big)\\
&+(-1)^{k+1}\big(\mathrm{ad}_L[\mathfrak{f}(x_1,\dots,x_k),x_{k+1}]_{\mathfrak{g}}+[\mathfrak{f}(x_1,\dots,x_k),x_{k+1}]_{\mathfrak{g}}\big)\\
&+\sum_{1\le i<j\le k+1}(-1)^i\big(\mathrm{ad}_L\mathfrak{f}(x_1,\dots,\widehat{x_i},\dots,x_{j-1},[x_i,x_j]_{\mathfrak{g}},x_{j+1},\dots,x_{k+1})\\
&\qquad\qquad +\mathfrak{f}(x_1,\dots,\widehat{x_i},\dots,x_{j-1},[x_i,x_j]_{\mathfrak{g}},x_{j+1},\dots,x_{k+1})\big)\\
=&\,\mathrm{ad}_L\partial\mathfrak{f}(x_1,\dots,x_{k+1})+\partial\mathfrak{f}(x_1,\dots,x_{k+1}).
\end{aligned}
$$

Thus, we have $\delta f=0$ if and only if $\partial\mathfrak{f}=0$. Similarly, we can prove that f is exact if and only if $\mathfrak{f}$ is exact. Thus, the corresponding cohomology groups are isomorphic. The proof is complete. □

At last, we consider an omni-representation ρ such that the image of ρ is contained in the graph $\mathcal{G}_\varphi$ for some linear map $\varphi: V\longrightarrow \mathfrak{gl}(V)$ satisfying Eq. (3.2). In this case, ρ is of the form $\rho=\varphi\circ\theta+\theta$, where $\theta:\mathfrak{g}\longrightarrow V$ is a linear map. A k-cochain $f:\otimes^k\mathfrak{g}\longrightarrow \mathrm{im}(\rho)$ is of the form $f=\varphi\circ\mathfrak{f}+\mathfrak{f}$, where $\mathfrak{f}:\otimes^k\mathfrak{g}\longrightarrow V$ is a linear map.

Define left and right actions in the sense of Definition 1.1 by

$$l_xu=\mathrm{pr}\{\rho(x),\varphi(u)+u\}=\varphi(\theta(x))u, \tag{4.3}$$

$$r_xu=\mathrm{pr}\{\varphi(u)+u,\rho(x)\}=\varphi(u)\theta(x), \tag{4.4}$$

where pr is the projection from $\mathfrak{gl}(V)\oplus V$ to V. Similar to Theorem 4.2, it is easy to prove that

Theorem 4.3. *Let ρ be an omni-representation such that the image of ρ is contained in the graph $\mathcal{G}_\phi$ for some linear map $\varphi: V\longrightarrow\mathfrak{gl}(V)$ satisfying Eq.* (3.2). *Then we have*

$$H^\bullet_{omni}(\mathfrak{g};\rho)=H^\bullet(\mathfrak{g};l,r),$$

where l and r are given by (4.3) *and* (4.4), *respectively.*

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