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Optimality Conditions for Constrained Minimax Optimization

Yu-Hong Dai[1,2] and Liwei Zhang[3,*]

[1] *LSEC, ICMSEC, AMSS, Chinese Academy of Sciences, Beijing 100190, China.*
[2] *School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China.*
[3] *School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, China.*

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Abstract. Minimax optimization problems arises from both modern machine learning including generative adversarial networks, adversarial training and multi-agent reinforcement learning, as well as from tradition research areas such as saddle point problems, numerical partial differential equations and optimality conditions of equality constrained optimization. For the unconstrained continuous nonconvex-nonconcave situation, Jin, Netrapalli and Jordan (2019) carefully considered the very basic question: what is a proper definition of local optima of a minimax optimization problem, and proposed a proper definition of local optimality called local minimax. We shall extend the definition of local minimax point to constrained nonconvex-nonconcave minimax optimization problems. By analyzing Jacobian uniqueness conditions for the lower-level maximization problem and the strong regularity of Karush-Kuhn-Tucker conditions of the maximization problem, we provide both necessary optimality conditions and sufficient optimality conditions for the local minimax points of constrained minimax optimization problems.

AMS subject classifications: 90C30

Key words: Constrained minimax optimization, value function, Jacobian uniqueness conditions, strong regularity, necessary optimality conditions, sufficient optimality conditions.

1 Introduction

Minimax optimization problems arise from both modern machine learning including generative adversarial networks, adversarial training and multi-agent reinforcement

*Corresponding author. *Email addresses:* `dyh@lsec.cc.ac.cn` (Y.-H. Dai), `lwzhang@dlut.edu.cn` (L. Zhang)

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learning, as well as from tradition research areas such as saddle point problems, numerical partial differential equations and optimality conditions of equality constrained optimization. Let m,n,m_1,m_2,n_1 and n_2 be positive integers, $f:\Re^n\times\Re^m\to\Re$, $h:\Re^n\times\Re^m\to\Re^{m_1}$, $g:\Re^n\times\Re^m\to\Re^{m_2}$, $H:\Re^n\to\Re^{n_1}$ and $G:\Re^n\to\Re^{n_2}$ be given functions. We are interested in the constrained minimax optimization problem of the form

$$\min_{x\in\Phi}\max_{y\in Y(x)} f(x,y), \tag{1.1}$$

where $f:\Re^n\times\Re^m\to\Re$, $\Phi\subset\Re^n$ is a feasible set of decision variable x defined by

$$\Phi=\{x\in\Re^n : H(x)=0,\ G(x)\le 0\} \tag{1.2}$$

and $Y:\Re^n\rightrightarrows\Re^m$ is a set-valued mapping defined by

$$Y(x)=\{y\in\Re^m : h(x,y)=0,\ g(x,y)\le 0\}. \tag{1.3}$$

For the unconstrained continuous nonconvex-nonconcave situation, Jin, Netrapalli and Jordan [10] carefully considered the very basic question: what is a proper definition of local optima of a minimax optimization problem, and proposed a proper definition of local optimality called local minimax. We shall extend this definition of local minimax point for the constrained minimax optimization problem (1.1).

Definition 1.1. *A point $(x^*,y^*)\in\Re^n\times\Re^m$ is said to be a local minimax point of Problem* (1.1) *if there exists $\delta_0>0$ and a function $\eta:(0,\delta_0]\to\Re_+$ satisfying $\eta(\delta)\to 0$ as $\delta\to 0$, such that for any $\delta\in(0,\delta_0]$ and any $(x,y)\in[\boldsymbol{B}_\delta(x^*)\cap\Phi]\times[Y(x^*)\cap\boldsymbol{B}_\delta(y^*)]$, we have*

$$f(x^*,y)\le f(x^*,y^*)\le \max_z\Big\{f(x,z): z\in Y(x)\cap\boldsymbol{B}_{\eta(\delta)}(y^*)\Big\}. \tag{1.4}$$

The minimax optimization problem is essentially a bi-level programming problem and the local minimax point is closely related to the so-called pessimistic solution of bi-level programming problem, see [5]. There have been many results about optimality conditions for bi-level programming. Dempe [4] demonstrated necessary optimality conditions and the sufficient optimality conditions for the bi-level programming when the lower level problem is a convex optimization problem satisfying the Mangasarian-Fromovitz constraint qualification, the second-order sufficient optimality condition and the constant rank constraint qualification. Falk [9] discussed the optimality conditions when the lower level has a local unique solution and the upper level problem is unconstrained. Ye and Zhu [15] established necessary optimality conditions for bi-level programming based on the generalized gradient of value function. Dempe et al. [6] derived necessary optimality conditions for bi-level programming when the solution set of the lower level problem satisfies some calmness property. Dempe and Zemkoho [7] also developed necessary optimality conditions based on the value function reformulation of bi-level programming and the assumption that the value function is locally convex. Dempe