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A Family of curl-curl Conforming Finite Elements on Tetrahedral Meshes

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Abstract. In [23], we, together with our collaborator, proposed a family of $H(\text{curl}^2)$-conforming elements on both triangular and rectangular meshes. The elements provide a brand new method to solve the quad-curl problem in 2 dimensions. In this paper, we turn our focus to 3 dimensions and construct $H(\text{curl}^2)$-conforming finite elements on tetrahedral meshes. The newly proposed elements have been proved to have the optimal interpolation error estimate. Having the tetrahedral elements, we can solve the quad-curl problem in any Lipschitz domain by the conforming finite element method. We also provide several numerical examples of using our elements to solve the quad-curl problem. The results of the numerical experiments show the correctness of our elements.

AMS subject classifications: 65N30, 35Q60, 65N15, 41A25

Key words: H^2(curl)-conforming, finite elements, tetrahedral mesh, quad-curl problems, interpolation errors, convergence analysis.

1 Introduction

The quad-curl problem is involved in various practical problems, such as inverse electromagnetic scattering theory [2, 16, 20] or magnetohydrodynamics [26]. As its name implies, this problem involves a fourth-order curl operator which makes it much more challenging to solve than the lower-order electromagnetic problem [7, 11–15, 22]. The regularity of this problem was studied by Nicaise [18], and Chen et al. [4]. As for the numerical methods, Zheng et al. developed a nonconforming finite element method for this problem in [26]. This method has low computational cost since it has small number of degrees of freedom (DOFs), but it bears the disadvantage of low accuracy. Based on

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Nédélec elements, a discontinuous Galerkin method and a weak Galerkin method were presented in [10] and [21], respectively. In addition, error estimates for discontinuous Galerkin methods based on a relatively low regularity assumption of the exact solution are proposed in [3, 4]. Another approach to deal with the quad-curl operator is to introduce an auxiliary variable and reduce the original problem to a second-order system [20]. Zhang proposed a different mixed scheme [24], which relaxes the regularity requirement in theoretical analysis.

However, the most natural way to solve this problem is the conforming finite element method. In [23], the authors and another collaborator constructed curl-curl conforming or $H(\mathrm{curl}^2)$-conforming elements in 2 dimensions (2D) to solve the quad-curl problem. In three dimensions (3D), the numerical solution derived by the existing H^2-conforming (or C^1-conforming) elements ($\boldsymbol{u}\in\boldsymbol{H}^1$ and $\nabla\boldsymbol{u}\in\boldsymbol{H}^1$) [25] converges to an H^2 projection of the exact solution. The distance between such a projection and the exact solution may be a positive constant since C_0^∞ may not be dense in $H(\mathrm{curl}^2)\cap H(\mathrm{div}^0)$ under a specific norm. Indeed, the treatment of boundary conditions is also an issue when using H^2-conforming elements to solve the quad-curl problem. Also, Neilan constructed a family of $\boldsymbol{H}^1(\mathrm{curl})$-conforming elements ($\boldsymbol{u}\in\boldsymbol{H}^1$ and $\nabla\times\boldsymbol{u}\in\boldsymbol{H}^1$) in [17] (see [9] for the 2D case). The family of elements can also lead to conforming approximations of the quad-curl problem. However, in this paper, we derive a conforming finite element space for $H(\mathrm{curl}^2;\Omega)$ ($\boldsymbol{u}\in\boldsymbol{L}^2$ and $\nabla\times\boldsymbol{u}\in\boldsymbol{H}^1$) where the function regularity is weaker than the space $\boldsymbol{H}^1(\mathrm{curl})$. Such types of elements, to the best of the authors' knowledge, are not available in the literature. Due to the large kernel space and the natural divergence-free property of the curl operator $\nabla\times$, the construction of $H(\mathrm{curl}^2)$-conforming elements is more difficult than the 2D case.

Our paper starts by describing the tetrahedral $H(\mathrm{curl}^2)$-conforming finite elements. The unisolvence and conformity of our $H(\mathrm{curl}^2)$-conforming finite elements can be verified by a rigorous mathematical analysis. Moreover, our new elements have been proved to possess good interpolation properties. Although the involvement of normal derivatives to edges render the DOFs on a general element failing to relate to those on the reference element, we constructed intermediate elements whose DOFs can be related to those on the reference element and are close to our elements. In this way, we prove the optimal error estimate of the finite element interpolation. In our construction, the number of the DOFs for the lowest-order element is 315. Because of the large number of DOFs, it's hard the compute the Lagrange-type basis functions by the traditional method. Hence we employ the method proposed in [8] to obtain the basis functions on a general element.

The rest of the paper is organized as follows. In Section 2 we list some function spaces and notations. Section 3 is the technical part, where we construct the $H(\mathrm{curl}^2)$- conforming finite elements on a tetrahedron. In Section 4 we give the error estimate for the interpolation. In Section 5 we use our newly proposed elements to solve the quad-curl problem and give some numerical results to verify the correctness of our method. Finally, some concluding remarks and possible future works are given in Section 6. We present how to implement the finite elements in Appendix and provide the code for it.