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An Efficient Online-Offline Method for Elliptic Homogenization Problems

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Abstract. We present a new numerical method for solving the elliptic homogenization problem. The main idea is that the missing effective matrix is reconstructed by solving the local least-squares in an offline stage, which shall be served as the input data for the online computation. The accuracy of the proposed method is analyzed with the aid of the refined estimates of the reconstruction operator. Two dimensional and three dimensional numerical tests confirm the efficiency of the proposed method, and illustrate that this online-offline strategy may significantly reduce the cost without loss of the accuracy.

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Key words: Numerical homogenization, heterogeneous multiscale method, online-offline computing, least-squares reconstruction, stability.

1 Introduction

We consider a prototypical elliptic boundary value problem

$$\begin{cases} -\mathrm{div}(a^\varepsilon(x)\nabla u^\varepsilon(x)) = f(x), & x\in D\subset\mathbb{R}^d, \\ u^\varepsilon(x)=0, & x\in\partial D, \end{cases} \tag{1.1}$$

where ε is a small parameter that signifies explicitly the multiscale nature of the problem. We assume that the coefficient a^ε, which is not necessarily symmetric, belongs to a set

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$\mathcal{M}(\alpha,\beta,D)$ that is defined by

$$\begin{aligned}\mathcal{M}(\alpha,\beta,D):=\{\mathcal{B}\in[L^\infty(D)]^{d^2}\,|\,(\mathcal{B}\xi,\xi)\geq\alpha|\xi|^2,\ |\mathcal{B}(x)\xi|\leq\beta|\xi|,\\ \text{for any } \xi\in\mathbb{R}^d \text{ and a.e. } x\in D\},\end{aligned}\tag{1.2}$$

where D is a bounded domain in $\mathbb{R}^d$ and $(\cdot,\cdot)$ denotes the inner product in $\mathbb{R}^d$, while $|\cdot|$ is the corresponding norm.

In the sense of H-convergence [53, 54], for every sequence $a^\varepsilon\in\mathcal{M}(\alpha,\beta,D)$ and $f\in H^{-1}(D)$, the sequence u^ε of the solution to (1.1) satisfies

$$\left\{\begin{array}{ll} u^\varepsilon\rightharpoonup u_0, & \text{weakly in } H_0^1(D),\\ a^\varepsilon\nabla u^\varepsilon\rightharpoonup\mathcal{A}\nabla u_0, & \text{weakly in } [L^2(D)]^d,\end{array}\right.\quad \text{as } \varepsilon\to 0,\tag{1.3}$$

where u_0 is the solution of the homogenization problem

$$\left\{\begin{array}{ll} -\mathrm{div}(\mathcal{A}(x)\nabla u_0(x))=f(x), & x\in D,\\ u_0(x)=0, & x\in\partial D,\end{array}\right.\tag{1.4}$$

and $\mathcal{A}\in\mathcal{M}(\alpha,\beta,D)$. Here $H_0^1(D), L^2(D)$ and $H^{-1}(D)$ are standard Sobolev spaces [4].

The quantities of interest for Problem (1.1) and Problem (1.4) are the homogenized solution u_0 over the whole domain and the solution u^ε at certain critical local region. The former stands for the information at the large scale, and the later mimics the information at small scale. There are lots of work devoted to efficiently compute such quantities during the last several decades; see, e.g., [6, 17, 20], among many others. Presently we are interested in the efficient way to compute u_0. A typical way that towards this is provided by the heterogeneous multiscale method (HMM) [3, 18], and the FE^2-method [40] commonly used in the engineering community that is also in the same spirit of HMM. The underlying idea of this approach is to extract $\mathcal{A}$ by solving the cell problems posed on the sampling points of the macoscopic solver. At each point, one needs to solve d cell problems with d the dimensionality. Therefore, the main computational cost comes from solving all these cell problems. The number of the cell problems grows rapidly when higher-order macroscopic solvers are employed. To reduce the cost, certain nonconventional quadrature schemes were proposed in [16] when finite element method is used as the macroscopic solver. The number of the cell problems reduces to one third compared to the standard mid-point quadrature scheme when $\mathbb{P}_2$ Lagrange finite element method is employed as the macroscopic solver. Unfortunately, it does not seem easy to extend such idea to even higher order macroscopic solvers because the quadrature nodes tend to accumulate in the interior of the element [48, 50, 51].

In [35], the authors presented a local least-squares reconstruction of the effective matrix using the solution of the cell problems posed on the vertices of the triangulation, which was dubbed as HMM-LS. The total number of the cell problems equals to the total number of the interior vertices of the triangulation, which is of $\mathcal{O}(h^{-d})$ with h the mesh size of the macroscopic solver. This method achieves higher-order accuracy with