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A Nonlinear Eigenvalue Problem Associated with the Sum-Of-Rayleigh-Quotients Maximization

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Abstract. Recent applications in the data science and wireless communications give rise to a particular Rayleigh-quotient maximization, namely, maximizing the sum-of-Rayleigh-quotients over a sphere constraint. Previously, it is shown that maximizing the sum of two Rayleigh quotients is related with a certain eigenvector-dependent nonlinear eigenvalue problem (NEPv), and any global maximizer must be an eigenvector associated with the largest eigenvalue of this NEPv. Based on such a principle for the global maximizer, the self-consistent field (SCF) iteration turns out to be an efficient numerical method. However, generalization of sum of two Rayleigh-quotients to the sum of an arbitrary number of Rayleigh-quotients maximization is not a trivial task. In this paper, we shall develop a new treatment based on the S-Lemma. The new argument, on one hand, handles the sum of two and three Rayleigh-quotients maximizations in a simple way, and also deals with certain general cases, on the other hand. Our result gives a characterization for the solution of this sum-of-Rayleigh-quotients maximization and provides theoretical foundation for an associated SCF iteration. Preliminary numerical results are reported to demonstrate the performance of the SCF iteration.

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Key words: Eigenvector-dependent nonlinear eigenvalue problem, self-consistent-field iteration, Rayleigh quotient maximization, S-Lemma.

1 Introduction

Given a symmetric matrix $B \in \mathbb{R}^{n\times n}$ and a symmetric positive definite matrix $W \in \mathbb{R}^{n\times n}$, the minimum (maximum) of the Rayleigh quotient $\frac{\mathbf{x}^\top B\mathbf{x}}{\mathbf{x}^\top W\mathbf{x}}$ is just the smallest (largest)

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eigenvalue of the generalized eigenvalue problem (GEP): $B\mathbf{x}=\lambda W\mathbf{x}$. The applications of GEP arise in various areas, and several state-of-the-art numerical methods for GEP have been developed in the numerical linear algebra (e.g., [2, 3, 9]) and widely-used in many real world applications. For optimizing the Rayleigh quotient $\frac{\mathbf{x}^\top B\mathbf{x}}{\mathbf{x}^\top W\mathbf{x}}$, the homogeneity implies that we can consider the associated eigenvectors $\mathbf{x}$ with unit length, and it turns out that, the associated optimization does not admit local but nonglobal maximizers. This also facilitates the application of certain optimization methods for finding the dominant eigenpairs.

Some applications in the data science and wireless communications give rise to certain generalizations of Rayleigh-quotient maximization. For example, in the sparse Fisher discriminant analysis [33, Section 2] and in jointly optimizing the receive beamforming in a hybrid cognitive radio system [20, Lemma 4], solution of the maximization

$$\max_{\|\mathbf{x}\|_2=1}\left\{\mathbf{x}^\top D\mathbf{x}+\frac{\mathbf{x}^\top B\mathbf{x}}{\mathbf{x}^\top W\mathbf{x}}\right\} \tag{1.1}$$

is required, where D is symmetric. The global maximizer of (1.1) has been characterized as an eigenvector associated with the largest eigenvalue of a special eigenvector-dependent nonlinear eigenvalue problem (NEPv), and an eigenvalue-based iteration, namely the self-consistent field (SCF) iteration [34], is suggested as an efficient way for (1.1). Other optimization methods for finding the global maximizer of (1.1) have also been proposed in [16, 23, 26].

Further extensions [13, 20, 35, 36] of (1.1) also appear in recent applications. For instance, in the downlink of a multi-user MIMO system [20, eq. 22], the objective function becomes the sum-of-Rayleigh-quotients $\mathbf{x}^\top D\mathbf{x}+\sum_{i=1}^m\frac{\mathbf{x}^\top B_i\mathbf{x}}{\mathbf{x}^\top W_i\mathbf{x}}$, where $B_i,W_i\in\mathbb{R}^{n\times n}$ are symmetric and W_i are positive definite. Moreover, this sum-of-Rayleigh-quotients maximization is also a model, namely the weighted harmonic mean of trace ratios (with projected low-dimension $d=1$), for multiclass discriminant analysis [13, Eq. 26]. Even though an SCF iteration is proposed to solve this maximization in [13], the characterization of the global maximizer with respect to certain eigenvector of the associated nonlinear eigenvalue problem has not been investigated for the general case $m>1$. It turns out that the generalization for $m=1$ made in [34] to $m>1$ is not a trivial task.

In this paper, we shall develop a new argument based on the S-Lemma (see [28] and a survey [19]) for the problem

$$\max_{\mathbf{x}\in\mathcal{M}}\left\{f(\mathbf{x}):=\mathbf{x}^\top D\mathbf{x}+\sum_{i=1}^m\frac{\mathbf{x}^\top B_i\mathbf{x}}{\mathbf{x}^\top W_i\mathbf{x}}\right\} \tag{1.2}$$

where $m<n$ and $\mathcal{M}=\{\mathbf{x}\in\mathbb{R}^n|\mathbf{x}^\top\mathbf{x}=1\}$ is the unit sphere. The new treatment, on one hand, handles the case $m=1$ in a much simpler way, and also deals with $m=2$ as well as certain cases of $m>1$, on the other hand. Our results shed some light on this sum-of-Rayleigh-quotients maximization and provide theoretical foundation for the associated SCF iteration.