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# Combined Second and Fourth-Order PDEs Model and Associated Variational Problems for Geometric Images Inpainting and Denoising

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**Abstract.** We consider a Partial Differential Equation model combining second and fourth-order operators for solving the geometry inpainting and denoising problems. The model allows the accurate recovery of curvatures and the singular set of the reconstructed image (edges, corners). The approach proposed permits a dynamical modelling by constructing a family of simple discrete energies that admit as a $\Gamma$-limit Mumford-Shah-Euler like functional. The approximation functionals are build within an adaptive strategy, based on two ingredients: a fine location of the singular set using mesh refinement, and second, a local choice of the diffusion coefficients which modify the reconstruction operator. Unlike the usual methods, mostly based on prior guess on the continuous solution and leading to complex and nonlinear systems of PDEs, our method consists in solving linear problems and updating the diffusion coefficients. The high order of the operator allows us to perform simultaneously efficient filtering of the data and the interpolation in the damaged regions. The method turns out to be superior to any second-order model in restoring large gap connections and curvy features. In order to validate this approach, we compare the results of our method with those of some existing one in the fields of geometry- oriented inpainting and we present several numerical examples.

**AMS subject classifications**: 35G15, 34K28, 65M32, 65M50, 94A08

**Key words**: Moving mesh method, conservative interpolation, iterative method, $l^2$ projection.

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# 1 Introduction

Image inpainting is a central problem in image processing. It refers to the problem of recovering damaged/missing parts of a digital image by interpolation from the valuable areas [12, 13, 22, 25] For the geometry inpainting, Partial Differential Equations and energy methods are extensively used and turn out to be very efficient (see [3, 11, 14, 21, 25, 29,30,33,35,43] and the references therein). The main difficulty in geometry inpainting is to accurately recover the *singularity set* components (edges, corners) and to preserve the curvy features of the level lines of the image. When there is no missing area, the problem reduces to a segmentation/filtering problem and the same difficulties may persist. Conceptually, the main difference between these two situations is that available information come only from one side in the inpainting case.

Second-order PDEs are used in this field and they led to to well established models, such as TV inpainting [25], anisotropic diffusion [49] and (weighted) harmonic method [9], that provide good interpolation properties in geometry inpainting. However, as they are low-order methods, they usually disconnect edges over large distances (violating the connectivity principle) and they fail to reproduce some geometric features of higher-order (curvature, some corners, see e.g. [1, 41]). These shortcomings gave rise to a new class of higher-order diffusion models which perform generally better both in image restoration and in geometry inpainting problems (see [3, 13, 15, 21, 26–28, 32, 35, 40, 41, 43, 44, 47]). In fact, they damp the oscillations and high frequencies (noise) in the homogeneous areas faster than any second-order model [34], and they preserve more efficiently the curvy features of the edges as well as the corners [16, 41]. Several models were proposed in the literature and most of them resort to a strong prior on how to interpolate the solutions with high order operators. One of the very advanced approaches is due to Masnou and Morel who adapted G. Kaniza principle to the interpolation of the level lines of BV-images [35]. It consists of minimizing an energy which combines a length term and a given power of the curvature (e.g. Willmore energy, [20, 35, 36, 42, 43]).

**Our contribution:** In a previous work [46], we considered a multiscale fourth-order model based on an adaptive method which uses a weighted bi-Laplace operator; the weight is a spatially varying diffusion function. The model is updated dynamically, by choosing the weight to fit the geometry of the computed solution and by allowing a tight location of the singularity set. We have shown that this approach is based on the construction of a family of discrete energies which $\Gamma$-converges to a Mumford Shah-$H^{-1}$ limit. The size of the singularity set is bounded by the Lebesgue measure of the set where the diffusion is minimal. The curvature feature is preserved thanks to an $H^{-1}$-filtering. In this article, we improve this approach in two directions: we add a length term to the functional (mimicking the Euler-elastica functional [20, 36, 42, 43]) and we extend the adaptive strategy to two diffusion functions, which allows us to balance more efficiently both shortness and the curvature of the level lines in the inpainted region. Moreover, the family of the discrete energies associated to this approach converges in the $\Gamma$-convergence