CSIAM Trans. Appl. Math.
doi: 10.4208/csiam-am.SO-2021-0008

Vol. 3, No. 1, pp. 57-81
March 2022

Efficient and Unconditional Energy Stable Schemes for the Micropolar Navier-Stokes Equations

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Received 31 January 2021; Accepted 14 June 2021

Abstract. We develop in this paper efficient numerical schemes for solving the micropolar Navier-Stokes equations by combining the SAV approach and pressure-correction method. Our first- and second-order semi-discrete schemes enjoy remarkable properties such as (i) unconditional energy stable with a modified energy, and (ii) only a sequence of decoupled linear equations with constant coefficients need to be solved at each time step. We also construct fully discrete versions of these schemes with a special spectral discretization which preserve the essential properties of the semi-discrete schemes. Numerical experiments are presented to validate the proposed schemes.

AMS subject classifications: 65M12, 65M70, 35Q30, 76A05

Key words: Micropolar Navier-Stokes, pressure-correction, scalar auxiliary variable, energy stability.

1 Introduction

We consider in this paper the so-called micropolar Navier-Stokes equations [16]:

$$\begin{cases} \mathbf{u}_t+\mathbf{u}\cdot\nabla\mathbf{u}-\nu_0\triangle\mathbf{u}+\nabla p=2\nu_r\nabla\times\mathbf{w}+\mathbf{f}, \\ \nabla\cdot\mathbf{u}=0, \\ j\mathbf{w}_t+j\mathbf{u}\cdot\nabla\mathbf{w}-c_1\triangle\mathbf{w}-c_2\nabla\nabla\cdot\mathbf{w}+4\nu_r\mathbf{w}=2\nu_r\nabla\times\mathbf{u}+\mathbf{g}, \end{cases} \tag{1.1}$$

where $\mathbf{u}$ is the linear velocity vector, p the pressure and $\mathbf{w}$ is the non-divergence free micro-rotation field (angular velocity of the rotation of particles of the fluid). The functions $\mathbf{f}(\mathbf{x})$ and $\mathbf{g}(\mathbf{x})$ denote external sources of linear and angular momentum, respectively [13], which depend upon external fields explicitly. The material constants j, ν_0, ν_r, c_1 and c_2 are all positive.

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This system is supplemented with the following initial and boundary conditions:

$$\begin{aligned} \mathbf{u}|_{t=0} = u_0, \quad \mathbf{u}|_{\partial\Omega} = 0, \\ \mathbf{w}|_{t=0} = w_0, \quad \mathbf{w}|_{\partial\Omega} = 0. \end{aligned} \tag{1.2}$$

When the micro-rotation effect is ignored (i.e., $\mathbf{w} = 0$), (1.1) reduce to the celebrated Navier-Stokes equations. The kinematic Newtonian viscosity ν_r is essential for the flow of micropolar fluids, so that the velocity and the micro-rotation are coupled and the global motion is affected by the micro-rotations. The micropolar fluid is less prone to instability than the classical fluid [5, 7, 22], and this model can describe some polymeric fluids and fluids containing certain additives in narrow films [6].

Mathematical properties of the micropolar Navier-Stokes equations have been studied in [8, 13, 18, 20]. In particular, the global existence of weak solutions and the local existence and uniqueness of strong solutions are all well understood. From a numerical point of view, there are some attempts in designing numerical schemes for the micropolar Navier-Stokes equations. For instance, a penalty projection method is proposed and suboptimal error estimates are proved in [17]; a semi-implicit fully discrete scheme is presented in [15] which requires to solve a saddle point problem for velocity and pressure at each time step; in a related work [21], a fractional time stepping technique is proposed to decouple the computation of pressure and velocity. The nonlinear terms in these work are treated either implicitly or semi-implicitly, so that a coupled linear or nonlinear system with variable coefficients has to be solved at each time step.

Recently, the so called scalar auxiliary variable (SAV) approach [24] is introduced for gradient flows. The SAV approach introduces an auxiliary variable and treat all nonlinear terms explicitly, making it possible to construct linear, decoupled, unconditionally energy stable schemes which only require solving linear systems with constant coefficients at each time step. The SAV approach has also been successfully applied to solve Navier-Stokes and related dissipative equations which are not gradient flows [11, 12]. In this paper, we combine the SAV approach with the pressure-correction method (see, e.g., [10]) to construct efficient numerical schemes for the micropolar Navier Stokes equations. More precisely, the semi-discrete schemes we construct using the SAV approach enjoy the following remarkable properties: (i) unconditionally energy stable with a modified energy; and (ii) at each time step, only a sequence of Poisson type equations need to be solved.

Unlike in the case of gradient flows for which any consistent Galerkin type spatial discretization will preserve the nice properties of the semi-discrete SAV schemes, special care has to be taken to choose compatible spatial discretization for pressure and velocity, which is different from the usual inf-sup compatible discretization in a coupled approach. We constructed fully discrete schemes with Galerkin spectral-discretization in space that full preserve the two essential properties of the semi-discrete SAV schemes, and present several numerical experiments to validate our results.

The reminder of the paper is organized as follows. In Section 2, we construct the semi-discrete first- and second-order SAV schemes with pressure-correction to decouple