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Adaptive H(div)-Conforming Embedded-Hybridized Discontinuous Galerkin Finite Element Methods for the Stokes Problems

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Abstract. In this paper, we propose a residual-based a posteriori error estimator of embedded–hybridized discontinuous Galerkin finite element methods for the Stokes problems in two and three dimensions. The piecewise polynomials of degree $k\,(k\geq 1)$ and $k-1$ are used to approximate the velocity and pressure in the interior of elements, and the piecewise polynomials of degree k are utilized to approximate the velocity and pressure on the inter-element boundaries. The attractive properties, named divergence-free and $\boldsymbol{H}(\text{div})$-conforming, are satisfied by the approximate velocity field. We prove that the a posteriori error estimator is robust in the sense that the ratio of the upper and lower bounds is independent of the mesh size and the viscosity. Finally, we provide several numerical examples to verify the theoretical results.

AMS subject classifications: 49M25, 65K10, 65M50

Key words: Stokes equations, HDG methods, E-HDG methods, a posteriori error estimator, divergence-free, $\boldsymbol{H}(\text{div})$-conforming.

1 Introduction

In this paper, we consider the following Stokes problems: find the velocity $\boldsymbol{u}$ and pressure p such that

$$\begin{cases} -\nu\Delta\boldsymbol{u}+\nabla p=\boldsymbol{f}, & \text{in } \Omega, \\ \nabla\cdot\boldsymbol{u}=\boldsymbol{0}, & \text{in } \Omega, \\ \boldsymbol{u}=\boldsymbol{g}, & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

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Here $\Omega \subset \mathbb{R}^d$ $(d=2,3)$ is an open, bounded, polygonal/polyhedral domain, $\nu > 0$ is the dynamic viscosity and $\boldsymbol{f}$ is the external body force. To keep compatibility, the boundary data $\boldsymbol{g}$ needs to satisfy

$$\int_{\partial\Omega} \boldsymbol{g} \cdot \boldsymbol{n} = 0, \tag{1.2}$$

where $\boldsymbol{n}$ is the unit vector normal to the boundary $\partial\Omega$.

Stokes problems are used to describe steady viscous incompressible flow, and the corresponding development of reliable and efficient a posteriori error analysis for finite element discretizations has been a very active field in recent decades, see [7,8,29,31,32,45, 46] and the references therein for more detail. We note that a framework for a posteriori error estimation for the Stokes problems has been provided in [29] based on the $\boldsymbol{H}^1$-conforming velocity reconstruction and the $\boldsymbol{H}(\text{div})$-conforming locally conservative flux (stress) reconstruction.

The hybridizable discontinuous Galerkin (HDG) method [14] provides a unifying strategy for hybridization of finite element methods for second order elliptic problems. By the local elimination of the unknowns defined in the interior of elements, the HDG method leads to a system where the unknowns are only the globally coupled degrees of freedom describing the introduced Lagrange multipliers. We refer to [3, 15, 17, 20, 21, 24, 27, 28, 33, 37, 38, 40, 47] for some developments and applications for the HDG method. As for a posteriori error analysis of the HDG method, the authors in [2, 18, 19] established a posteriori error estimates for second-order elliptic equations. For Stokes and Brinkman equations, the residual-based a posterior error estimator based on the gradient-velocity-pressure, velocity-pseudostress and velocity-pressure formulations were proved by [3, 23, 30]. Other related studies can be found in [9, 12, 13, 36].

The embedded discontinuous Galerkin (EDG) methods were proposed in [16,26]. The main difference between EDG and HDG methods is that the approximation spaces for Lagrange multipliers are continuous or not. Some developments of EDG methods could been found in [11,22,35,39,48]. On the other hand, we mention [10,43] for an embedded-hybridized discontinuous Galerkin method for Stokes and Stokes-Darcy problems. It is worth noting that the approximate velocity field obtained by the embedded-hybridized discontinuous Galerkin method is $\boldsymbol{H}(\text{div})$-conforming and divergence-free. To the best of authors knowledge, there is no literature for a posteriori error analysis for this method.

In this paper, we develop a residual-based a posteriori error estimator for hybridized discontinuous Galerkin (HDG) and embedded-hybridized discontinuous Galerkin (E-HDG) methods for the velocity-pressure formulation of Stokes problems in two and three dimensions. We have the following features:

- The piecewise polynomials of degree k $(k \geq 1)$ and $k-1$ are used to approximate the velocity and pressure in the interior of elements, and the piecewise polynomials of degree k are adopted to approximate the velocity and pressure on the inter-element boundaries. Certainly, the approximate velocity field is divergence-free and $\boldsymbol{H}(\text{div})$-conforming.