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On Symmetry Breaking of Allen-Cahn

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Abstract. This paper is concerned with numerical solutions for the Allen-Cahn equation with standard double well potential and periodic boundary conditions. Surprisingly it is found that using standard numerical discretizations with high precision computational solutions may converge to completely incorrect steady states. This happens for very smooth initial data and state-of-the-art algorithms. We analyze this phenomenon and showcase the resolution of this problem by a new symmetry-preserving filter technique. We develop a new theoretical framework and rigorously prove the convergence to steady states for the filtered solutions.

AMS subject classifications: 35K55, 65M12

Key words: Allen-Cahn equation, symmetry breaking, steady state.

1 Introduction

A curious experiment. Consider the following 1D Allen–Cahn on the periodic torus $\mathbb{T}=[-\pi,\pi]$:

$$\begin{cases} \partial_t u = \kappa^2 \partial_{xx} u - f(u), \\ u\big|_{t=0} = u_0, \end{cases} \tag{1.1}$$

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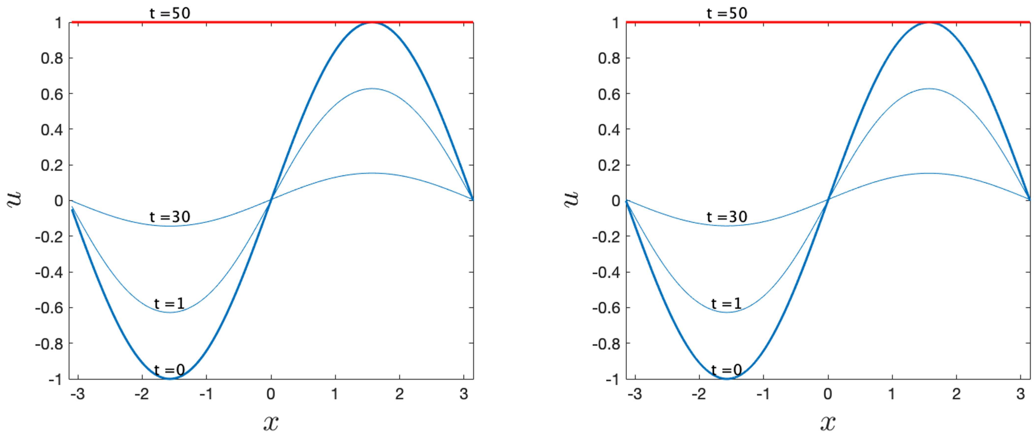

(a) (b)

Figure 1: Dynamics of the Allen-Cahn equation using the first-order implicit-explicit scheme with different discretization parameters. (a): $\kappa=1$, $N_p=128$ and $\tau=0.01$, and (b): $\kappa=1$, $N_p=4096$ and $\tau=10^{-5}$.

where $f(u)=u^3-u$ corresponds to the usual double-well potential, $\kappa^2>0$ is the diffusion coefficient. The scalar function $u: \mathbb{T}\to\mathbb{R}$ represents the concentration of a phase in an alloy and typically has values in the physical range $[-1,1]$. There is by now an extensive literature on the theoretical analysis and numerical simulation of the Allen-Cahn equation and related phase field models (cf. [1–5, 7, 8, 14]). For the system (1.1) we take the initial data u_0 as an odd function of x (here we tacitly "lift" the periodic function u to be defined on the whole real axis so that oddness can be defined in the usual way). For example, one can take $u_0(x)=\sin(x)$. Denote

$$u_\infty(x)=\lim_{t\to\infty} u(x,t). \tag{1.2}$$

Since u_0 is odd and smooth, it is clear that the odd symmetry should be preserved for all time. In particular, the final state u_∞ must be a periodic odd function of x. However to our surprise, standard numerical experiments show that this parity property may be lost in not very long time simulations. For example, by using the finite difference method with $N_p=128$ nodes for space discretization and the first-order implicit-explicit method with time step $\tau=0.01$ for time discretization, we find that u tends to ±1 when t is suitably large (see the left-hand side of Fig. 1). Obviously, ±1 are not the correct steady states as the oddness is not preserved.

To check the fidelity of the numerical scheme and rule out the issues connected with inaccurate numerical discretization, we first test larger $N=4096$ and smaller $\tau=10^{-5}$. It turns out that for this case, the oddness is preserved up to around $t\approx32$, after which the solution u lost its oddness apparently and tends to 1 quickly. See the right-hand side of Fig. 1.