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Numerical Research for the 2D Vorticity-Stream Function Formulation of the Navier-Stokes Equations and its application in Vortex Merging at High Reynolds Numbers

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Abstract. This paper is concerned with the development of numerical research for the 2D vorticity-stream function formulation and its application in vortex merging at high Reynolds numbers. A novel numerical method for solving the vorticity-stream function formulation of the Navier-Stokes equations at high Reynolds number is presented. We implement the second-order linear scheme by combining the finite difference method and finite volume method with the help of careful treatment of nonlinear terms and splitting techniques, on a staggered-mesh grid system, which typically consists of two steps: prediction and correction. We show in a rigorous fashion that the scheme is uniquely solvable at each time step. A verification algorithm that has the analytical solution is designed to demonstrate the feasibility and effectiveness of the proposed scheme. Furthermore, the proposed scheme is applied to study the vortex merging problem. Ample numerical experiments are performed to show some essential features of the merging of multiple vortices at high Reynolds numbers. Meanwhile, considering the importance of the inversion for the initial position of the vorticity field, we present an iteration algorithm for the reconstruction of the initial position parameters.

AMS subject classifications: 35R30, 65M12, 76D05

Key words: Navier-Stokes equation, inverse problem, stability, convergence, finite difference method, finite volume method, high Reynolds numbers.

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1 Introduction

The Navier-Stokes (N-S) equations that are often used to describe the complexity and variety of fluid dynamical phenomena. Mathematicians and physicists always focus on the solutions of the N-S equations and try to understand the dynamical phenomena such as turbulence, vortex, etc. The classical primitive variables formulation of the incompressible N-S equations on a domain Ω are given by

$$\rho(\boldsymbol{u}_t+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u})+\nabla p=\nu\Delta\boldsymbol{u},\quad \boldsymbol{x}\in\Omega,\quad t\in(0,T], \tag{1.1}$$

and the continuity equation

$$\nabla\cdot\boldsymbol{u}=0,\quad \boldsymbol{x}\in\Omega,\quad t\in[0,T], \tag{1.2}$$

where $\boldsymbol{u}(\boldsymbol{x},t)$ is the fluid velocity, $p(\boldsymbol{x},t)$ is the pressure, $\nu>0$ is the kinematic viscosity of the fluid and $\rho>0$ is the density of the fluid that is constant. Usually we set $\rho=1$. Alternatively, Equation (1.1) (with $\rho=1$) can be considered as the nondimensionalized form of the N-S equations. The Dirichlet boundary condition is imposed on the solid boundary $\partial\Omega$,

$$\boldsymbol{u}|_{\partial\Omega}=0,\quad t\in[0,T]. \tag{1.3}$$

The initial condition

$$\boldsymbol{u}|_{t=0}=\boldsymbol{u}_0,\quad \boldsymbol{x}\in\Omega. \tag{1.4}$$

There are some existence and uniqueness results for N-S equations, we refer the reader to [1] and references therein. In order to accurately predict the variety of fluid dynamical phenomena, the numerical solution of the Navier-Stokes equations is required, e.g., some viscosity-splitting schemes for the Navier-Stokes equations were considered in [2–4], the spectral methods were used to simulate the incompressible viscous flow (e.g., [5–8]), the finite difference (FDM) and finite element methods (FEM) have recently been used to solve the complex flow fields with multi-scale structures (e.g., [9–13]). The finite volume method (FVM) holds a significant place in the computational fluid dynamics community for the intrinsic conservation properties (see, e.g., [14]). The relationship between compact schemes and FVM were presented in [15–18]. However, there are still lots of challenging problems that need to be studied, eg., the fundamental aspects related to the interaction of co-rotating vortices and the phenomenon of vortex merging. Vortex merging is an ingredient of fluid motion and plays a major role in a variety of fields such as geophysics, meteorology, decaying two-dimensional turbulence, mixing layers, and aircraft trailing wake, to name a few. Numerical methods that are used to simulate incompressible viscous flows can be classified into three major categories, depending on the choice of dependent variables, namely velocity-pressure formulation (e.g., [19–22]), stream function vector or vorticity-vector-potential formulation (e.g., [23–25]) and velocity-vorticity formulation (e.g., [26–30]). Now, most of the numerical simulations have been confined to low or moderate Reynolds numbers. The grid resolution is one of the difficulties which