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Boundedness and Asymptotic Behavior in a 3D Keller-Segel-Stokes System Modeling Coral Fertilization with Nonlinear Diffusion and Rotation

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Abstract. This paper deals with the four-component Keller-Segel-Stokes model of coral fertilization

$$\begin{cases} n_t+u\cdot\nabla n=\Delta n^m-\nabla\cdot(nS(x,n,c)\cdot\nabla c)-n\rho, \\ c_t+u\cdot\nabla c=\Delta c-c+\rho, \\ \rho_t+u\cdot\nabla\rho=\Delta\rho-n\rho, \\ u_t+\nabla P=\Delta u+(n+\rho)\nabla\phi, \quad \nabla\cdot u=0 \end{cases}$$

in a bounded and smooth domain $\Omega\subset\mathbb{R}^3$ with zero-flux boundary for n, c, ρ and no-slip boundary for u, where $m>0$, $\phi\in W^{2,\infty}(\Omega)$, and $S:\bar{\Omega}\times[0,\infty)^2\to\mathbb{R}^{3\times 3}$ is given sufficiently smooth function such that $|S(x,n,c)|\le S_0(c)(n+1)^{-\alpha}$ for all $(x,n,c)\in\bar{\Omega}\times[0,\infty)^2$ with $\alpha\ge 0$ and some nondecreasing function $S_0:[0,\infty)\mapsto[0,\infty)$. It is shown that if $m>1-\alpha$ for $0\le\alpha\le\frac{2}{3}$, or $m\ge\frac{1}{3}$ for $\alpha>\frac{2}{3}$, then for any reasonably regular initial data, the corresponding initial-boundary value problem admits at least one globally bounded weak solution which stabilizes to the spatially homogeneous equilibrium $(n_\infty,\rho_\infty,\rho_\infty,0)$ in an appropriate sense, where $n_\infty:=\frac{1}{|\Omega|}\left\{\int_\Omega n_0-\int_\Omega\rho_0\right\}_+$ and $\rho_\infty:=\frac{1}{|\Omega|}\left\{\int_\Omega\rho_0-\int_\Omega n_0\right\}_+$. These results improve and extend previously known ones.

AMS subject classifications: 35B40, 35K55, 35Q92, 35Q35, 92C17

Key words: Keller-Segel-Stokes, nonlinear diffusion, tensor-valued sensitivity, boundedness, asymptotic behavior.

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1 Introduction

In this paper, we consider the following four-component Keller-Segel-Stokes system with nonlinear diffusion and rotation

$$\begin{cases} n_t+u\cdot\nabla n=\Delta n^m-\nabla\cdot(nS(x,n,c)\cdot\nabla c)-n\rho, & x\in\Omega,\quad t>0,\\ c_t+u\cdot\nabla c=\Delta c-c+\rho, & x\in\Omega,\quad t>0,\\ \rho_t+u\cdot\nabla\rho=\Delta\rho-n\rho, & x\in\Omega,\quad t>0,\\ u_t+\nabla P=\Delta u+(n+\rho)\nabla\phi,\quad \nabla\cdot u=0, & x\in\Omega,\quad t>0,\\ (\nabla n^m-nS(x,n,c)\cdot\nabla c)\cdot\nu=\nabla c\cdot\nu=\nabla\rho\cdot\nu=0,\quad u=0, & x\in\partial\Omega,\quad t>0,\\ n(x,0)=n_0(x),\quad c(x,0)=c_0(x),\quad \rho(x,0)=\rho_0(x),\quad u(x,0)=u_0(x), & x\in\Omega \end{cases} \tag{1.1}$$

in a bounded physical domain $\Omega\subset\mathbb{R}^3$ with smooth boundary $\partial\Omega$, where ν denotes the outward normal vector on $\partial\Omega$, and ϕ symbolizes the gravitational potential. System (1.1) was recently proposed by Espejio and Winkler [9] to characterize the process of coral fertilization occurring in ocean flow. Specifically, the unfertilized sperm (with density $n=n(x,t)$) chemotactically moves toward the higher concentration of a chemical signal (with concentration $c=c(x,t)$) released by the unfertilized egg with density $\rho=\rho(x,t)$. The evolutions of the unfertilized sperm and egg density are influenced by random diffusion, fluid-driven transport and the degradation during the period of fertilization. The velocity field $u=u(x,t)$ of the ambient ocean flow is supposed to be governed by an incompressible Stokes equation with associated pressure P and buoyant force of sperm and egg $(n+\rho)\nabla\phi$, which is justified by the observation that the motion of fluid flow is relatively slow compared with the movements of the sperm and egg. We note that (1.1) involves possibly nonlinear diffusion measured by n^{m-1} (degenerate diffusion or porous medium diffusion if $m>1$, and singular diffusion or fast diffusion if $0<m<1$), and off-diagonal cross-diffusion mechanisms with the tensor-valued chemotactic sensitivity $S(x,n,c)=(S_{ij}(x,n,c))_{i,j\in\{1,2,3\}}$ [66,67].

To motivate our study, we shall mention some previous contributions in the chemotaxis system and chemotaxis-fluid system which are closely related to (1.1).

Keller-Segel model. In order to better characterize chemotaxis of cell populations, on the basis of volume-filling and quorum-sensing effect assumptions, Painter and Hillen [31] modified the classical Keller-Segel model proposed by Keller and Segel [15] to the following quasilinear chemotaxis system

$$\begin{cases} n_t=\nabla\cdot(D(n)\nabla n)-\nabla\cdot(S(n)\nabla c), & x\in\Omega,\quad t>0,\\ c_t=\Delta c-c+n, & x\in\Omega,\quad t>0 \end{cases} \tag{1.2}$$

in a bounded and smooth domain $\Omega\subset\mathbb{R}^N$ $(N\geq 1)$, where the unknown variables n and c represent the density of cells (or organisms) and the concentration of chemical signal (or chemoattractant), respectively; the functions $D(n)$ and $S(n)$ measure the diffusivity