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On the Generalized Calderón Formulas for Closed- and Open-Surface Elastic Scattering Problems

Liwei Xu[1] and Tao Yin[2,3,*]

[1] *School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu, Sichuan 611731, China.*
[2] *LSEC, Institute of Computational Mathematics and Scientific/Engineering Computing, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China.*
[3] *School of Mathematical Science, University of Chinese Academy of Sciences, Beijing 100049, China.*

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Abstract. The Calderón formulas have been recently utilized in the process of constructing valid boundary integral equation systems which possess highly favorable spectral properties. This work is devoted to studying the theoretical properties of elastodynamic Calderón formulas which provide us with a solid basis for the design of fast boundary integral equation methods solving elastic wave problems defined on a close- or open-surface in two dimensions. For the closed-surface case, it is proved that the Calderón formula is a Fredholm operator of second-kind except for certain circumstances. For the open-surface case, we investigate weighted integral operators instead of the original integral operators which are resulted from dealing with edge singularities of potentials corresponding to the elastic scattering problems by open-surfaces, and show that the Calderón formula is a compact perturbation of a bounded and invertible operator whose spectrum enjoys the same accumulation points as the Calderón formula in the closed-surface case.

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Key words: Elastic wave, open-surface, Calderón relation, Fredholm operator.

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1 Introduction

As one of the most fundamental numerical methods, the boundary integral equation (BIE) method [28] has been extensively developed for numerical solutions of partial differential equations problems with various structures including bounded closed-surface [5,

*Corresponding author. *Email addresses:* `xul@uestc.edu.cn` (L. Xu), `yintao@lsec.cc.ac.cn` (T. Yin)

9, 13], open screens [10–12, 31, 39], period or non-period infinite surface [14], and so on. The BIE method has a feature of discretization of domains with lower dimensionality, and it is also a feasible method for the numerics of high frequency scattering problems. For large-scale problems with high-frequencies or three-dimensional complicated geometries, such iterative algorithms [20] as the Krylov-subspace linear algebra solver GMRES, together with adequate acceleration techniques [6, 9, 32], are generally required for fast solving the resulting linear system whose coefficient matrix is dense. The efficiency of the GMRES iteration is highly related to the spectral features of the coefficient matrix of the linear system [38] and therefore, appropriate preconditioning, such as the analytical preconditioning based on the Calderón formulas [8, 15] and the algebraic preconditioning strategies [7], are usually employed. However, only a few theoretical properties of the Calderón formulas (also called the Calderón relation in this work) for acoustic/electromagnetic closed-surface problems [8, 15], two-dimensional acoustic open-surface problems [10,31] and elastic closed-surface problems with the standard traction operator [12, 14], have been studied in open literatures. We also refer to [24–27] and the references therein for the study of inverses of integral operators on disks and the corresponding preconditioning associated with boundary element Galerkin discretizations.

This work is devoted to studying the theoretical properties of the Calderón formulas related to the two-dimensional problems of elastic scattering by closed- or open-surfaces which have many significant applications in science and engineering [35, 40], including geophysics, non-destructive testing of solids materials, mining and energy production, etc. A fundamental purpose of utilizing the Calderón formulas is to construct BIEs, for example, the second-kind Fredholm integral equations, with the highly favorable spectral properties that the eigenvalues of the BIEs are bounded away from zero and infinity. One can refer to the methodologies discussed in [8] for the acoustic case and those in [16] for the electromagnetic case. Although for the acoustic and elastodynamic problems, the Calderón formulas are indeed the composition of the single-layer integral operator S and the hyper-singular integral operator N, the extension of the theoretical analysis on the Calderón formulas in acoustics to that on the elastodynamic cases, however, encounters additional challenges. More precisely, for the smooth closed-surface case, the acoustic Calderón formula reads $NS=-I/4+(D^*)^2$ where D^* represents the transpose of the double-layer boundary integral operator, and is compact in appropriate Sobolev spaces. This fact ensures that the acoustic Calderón formula is of the second-kind Fredholm type. However, the corresponding operator D^* in the elastic case is not compact, see for example [1, 2]. In addition, the highly singular character of the associated integral kernel in elastodynamic hyper-singular operator is much more complicated than that in the acoustic case.

For the closed-surface case, by applying the polynomial compactness of the statistic elastic Neumann-Poincaré double-layer operator D_0 and its transpose D_0^*, it has been shown in [12, 13] that the elastodynamic Calderón formula involving the standard traction operator (2.1) is exactly a second-kind Fredholm operator (see [42] for two dimensional poroelastic case) whose eigenvalues are bounded away from zero and infinity