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Unconditionally Maximum-Principle-Preserving Parametric Integrating Factor Two-Step Runge-Kutta Schemes for Parabolic Sine-Gordon Equations

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Abstract. We present a systematic two-step approach to derive temporal up to the eighth-order, unconditionally maximum-principle-preserving schemes for a semilinear parabolic sine-Gordon equation and its conservative modification. By introducing a stabilization term to an explicit integrating factor approach, and designing suitable approximations to the exponential functions, we propose a unified parametric two-step Runge-Kutta framework to conserve the linear invariant of the original system. To preserve the maximum principle unconditionally, we develop parametric integrating factor two-step Runge-Kutta schemes by enforcing the non-negativeness of the Butcher coefficients and non-decreasing constraint of the abscissas. The order conditions, linear stability, and convergence in the L^∞-norm are analyzed. Theoretical and numerical results demonstrate that the proposed framework, which is explicit and free of limiters, cut-off post-processing, or exponential effects, offers a concise, and effective approach to develop high-order inequality-preserving and linear-invariant-conserving algorithms.

AMS subject classifications: 65L06, 65M12, 35B50, 35K55

Key words: Parabolic sine-Gordon equation, linear-invariant-conserving, unconditionally maximum-principle-preserving, parametric two-step Runge-Kutta method.

1 Introduction

Many differential equations in fluid dynamics, physics, chemistry, biology, engineering, and material science are naturally equipped with inequality constraints on the solution

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components, such as strong stability [30], positivity [78], maximum principle [19,54], and contractivity [48] constraints. It is recognized that preserving these inequality structures is not only important for solutions to be physically meaningful but also relevant for the numerical stability of time integration methods. In the last three decades, the development of high-order accurate and efficient algorithms that can preserve such inequalities has been a serious research objective [10,36,39,40,55,66]. However, it still remains an outstanding open problem [3] to develop numerical methods that are both (i) of high-order accuracy and (ii) capable of preserving inequality structures for any time-step size. To the best of our knowledge, the high-order methods that can unconditionally guarantee these properties are very limited [4, 19, 28, 46]. This motivated us to pursue high-order-accurate and stable explicit schemes that can preserve the above inequality structures for any time step. In particular, we selected the preservation of the maximum principle for newly developed semilinear parabolic sine-Gordon equations as an example.

Recently, starting from the semilinear parabolic Allen-Cahn (AC) equation [1] and hyperbolic sine-Gordon equation [23], Cheng *et al.* [11] proposed and analyzed a parabolic sine-Gordon (pSG) equation. They rigorously analyzed the existence of a maximum principle, bounded steady states, a conditionally maximum-principle-preserving (MPP) first-order implicit-explicit (IMEX) scheme, and an energy-stable second-order backward differentiation formula (BDF2) scheme. The pSG equation is interesting for a number of reasons. First and foremost, Cheng *et al.* [11] demonstrated that the pSG model exhibits striking similarity with the classical AC equation (Ginzburg-Landau potential). Because of its very benign nonlinear structure, one can develop unconditionally MPP schemes. Thus, it is of purely mathematical interest as a suitable testbed for phase field simulations, and is expected to have a ubiquitous presence [11]. Moreover, from a physical perspective, the pSG equation (with a white noise term and suitable parameters) is closely related to models of a globally neutral gas of interacting charges [31], most directly it is the natural (Langevin) dynamics for the sine-Gordon (Euclidean) quantum filed theory [7]. Hairer and Shen [31] showed that the pSG equation with white noise also arises naturally from a class of equilibrium interface fluctuation models with periodic nonlinearities. In addition, it was also proposed as a model for the dynamics of crystal-vapour interfaces at the roughening transition [56], and has attracted much attention [32] in recent years. Therefore, the development of efficient and stable schemes for the pSG equation has practical significance. Consider the semilinear pSG equation [11]

$$\begin{cases} u_t = \epsilon^2 \Delta u + f(u), & x \in \Omega, \quad t \in (0,T], \\ u(x,0) = u^0(x), & x \in \bar{\Omega}, \end{cases} \tag{1.1}$$

where the unknown function u represents the difference between the concentrations of the two components, $\Omega \in \mathbb{R}^d$ is an open, connected, and bounded region with a Lipschitz boundary $\partial\Omega$, and periodic or homogeneous Neumann boundary conditions. The nonlinear function $f(u) = \sin(u)$ is the negative derivative of a cosine potential function $F(u) = \cos(u)$.