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Computation of Transmission Eigenvalues by the Regularized Schur Complement for the Boundary Integral Operators

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Abstract. This paper is devoted to the numerical computation of transmission eigenvalues arising in the inverse acoustic scattering theory. This problem is first reformulated as a two-by-two system of boundary integral equations. Next, we develop a Schur complement operator with regularization to obtain a reduced system of boundary integral equations. The Nyström discretization is then used to obtain an eigenvalue problem for a matrix. In conjunction with the recursive integral method, the numerical computation of the matrix eigenvalue problem produces the indicator for finding the transmission eigenvalues. Numerical implementations are presented and archetypal examples are provided to demonstrate the effectiveness of the proposed method.

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Key words: Transmission eigenvalues, inverse scattering, boundary integral equations, Nyström method, Schur complement, spectral projection.

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1 Introduction

We consider the calculation of the transmission eigenvalue problems, which play an important role in scattering theory for inhomogeneous media. Transmission eigenvalues are not only related to the validity of the linear sampling method [5], but also shed light on

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the material properties of the scattering object [3, 6]. The transmission eigenvalue problem (also known as the interior transmission eigenvalue problem) is a boundary value problem for a coupled pair of partial differential equations in a bounded domain. In fact, the analysis of the transmission eigenvalue problem cannot be covered by the standard theory of elliptic partial differential equations since it is neither elliptic nor self-adjoint. Hence, the relevant studies are widely perceived as challenging issues. Calculating the transmission eigenvalues requires special effort.

In general, concerns associated with the general theory of transmission eigenvalues are the solvability, discreteness and existence, and the spectral properties of the transmission eigenvalue problems [3, 6, 10, 25]. The mathematical methods for studying these problems include the variational methods [27] and boundary integral equation methods [9]. The variational problems are derived by formulating the transmission problems as a fourth-order partial differential equation. Under appropriate assumptions on the contrast in the media, the solvability and existence of the transmission eigenvalues could be obtained from the variational framework [3,7]. The discreteness of transmission eigenvalues can be proven using the analytical Fredholm theory. An alternative approach to study the transmission problems is the boundary integral method. A system of boundary integral equations equivalent to the interior transmission problem is arrived based on Green's representation formula [9]. One single boundary integral equation is proposed in [4] to characterize transmission eigenvalues in terms of Dirichlet-to-Neumann or Robin-to-Dirichlet operators. This results in a noticeable reduction of computational costs, since the system is much smaller than the former method.

As an entry to the vast literature on numerical computations of the transmission eigenvalues, several typical techniques for solving the transmission eigenvalues are the finite element [14, 29], boundary element methods [4, 9, 17, 22] and radial basis functions [18]. The finite element methods are developed based on a fourth-order reformulation of the transmission eigenvalues [14, 28], or a related quadratic eigenvalue problem [24]. Those lead to eigenvalue problems of sparse matrices. The boundary element methods [30] are proposed upon the boundary integral equations and lead to eigenvalue problems of dense matrices. The method of radial basis functions is a discrete equivalent of an integral operator realization generated by the fundamental solution to the Helmholtz equation. Note that the transmission eigenvalue problem is non-linear and non-selfadjoint. Hence, the numerical discretization usually leads to a non-Hermitian and nonlinear matrix eigenvalue problem, which is very challenging in numerical linear algebra. Efficient methods for computing those matrix eigenvalues include the Arnoldi method [23], Jacobi-Davidson method [1], and the integral based methods [2, 12, 13]. Recently, the integral equation based methods become popular in many areas. The classical spectral perturbation theory [15] provides the theoretical basis for integral based methods. An approximation to the eigenvalue in a given simple closed curve in the complex plane is found by spectral projection using counter integral of the resolvent [13, 30].

In this paper, we aim to develop a novel integral equation formulation built upon the Schur complement to a two by two system of boundary integral equations. If one