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# Determination of Source Terms in Diffusion and Wave Equations by Observations After Incidents: Uniqueness and Stability

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**Abstract.** We consider a diffusion and a wave equations

$$\partial_t^k u(x,t) = \Delta u(x,t) + \mu(t) f(x), \quad x \in \Omega, \quad t > 0, \quad k = 1,2$$

with the zero initial and boundary conditions, where $\Omega \subset \mathbb{R}^d$ is a bounded domain. We establish uniqueness and/or stability results for inverse problems of determining $\mu(t)$, $0 < t < T$ with given $f(x)$, determining $f(x)$, $x \in \Omega$ with given $\mu(t)$, by data of $u$: $u(x_0,\cdot)$ with fixed point $x_0 \in \Omega$ or Neumann data on subboundary over time interval. In our inverse problems, data are taken over time interval $T_1 < t < T_2$, by assuming that $T < T_1 < T_2$ and $\mu(t) = 0$ for $t \geq T$, which means that the source stops to be active after the time $T$ and the observations are started only after $T$. This assumption is practical by such a posteriori data after incidents, although inverse problems had been well studied in the case of $T = 0$. We establish the non-uniqueness, the uniqueness and conditional stability for a diffusion and a wave equations. The proofs are based on eigenfunction expansions of the solutions $u(x,t)$, and we rely on various knowledge of the generalized Weierstrass theorem on polynomial approximation, almost periodic functions, Carleman estimate, non-harmonic Fourier series.

**AMS subject classifications**: 35R30, 35R25, 35K20, 35L20

**Key words**: Inverse source problem, data after incidents, diffusion equation, wave equation.

## 1 Introduction

In this article, we consider initial-boundary value problems for a diffusion and a wave equations

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$$\begin{cases} \partial_t u(x,t) = \Delta u(x,t) + \mu(t) f(x), & x \in \Omega, \quad t > 0, \\ u(x,0) = 0, & x \in \Omega, \\ u(x,t) = 0, & x \in \partial\Omega, \quad t > 0, \end{cases} \tag{1.1}$$

$$\begin{cases} \partial_t^2 u(x,t) = \Delta u(x,t) + \mu(t) f(x), & x \in \Omega, \quad t > 0, \\ u(x,0) = \partial_t u(x,0) = 0, & x \in \Omega, \\ u(x,t) = 0, & x \in \partial\Omega, \quad t > 0. \end{cases} \tag{1.2}$$

Here and henceforth $\Omega \subset \mathbb{R}^d$ is a bounded domain with smooth boundary $\partial\Omega$ and we set

$$x = (x_1, \ldots, x_d) \in \mathbb{R}^d, \quad \partial_j = \frac{\partial}{\partial x_j}, \quad \Delta = \sum_{j=1}^{d} \partial_j^2.$$

Let $\nu = \nu(x)$ be the unit outward normal vector to $\partial\Omega$ and let $\partial_\nu u = \nabla u \cdot \nu$. We mainly consider the zero Dirichlet boundary condition and can treat the Neumann boundary condition similarly but we omit the details. Moreover, we can consider the inverse problems for (1.1) and (1.2) where $\Delta$ is replaced by a suitable elliptic operator with time independent coefficients, but for simplicity, we mainly argue for $\Delta$.

The source is assumed to be represented in the form of $\mu(t)f(x)$ where $\mu(t)$ and $f(x)$ describe changes in the time $t$ and the spacial variable $x$ respectively. Such a form of separation of variables is frequently used in modelling diffusion and wave phenomena.

The unique existence of solutions to (1.1) and (1.2) are standard results (e.g., Evans [10], Lions and Magenes [17], Pazy [18]), but we need more regularity of solutions. We sum up these results as Lemmas 1.1 and 1.2. We arbitrarily fix $T_0 > 0$.

**Lemma 1.1.** *Let $f \in C_0^\infty(\Omega)$ and $\mu \in H^1(0,T_0)$.*

*(i) To* (1.1)*, there exists a unique solution*

$$u \in C\big([0,T_0]; H^2(\Omega) \cap H_0^1(\Omega)\big) \cap C^1\big([0,T_0]; L^2(\Omega)\big),$$

*and we can choose a constant $C > 0$, dependent on $f$, such that*

$$\|u\|_{C(\Omega \times [0,T_0])} \le C \|\mu\|_{L^2(0,T_0)}. \tag{1.3}$$

*(ii) To* (1.2)*, there exists a unique solution*

$$u \in C\big([0,T_0]; H^2(\Omega)\big) \cap C^1\big([0,T_0]; H_0^1(\Omega)\big) \cap C^2\big([0,T_0]; L^2(\Omega)\big) \cap C\big(\overline{\Omega} \times [0,T_0]\big)$$

*such that $\partial_\nu u \in H^1(0,T_0; L^2(\partial\Omega))$ and* (1.3) *holds.*

**Lemma 1.2.** *Let $f \in L^2(\Omega)$ and $\mu \in C^1[0,T_0]$.*

*(i) To* (1.1)*, there exists a unique solution*

$$u \in H^1\big(0,T_0; L^2(\Omega)\big) \cap L^2\big(0,T_0; H_0^1(\Omega)\big).$$