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Robust Convergence of Parareal Algorithms with Arbitrarily High-Order Fine Propagators

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Abstract. The aim of this paper is to analyze the robust convergence of a class of parareal algorithms for solving parabolic problems. The coarse propagator is fixed to the backward Euler method and the fine propagator is a high-order single step integrator. Under some conditions on the fine propagator, we show that there exists some critical J_* such that the parareal solver converges linearly with a convergence rate near 0.3, provided that the ratio between the coarse time step and fine time step named J satisfies $J \geq J_*$. The convergence is robust even if the problem data is nonsmooth and incompatible with boundary conditions. The qualified methods include all absolutely stable single step methods, whose stability function satisfies $|r(-\infty)|<1$, and hence the fine propagator could be arbitrarily high-order. Moreover, we examine some popular high-order single step methods, e.g., two-, three- and four-stage Lobatto IIIC methods, and verify that the corresponding parareal algorithms converge linearly with a factor 0.31 and the threshold for these cases is $J_* = 2$. Intensive numerical examples are presented to support and complete our theoretical predictions.

AMS subject classifications: 65M12, 65M60, 65L06

Key words: Parareal algorithm, parabolic problems, arbitrarily high-order, single step integrator, convergence factor.

1 Introduction

The main focus of this paper is to study the convergence of a class of parareal solver for the parabolic problems. Specifically, we let $T > 0$, $u^0 \in H$, and consider the initial value

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problem of seeking $u\in C((0,T];D(A))\cap C([0,T];H)$ satisfying

$$\begin{cases} u'(t)+Au(t)=f(t), & 0<t<T, \\ u(0)=u^0, \end{cases} \tag{1.1}$$

where A is a positive definite, selfadjoint, linear operator with a compact inverse, defined in Hilbert space $(H,(\cdot,\cdot))$ with domain $D(A)$ dense in H. Here $u^0\in H$ is a given initial condition and $f:[0,T]\to H$ is a given forcing term. Throughout the paper, $\|\cdot\|$ denotes the norm of the space H.

Parallel-in-time (PinT) methods, dating back to the work of Nievergelt in 1964 [25], have attracted a lot of interest in the last several decades. The parareal method, introduced in 2001 [22], is perhaps one of the most popular PinT algorithms. This method is relatively simple to implement, and can be employed for any single step integrators. In recent years, the parareal algorithm and some relevant algorithms, have been applied in many fields, such as turbulent plasma [27,28], structural (fluid) dynamics [9,13], molecular dynamics [4], optimal control [23,24], Volterra integral equations and fractional models [21, 37], etc. We refer the interested reader to survey papers [15, 26] and references therein.

The parareal algorithm is defined by using two time propagators, $\mathcal{G}$ and $\mathcal{F}$, associated with the large step size ΔT and the small step size Δt respectively, where we assume that the ratio $J=\Delta T/\Delta t$ is an integer greater than 1. The fine time propagator $\mathcal{F}$ is operated with small step size Δt in each coarse sub-interval parallelly, after which the coarse time propagator $\mathcal{G}$ is operated with large step size ΔT sequentially for corrections. In general, the coarse propagator $\mathcal{G}$ is assumed to be much cheaper than the fine propagator $\mathcal{F}$. Therefore, throughout this paper, we fix $\mathcal{G}$ to the backward-Euler method and study the choices of $\mathcal{F}$. Then a natural question arises related to convergence of the parareal algorithm. For parabolic type problems, in the pioneer work [5], Bal proved a fast convergence of the parareal method with a strongly stable coarse propagator and the exact fine propagator, provided some regularity assumptions on the problem data. The analysis works for both linear and nonlinear problems. This convergence behavior is clearly observed in numerical experiments, see e.g. Fig. 2. However, without those regularity assumptions, the convergence observed from the empirical experiments will be much slower than expected, cf. Fig. 3. See also some rigorous analysis in [10, 14, 31].

This interesting phenomenon motivates the current work, where we aim to study the convergence of parareal algorithm which is expected to be robust in the case of nonsmooth/incompatible problem data, that is related to various applications, e.g., optimal control, inverse problems, and stochastic models. There have existed some case studies. Mathew *et al.* [24] considered the backward Euler method as the fine propagator and proved the robust convergence of the parareal algorithm with a convergence factor 0.298 (for all $J\geq 2$); see also [16, 32] for some related discussion. Wu [35] showed that the convergence factors for the second-order diagonal implicit Runge-Kutta method and a single step TR/BDF2 method (i.e., the `ode23tb` solver for ODEs in MATLAB) are 0.316