CSIAM Trans. Appl. Math.
doi: 10.4208/csiam-am.SO-2022-0047

Vol. **4**, No. 3, pp. 619-652
September 2023

Hopf Bifurcation and Its Normal Form of Reaction Diffusion Systems Defined on Directed Networks

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Received 26 November 2022; Accepted 3 March 2023

Abstract. Compared with the real Laplacian eigenvalues of undirected networks, the ones of asymmetrical directed networks might be complex, which is able to trigger additional collective dynamics, including the oscillatory behaviors. However, the high dimensionality of the reaction-diffusion systems defined on directed networks makes it difficult to do in-depth dynamic analysis. In this paper, we strictly derive the Hopf normal form of the general two-species reaction-diffusion systems defined on directed networks, with revealing some noteworthy differences in the derivation process from the corresponding on undirected networks. Applying the obtained theoretical framework, we conduct a rigorous Hopf bifurcation analysis for an SI reaction-diffusion system defined on directed networks, where numerical simulations are well consistent with theoretical analysis. Undoubtedly, our work will provide an important way to study the oscillations in directed networks.

AMS subject classifications: 34C23, 37G15, 92D30

Key words: Directed network, reaction-diffusion system, Hopf bifurcation, normal form, SI epidemic system.

1 Introduction

As early as 1952, Turing [26] found that the reaction-diffusion system can be used to describe the differentiation and the formation of spatial patterns in biological systems. He

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laid down the mathematical basis for analysing the diffusion driven instability of the homogeneous equilibria in one reaction-diffusion system. This celebrated Turing instability has been as a universally accepted paradigm to understand the self-organizing patterns in physic, chemistry, biology, ecology and so on [15, 18, 23].

In the 1970s, Othmer and Scriven [21, 22] extended the Turing theory of pattern formation in continuous space with putting forward a general mathematical framework for the Turing instability in reaction-diffusion systems defined on several discrete lattices. During the following decades, however, the early researches are limited to regular lattices or small-scale networks [12, 17, 25]. Until 2010, Nakao and Mikhailov [19] studied Turing patterns on large-scale random complex networks in detail. They revealed striking differences from the classical behaviours in macroscopic segregation of activator-rich and activator-poor nodes. Indeed, this landmark work has virtually opened a new era for more extensive research of self-organizations in networks.

In the past decade, there appeared a large number of achievements which has broadened our understanding of patterns in various networks. For example, Mimar *et al.* [16] reported that the Turning instability in undirected (symmetry) networks can be induced by mediating network topology such as network connectivity, and Guo *et al.* [10] presented an exponential decay of Turning patterns on undirected networks, providing a way to quantitatively characterize the influence of network topology on pattern formations. Generally, the topology of undirected networks might determine the relevant directions for the spreading of the perturbation, but the onset of Turning instability will emerge under the very same conditions that apply when the system is defined on a continuous support. For multiplex networks, the interlayer diffusion could seed the instability of a homogeneous fixed point, yielding self-organized patterns which were instead impeded in the limit of decoupled layers [1]. It was also founded that inserting just one additional link to differentiate single-layer networks could induce or destroy the Turing pattern [2]. Kouvaris *et al.* [13] found that diffusion-induced instability could occur even if the mobility rates of two species were equal, which could never happen in single-layer networks under this condition. Besides, the pattern formations in the time varying networks also has been studied by some researchers [24, 27].

Particularly, the self-regulating patterning in directed network yet received special attention. Compared with undirected networks, the Laplacian matrix of directed networks is asymmetric, and its eigenvalues could be complex, which undoubtedly complicates the dynamic behavior of directed network reaction-diffusion systems. Asllani *et al.* [4] for the first time showed the oscillatory Turing instability producing traveling or standing waves in the systems defined on directed networks with even just two species. This result was considered impossible in systems defined on undirected networks with only two species, as it is previously believed that at least three species are needed for the oscillatory instability [11]. While they further considered the general framework to single out the conditions for pattern formation with asymmetric diffusion [3], but the characterization of the subsequent nonlinear evolution solely relies on numerical methods. Therefore, Contemori and Di Patti and their colleagues [6, 7] exploited a multiple time-scale analy-