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A Multi-Frequency Sampling Method for the Inverse Source Problems with Sparse Measurements

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Abstract. We consider the inverse source problems with multi-frequency sparse near field measurements. In contrast to the existing near field operator based on the integral over the space variable, a multi-frequency near field operator is introduced based on the integral over the frequency variable. A factorization of this multi-frequency near field operator is further given and analyzed. Based on such a factorization, we introduce a single-receiver multi-frequency sampling method to reconstruct a shell support of the source. Its theoretical foundation is derived from the properties of the factorized operators and a properly chosen point spread function. Numerical examples are provided to illustrate the multi-frequency sampling method with sparse near field measurements. Finally we briefly discuss how to extend the near field case to the far field case.

AMS subject classifications: 35P25, 45Q05, 78A46, 74B05

Key words: Sampling method, multi-frequency, sparse data, inverse source problems.

1 Introduction

Inverse source problems merit numerous applications in medical imaging, geophysics, non-destructive testing, and many others. A survey on the state of the art of the mathematical theory and numerical approaches can be found in the monograph [22] and the review paper [6]. In the full aperture case when the multi-frequency measurements are available all around the unknown source, the uniqueness of the source with compact support is a direct consequence of the Fourier theory, see for instance [7, 18]. Several numerical methods were proposed for reconstructing the source, see e.g., an iterative method [9] and Fourier method [38,39]. We also refer to [11,40] on direct sampling methods for imaging point sources. However, in applications where measurements are only

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available at a limited number of sparse receivers, we are led to the inverse source problem with sparse multi-frequency measurements. Generally, neither the exact source nor its exact support is uniquely determined by the sparse data. Some lower bounds for the support of an extended source was established in [37] in terms of multi-frequency sparse far field measurements. For extended sources we refer to [20] and [1] for a factorization method and a direct sampling method, respectively, using multi-frequency sparse far field measurements; in these work they show that a strip support of the source can be uniquely determined by multi-frequency measurements at a single observation direction and provide sampling methods to image such a strip. For the multi-frequency sparse near field measurements, the corresponding uniqueness and direct sampling method can be found in a recent work [27]. For point sources, the uniqueness on the locations as well as the source strengths was established in [24, 26, 27] using multi-frequency sparse far field or near field measurements. In particular, the smallest number of sensors needed was given in terms of the number of point sources. We also refer to [34] regarding a study on multi-frequency sampling methods in waveguides.

Sampling methods have attached a lot of attentions in the last 30 years. Classical sampling methods such as the linear sampling method [14] and the factorization method [28] are independent of certain a priori information on the geometry and physical properties of the unknown scatterers. Based on a factorization of the far field or the near field operator, one may derive a criteria to reconstruct the unknown object and design an indicator function which is large inside the underlying object and relatively small outside. We refer to the monographs [12, 15, 29] for a comprehensive introduction. There have been recent efforts on other types of sampling methods such as orthogonality sampling method [19, 21, 35], direct sampling method [4, 5, 23, 25, 31], single-shot method [30], reverse time migration [13], and other direct reconstruction methods [2,3]. These sampling methods inherit many advantages of the classical ones. The main feature of these sampling methods is that only inner product of the measurements with some suitably chosen functions is involved in the imaging function and thus these sampling methods seem very robust to noises.

In two recent papers [1,27], under certain assumption, the shell support of the source can be uniquely determined by multi-frequency measurement at a given receiver. However, the theoretical basis is far less developed. In particular, the indicator function given in [27] is merely based on an observation that the scattered field due to an extend source can be considered as the one from a superposition of monopoles. This work contributes to a novel multi-frequency sampling method. Different from [1,27], in this work we introduce a multi-frequency near field operator $\mathcal{N}_x$ based on the integral over the frequency range K at a single receiver x and study its factorization

$$\mathcal{N}_x = \mathcal{P}_x \mathcal{T}_x \mathcal{P}_x^*.$$

Based on the coercivity of the middle operator, we show that for any sampling point z,

$$C_1 \left\| \mathcal{P}_x^* g_{xz} \right\|^2_{L^2(D)} \le \left| (\mathcal{N}_x g_{xz}, g_{xz})_{L^2(K)} \right| \le C_2 \left\| \mathcal{P}_x^* g_{xz} \right\|^2_{L^2(D)}$$