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Hearing the Triangles: A Numerical Perspective

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Abstract. We introduce a two-step numerical scheme for reconstructing the shape of a triangle by its Dirichlet spectrum. With the help of the asymptotic behavior of the heat trace, the first step is to determine the area, the perimeter, and the sum of the reciprocals of the angles of the triangle. The shape is then reconstructed, in the second step, by an application of the Newton's iterative method or the Levenberg-Marquardt algorithm for solving a nonlinear system of equations on the angles. Numerically, we have used only finitely many eigenvalues to reconstruct the triangles. To our best knowledge, this is the first numerical simulation for the classical inverse spectrum problem in the plane. In addition, we give a counter example to show that, even if we have infinitely many eigenvalues, the shape of a quadrilateral may not be heard.

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1 Introduction

Since the landmark paper by Marc Kac in 1966 [15], the question "Can one hear the shape of a drum?" has attracted and inspired many mathematicians. This forms the subject of the mathematical discipline called spectral geometry.

More exactly, for a bounded simply connected domain $D\subset\mathbb{R}^2$, the vibration of a drum (membrane) which spans D, is governed by the wave equation

$$v_{tt}-\Delta v=0,$$

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where $v=v(x,t)$ denotes the displacement in some direction of a point $x\in D$ at time $t>0$ and the Laplacian Δ is taken with respect to the spatial variables $x=(x_1,x_2)$. Of particular interest are the time harmonic solutions in the form

$$v=e^{i\omega t}u(x),$$

where the spatial part u solves the stationary equation

$$\Delta u+\omega^2 u=0 \quad \text{in } D \tag{1.1}$$

with the Dirichlet boundary condition

$$u=0 \quad \text{on } \partial D, \tag{1.2}$$

corresponding to the drum being fixed along its boundary.

We call $\lambda:=\omega^2>0$ a Dirichlet eigenvalue of D if there is a nontrivial solution $u\neq 0$ of (1.1)-(1.2). For a fixed domain D, it is well known that, if we repeat each eigenvalue according to its (finite) multiplicity, we have

$$0<\lambda_1<\lambda_2\leq\lambda_3\leq\cdots$$

and

$$\lambda_n \to \infty \quad \text{as} \quad n \to \infty.$$

The distribution of the eigenvalues has been thoroughly investigated and a considerable amount of information is available. For survey articles on this subject we refer to [11,16].

Conversely, we are interested in the inverse problem: Is the domain D determined uniquely by its Dirichlet eigenvalues. In 1992, Gordon *et al.* [10] had constructed two concave polygons which are isospectral, but not congruent. However, it is still open for a general convex polygon or a domain with smooth boundary. We refer to [6,23] for the uniqueness results if the boundary is analytic with some symmetry. In a recent work, the uniqueness has also been established for ellipses of small eccentricity [14].

This work focuses on a numerical scheme for triangles. There have been some papers giving numerical illustrations on various conjectures on spectral geometry as seen, e.g. [1–3]. However, to our best knowledge, this is the first numerical algorithm for reconstructing the shape of a domain from its spectrum. We would like to remark that although the triangle is very specific and simple, it has recently become apparent that triangles do play an important role in both the shape optimization problems [2,17] and the spectral properties related to isoperimetric inequalities [1,21]. The uniqueness is first proved by Durso [7]. It is conjectured by Laugesen and Siudeja [17] that the first three eigenvalues are enough to determine the shape of a triangle. However, this is still open up to now. We refer to [3,5,9] for some recent study by knowing just finitely many eigenvalues. Our algorithm is motivated by a recent and simpler proof by Grieser and Maronna [12]. The proof is divided into two steps: