CSIAM Trans. Appl. Math.
doi: 10.4208/csiam-am.SO-2023-0022

Vol. **5**, No. 2, pp. 221-233
June 2024

On the Optimal Order Approximation of the Partition of Unity Finite Element Method

Yunqing Huang[1] and Shangyou Zhang[2,*]

[1] *Hunan Key Laboratory for Computation and Simulation in Science and Engineering, Xiangtan University, Hunan 411105, China.*
[2] *Department of Mathematical Sciences, University of Delaware, Newark, DE 19716, USA.*

Received 17 June 2022 ; Accepted 11 December 2023

Abstract. In the partition of unity finite element method, the nodal basis of the standard linear Lagrange finite element is multiplied by the P_k polynomial basis to form a local basis of an extended finite element space. Such a space contains the P_1 Lagrange element space, but is a proper subspace of the P_{k+1} Lagrange element space on triangular or tetrahedral grids. It is believed that the approximation order of this extended finite element is k, in H^1-norm, as it was proved in the first paper on the partition of unity, by Babuska and Melenk. In this work we show surprisingly the approximation order is $k+1$ in H^1-norm. In addition, we extend the method to rectangular/cuboid grids and give a proof to this sharp convergence order. Numerical verification is done with various partition of unity finite elements, on triangular, tetrahedral, and quadri-lateral grids.

AMS subject classifications: 65N15, 65N30

Key words: Finite element, partition of unity, triangular grid, tetrahedral grid, rectangular grid.

1 Introduction

The partition of unity finite element was proposed in 1996 [10]. The method is based on the P_1 Lagrange finite element

$$u_h(\mathbf{x}) = \sum_{\mathbf{v}_i \in \mathcal{V}_h} u_i \phi_i(\mathbf{x}), \tag{1.1}$$

where u_i is the nodal value of a continuous function u_h at a vertex, $u_h(\mathbf{v}_i), \mathcal{V}_h$ is the index set of vertices in a triangulation $\mathcal{T}_h$, and ϕ_i is a piecewise P_1 function on the grid $\mathcal{T}_h$ assuming value 1 at one vertex $\mathbf{v}_i$ and zero at the rest vertices. Instead of multiplied by the P_0

*Corresponding author. *Email addresses:* huangyq@xtu.edu.cn (Y. Huang), szhang@udel.edu (S. Zhang)

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polynomial in (1.1), in one partition of unity method each nodal basis $\phi_i(\mathbf{x})$ is multiplied by the P_k polynomial basis, cf. [1, 8, 10],

$$u_h(\mathbf{x})=\sum_{\mathbf{v}_i\in\mathcal{V}_h} u_i(\mathbf{x})\phi_i(\mathbf{x}),\quad u_i(\mathbf{x})=\sum_{|\alpha|\le k} u_{i,\alpha}(\mathbf{x}-\mathbf{v}_i)^\alpha, \tag{1.2}$$

where α is a multi-index, e.g. when $k=2$ in 2D, $\mathbf{x}^\alpha\in\{1,x,y,x^2,xy,y^2\}$.

Obviously, the extended finite element space contains the P_1 Lagrange finite element as a subspace, by letting $u_i(\mathbf{x})\in\mathbb{R}$ in (1.2). On the other side, because the sum of three P_1 basis functions at the three vertices of a triangle K is a constant function 1, the extended finite element space also contains the $P_k(K)$ Lagrange element space as a subspace locally, on this triangle K only. But globally, on a triangular grid in 2D, the dimension of $P_1\times P_k$ finite element space (P_1 Lagrange basis multiplied by P_k polynomials) is $C(k+1)(k+2)/2\sim Ck^2/2$ while that of P_k Lagrange finite element space is Ck^2, by the Euler formula, where C is about the number of vertices. For large k, C^0-$(P_1\times P_k)\not\supset C^0$-$P_k$. Nevertheless, the first partition of unity paper [10] proved an $\mathcal{O}(h^k)$ H^1-convergence and an $\mathcal{O}(h^{k+1})$ L^2-convergence for this partition of unity finite element method. This is not trivial, to prove a smaller space having the same order of approximation.

We may compare the $P_1\times P_k$ finite element space with the P_{k+1} Lagrange element space. Each extended finite element function u_h is a $p_1\times p_k=p_{k+1}$ polynomial, on each element. The partition of unity finite element space is clearly a subspace of the P_{k+1} Lagrange space. In 1D, because the number of elements is the same as the number of vertices (one less), from a dimension counting, the extended finite element space is precisely the P_{k+1} Lagrange space in 1D. In [7] proved an one-order higher convergence than that of [10] in 1D.

But the problem is less trivial in 2D and 3D, and remains open for twenty some years. For example, for the P_2 triangular element in 2D, the finite element dimension is the sum of the number of vertices and the number of edges. For the $P_1\times P_1$ partition of unity finite element, the space dimension is 3 times the number of vertices. By the Euler formula, the number of edges is about three times of the number of vertices. The dimension of the $P_1\times P_1$ space is about 3/4 of that of the P_2 Lagrange space. Similarly, the dimensions of $P_1\times P_k$ partition of unity finite element space and P_{k+1} Lagrange finite element space are about the number of vertices times $(k+1)(k+2)/2$ and $(k+1)^2$, respectively, on 2D triangular grids. For large k, the former is about half of the latter. On 3D tetrahedral grids, the dimensions of $P_1\times P_k$ partition of unity finite element space is about the number of vertices times $(k+1)(k+2)(k+3)/6$ while that of P_{k+1} Lagrange finite element space is about the number of vertices times $(k+1)^3$. The ratio is about 1/6 for large k. These ratios become even smaller for the $Q_1\times P_k$ partition of unity finite element space and the Q_{k+1} Lagrange finite element space, on 2D and 3D rectangular grids. We also extend this method to rectangular/cuboid grids in this paper.

Though the $P_1\times P_k$ partition of unity finite element space is a proper subspace of the P_{k+1} Lagrange finite element space, we prove both have the same order of convergence in this paper. That is, we show that the $P_1\times P_k$ partition of unity finite element