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A Receptance-Based Optimization Approach for Minimum Norm and Robust Partial Quadratic Eigenvalue Assignment

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Abstract. This paper is concerned with finding a minimum norm and robust solution to the partial quadratic eigenvalue assignment problem for vibrating structures by active feedback control. We present a receptance-based optimization approach for solving this problem. We provide a new cost function to measure the robustness and the feedback norms simultaneously, where the robustness is measured by the unitarity or orthogonalization of the closed-loop eigenvector matrix. Based on the measured receptances, the system matrices and a few undesired open-loop eigenvalues and associated eigenvectors, we derive the explicit gradient expression of the cost function. Finally, we report some numerical results to show the effectiveness of our method.

AMS subject classifications: 65F18, 93B55, 93C15

Key words: Partial quadratic eigenvalue assignment, robustness, optimization method, receptance measurements.

1 Introduction

The active vibration control is often needed in many vibrating structures in structural engineering, including structural dynamics [14, 16, 30], earthquake engineering control [13], damped-gyroscopic system control [17], large flexible space structure control theory [6, 7, 19, 20], control of mechanical descriptor systems [21]. In practice, by using the

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finite element technique and feedback vibration control, a vibrating structure is often discretized as a second-order feedback control system as follows [16]:

$$M\ddot{\mathbf{x}}(t)+C\dot{\mathbf{x}}(t)+K\mathbf{x}(t)=B\mathbf{u}(t), \quad \mathbf{u}(t)=F^T\dot{\mathbf{x}}(t)+G^T\mathbf{x}(t), \tag{1.1}$$

where $M,C,K\in\mathbb{R}^{n\times n}$ stand for the mass, damping and stiffness matrices accordingly, t means time, $\mathbf{x}(t),\dot{\mathbf{x}}(t),\ddot{\mathbf{x}}(t)$ denote the displacement, velocity and acceleration vectors accordingly, $B\in\mathbb{R}^{n\times m}$ is a control matrix ($m\le n$), and $\mathbf{u}(t)\in\mathbb{C}^m$ is the control vector with the unknown feedback matrices $F,G\in\mathbb{R}^{n\times m}$. In many engineering applications, M, C and K are all real symmetric with M being positive definite and K being positive semi-definite [16].

The dynamics of a vibrating structure as (1.1) is represented as the natural frequencies and mode shapes. In fact, by the separation of variables, $\mathbf{x}(t)=\mathbf{x}e^{\lambda t}$, where $\mathbf{x}$ is a constant vector, the general solution to the homogeneous system of (1.1) is determined by the quadratic eigenvalue problem: $P(\lambda)\mathbf{x}\equiv(\lambda^2M+\lambda C+K)\mathbf{x}=0$, where $P(\lambda)$ is called the open-loop quadratic matrix pencil and λ is called an eigenvalue of $P(\lambda)$ with associated right eigenvector $\mathbf{x}\neq\mathbf{0}$. If M is nonsingular, then $P(\lambda)$ has $2n$ finite right eigenpairs $\{(\lambda_j,\mathbf{x}_j)\}_{j=1}^{2n}$ [35]. From (1.1) we have the following closed-loop quadratic matrix pencil:

$$P_c(\lambda)\equiv\lambda^2M+\lambda(C-BF^T)+(K-BG^T).$$

In this paper, we consider the following partial quadratic eigenvalue assignment problem (PQEAP) for the second-order control system (1.1): find the feedback matrices $F,G\in\mathbb{R}^{n\times m}$ such that the closed-loop pencil $P_c(\lambda)$ has the assigned eigenvalues $\{\mu_j\}_{j=1}^p$, which replace the undesired open-loop eigenvalues $\{\lambda_j\}_{j=1}^p$ ($p\ll n$) while the remaining large number of open-loop right eigenpairs $\{(\lambda_j,\mathbf{x}_j)\}_{j=p+1}^{2n}$ are retained (i.e., the no spill-over property is preserved).

In addition, a robust and minimum norm solution to the PQEAP is expected so that the sensitivity of the closed-loop eigenvalues and the feedback norms are minimized simultaneously. The minimum norm solution may reduce the energy consumption and noise influence while the robust solution reduce the sensitivity to the vibrating structure perturbation and thus improve the reliability.

The classical methods for solving the PQEAP and the robust and minimum norm PQEAP include constructive methods and optimization method based on the solution of Sylvester equation. Recently, some receptance-based methods have been proposed for the pole assignment problem and the PQEAP (see for instance [4, 22–24, 27–29, 32, 33, 37]). The robust and minimum norm PQEAP is also considered based on the measured receptances and system matrices [1, 3, 5, 34, 37].

In this paper, we propose a new receptance-based optimization approach for solving the robust and minimum norm PQEAP. This is motivated by [8, 10] and [12]. In [12], an optimization method was presented for the robust pole assignment for the first-order model such that the closed-loop eigenvector matrix is as orthogonalized as possible.

In [8, 10], via Sylvester equation, a gradient-based optimization method was given for finding a robust and minimum norm solution to the PQEAP in the sense that the closed-loop eigenvector matrix Y (see (2.4) below) was as orthogonalized as possible, that is, the quantity $\|(I-Y^HY)^2\|_F$ was minimized. In this paper, we propose a receptance-based optimization method for solving the robust and minimum norm PQEAP, where the feedback norms are minimized and the closed-loop eigenvector matrix Y is orthogonalized as much as possible. Based on the measured receptances, the system matrices M,C,K and a few undesired open-loop eigenvalues and associated eigenvectors and using a small linear system-based parametric solution to the PQEAP in [4], we derive the explicit expression of the gradient of the new proposed cost function. To further improve the effectiveness of our method, we also present the real form of our method. Finally, some numerical examples are given to show our method is effective by comparing with the methods in [3, 4].

2 Preliminaries

In what follows, we use the following notation.

- $\mathbb{C}^{n\times m}$ and $\mathbb{R}^{n\times m}$ denote the set of all $n\times m$ complex matrices and the set all $n\times m$ real matrices, respectively. Here, $\mathbb{C}=\mathbb{C}^{1\times 1}$ and $\mathbb{R}=\mathbb{R}^{1\times 1}$.
- I is the identity matrix of appropriate dimension.
- A^T, $\overline{A}$, and A^H stand for the transpose, the conjugate and the complex conjugate transpose of a matrix or vector, respectively.
- A^{-1} is the inverse of a nonsingular matrix A.
- Let f be a functional on a functional space $\mathbb{V}$. For fixed $x,h\in\mathbb{V}$, the first variation $\Delta f(x,h)$ of f at x is the derivative of $f(x+th)$ with respect to (w.r.t.) $t\in\mathbb{R}$ evaluated at $t=0$, i.e., $\Delta f(x,h)=\lim_{t\to 0}(f(x+th)-f(x))/t=\frac{d}{dt}f(x+th)|_{t=0}$.
- $|\cdot|$ means the absolute value of a real or complex number.
- $\mathrm{vec}(\cdot)$ is a vector obtained by stacking the columns of a matrix on top of one another.
- $\|\cdot\|_2$ is the matrix 2-norm and $\kappa(A)=\|A\|_2\|A^{-1}\|_2$ is the condition number of a nonsingular matrix A.
- $\|\cdot\|_F$ denotes the matrix Frobenius norm.
- $\mathrm{tr}(\cdot)$ means the trace of a square matrix.
- $\mathcal{R}(A)$ denotes the subspace spanned by the column vectors of a matrix A.

Next, as in [3, 4], we give the basic assumptions, which are needed in this paper.

Assumption 2.1. Suppose $M,C,K\in\mathbb{R}^{n\times n}$ are all symmetric with M being nonsingular. Let $\{(\lambda_j,\mathbf{x}_j)\}_{j=1}^p$ be the available open-loop right eigenpairs, where the undesired open-loop eigenvalues $\{\lambda_j\}_{j=1}^p$ are all distinct and a conjugate pair of complex eigenvalues are involved (if exists) and the remaining open-loop right eigenpairs $\{(\lambda_j,\mathbf{x}_j)\}_{j=p+1}^{2n}$ are unavailable. Suppose the set of assigned closed-loop distinct eigenvalues $\{\mu_j\}_{j=1}^p$ is closed under complex conjugation with the set of associated unknown closed-loop right eigenvectors $\{\mathbf{y}_j\}_{j=1}^p$. We also assume that $\{\mu_1,\cdots,\mu_p\}\cap\{\lambda_1,\cdots,\lambda_{2n}\}=\emptyset$, $\{\lambda_1,\cdots,\lambda_p\}\cap\{\lambda_{p+1},\cdots,\lambda_{2n}\}=\emptyset$, the control matrix B is full column rank, and $(P(\lambda),B)$ is partially controllable w.r.t. $\lambda_1,\cdots,\lambda_p$, i.e.,

$$\mathrm{rank}\big([P(\lambda_j),B]\big)=n,\quad j=1,\cdots,p.$$

Let

$$\begin{cases}\Lambda_1=\mathrm{diag}(\lambda_1,\cdots,\lambda_p), & X_1=[\mathbf{x}_1,\cdots,\mathbf{x}_p],\\ \Lambda_2=\mathrm{diag}(\lambda_{p+1},\cdots,\lambda_{2n}), & X_2=[\mathbf{x}_{p+1},\cdots,\mathbf{x}_{2n}]\end{cases}\tag{2.1}$$

and

$$P=MX_1\quad\text{and}\quad Q=MX_1\Lambda_1+CX_1.\tag{2.2}$$

A first-order model of the second-order feedback control system (1.1) is given by

$$\dot{\mathbf{q}}(t)=\widetilde{A}\mathbf{q}(t)+\widetilde{B}\mathbf{u}(t),\tag{2.3}$$

where

$$\widetilde{A}=\begin{bmatrix} O & I\\ -M^{-1}K & -M^{-1}C\end{bmatrix},\quad \widetilde{B}=\begin{bmatrix}0\\ M^{-1}B\end{bmatrix},\quad \mathbf{q}(t)=\begin{bmatrix}\mathbf{x}(t)\\ \dot{\mathbf{x}}(t)\end{bmatrix},\quad \mathbf{u}(t)=[G^T,F^T]\mathbf{q}(t).$$

Hence, $(\lambda,\mathbf{x})$ is an eigenpair of the open-loop pencil $P(\lambda)$ if and only if $(\lambda,[\mathbf{x}^T,\lambda\mathbf{x}^T]^T)$ is an eigenpair of $\widetilde{A}$ and $(\mu,\mathbf{y})$ is an eigenpair of the closed-loop pencil $P_c(\lambda)$ if and only if $(\mu,[\mathbf{y}^T,\mu\mathbf{y}^T]^T)$ is an eigenpair of $\widetilde{A}+\widetilde{B}[G^T,F^T]$ [8,10]. In fact, the PQEAP aims to find the feedback matrix $[G^T,F^T]\in\mathbb{R}^{m\times 2n}$ such that the matrix $\widetilde{A}+\widetilde{B}[G^T,F^T]$ has the desired eigenvalues $\mu_1,\cdots,\mu_p$ and the eigenpairs $\{(\lambda_j,\mathbf{x}_j)\}_{j=p+1}^{2n}$. In this case, the matrix of right eigenvectors of $\widetilde{A}+\widetilde{B}[G^T,F^T]$ is given by

$$Y=\begin{bmatrix} Y_1 & X_2\\ Y_1\Lambda_c & X_2\Lambda_2\end{bmatrix}\equiv[\widetilde{Y}_1,\widetilde{X}_2],\quad \widetilde{Y}_1\in\mathbb{C}^{(2n)\times p},\tag{2.4}$$

where

$$\Lambda_c=\mathrm{diag}(\mu_1,\cdots,\mu_p)\quad\text{and}\quad Y_1=[\mathbf{y}_1,\cdots,\mathbf{y}_p]$$

with $\mathbf{y}_j$ being the right eigenvector of $P_c(\lambda)$ corresponding to the eigenvalue μ_j.

Define

$$\widetilde{X}_1=\begin{bmatrix}X_1\\ X_1\Lambda_1\end{bmatrix}\quad\text{and}\quad \widehat{X}_1=\widetilde{X}_1\Omega_1^{-1}\equiv\begin{bmatrix}\widehat{X}_{11}\\ \widehat{X}_{12}\end{bmatrix},\tag{2.5}$$

where $\widehat{X}_{11}\in\mathbb{C}^{n\times p}$ and $\Omega_1=(\widetilde{X}_1^H\widetilde{X}_1)^{\frac{1}{2}}$. We have $\widehat{X}_1^H\widehat{X}_1=I$. Let

$$P_1=M\widehat{X}_{11}\quad\text{and}\quad Q_1=M\widehat{X}_{12}+C\widehat{X}_{11}. \tag{2.6}$$

Finally, for any $s=\sigma+\mathrm{i}\omega\in\mathbb{C}$ with $\mathrm{i}=\sqrt{-1}$, the receptance matrix of the open-loop pencil $P(\lambda)$ is given by

$$H(s)=(s^2M+sC+K)^{-1},$$

which can be derived from the measured $H(\mathrm{i}\omega)$ by fitting rational fraction polynomials without the knowledge of the system matrices [15,22].

3 A receptance-based optimization approach

In this section, we propose a receptance-based optimization approach for finding a minimum norm and robust solution to the PQEAP, which is such that the eigenvector matrix Y is orthogonalized as much as possible and the feedback norms are also minimized. Based on the measured receptances, the system matrices M,C,K and a few undesired open-loop eigenvalues and associated eigenvectors and using a small linear system-based parametric solution to the PQEAP in [4], we establish the explicit expression of the gradient of the new cost function. Then we present a gradient-based optimization method for solving the minimum norm and robust PQEAP. To further improve performance, a real form of the proposed method is also presented.

We first recall the following orthogonality relation of eigendata of $P(\lambda)$ in [2, Lemma 4.1], where the symmetry of M,C,K and the assumption that $\{\lambda_1,\cdots,\lambda_p\}\cap\{\lambda_{p+1},\cdots,\lambda_{2n}\}=\emptyset$ were employed.

Lemma 3.1. *Let Λ_1, X_1, Λ_2, X_2, P, Q, P_1, Q_1 be defined by* (2.1), (2.2), *and* (2.6) *accordingly. Then we have*

$$\begin{bmatrix}\overline{Q}\\ \overline{P}\end{bmatrix}^H\begin{bmatrix}X_2\\ X_2\Lambda_2\end{bmatrix}=0\quad\text{and}\quad\begin{bmatrix}\overline{Q}_1\\ \overline{P}_1\end{bmatrix}^H\begin{bmatrix}X_2\\ X_2\Lambda_2\end{bmatrix}=0.$$

Next, we recall the parameterized solution to the PQEAP in [4, Theorem 2.5].

Lemma 3.2. *Suppose Assumption* 2.1 *is satisfied. Let $\Gamma=[\gamma_{\mu_1},\cdots,\gamma_{\mu_p}]\in\mathbb{C}^{m\times p}$ be a nontrivial matrix such that $\gamma_{\mu_i}=\overline{\gamma_{\mu_j}}$ whenever $\mu_i=\overline{\mu_j}$ and $\{H(\mu_j)\}_{j=1}^p$ be the measured receptances. Define P and Q by* (2.2). *Set $Z=[\mathbf{z}_1,\cdots,\mathbf{z}_p]\in\mathbb{C}^{p\times p}$, where*

$$\mathbf{z}_j=(\mu_jP^T+Q^T)\mathbf{y}_j\in\mathbb{C}^p,\quad j=1,\cdots,p$$

with $\mathbf{y}_j=R_{\mu_j}\gamma_{\mu_j}$ and $R_{\mu_j}=H(\mu_j)B$. Let $\Phi\in\mathbb{C}^{m\times p}$ be a solution to the linear system

$$\Phi Z=\Gamma.$$

Then the feedback matrices F and G defined by

$$F=P\Phi^T\quad\text{and}\quad G=Q\Phi^T$$

are real and solve the PQEAP for the multiple-input control system (1.1).

By using (2.2), (2.5), (2.6), we have the following corollary.

Corollary 3.1. *Suppose Assumption* 2.1 *is satisfied. Let* $\Gamma=[\gamma_{\mu_1},\cdots,\gamma_{\mu_p}]\in\mathbb{C}^{m\times p}$ *be a nontrivial matrix such that* $\gamma_{\mu_i}=\overline{\gamma_{\mu_j}}$ *whenever* $\mu_i=\overline{\mu_j}$ *and* $\{H(\mu_j)\}_{j=1}^p$ *be the measured receptances. Define* P_1 *and* Q_1 *by* (2.6)*. Set* $Z_1=[\mathbf{z}_1,\cdots,\mathbf{z}_p]\in\mathbb{C}^{p\times p}$*, where*

$$\mathbf{z}_j=(\mu_j P_1^T+Q_1^T)\mathbf{y}_j\in\mathbb{C}^p,\quad j=1,\cdots,p$$

with $\mathbf{y}_j=R_{\mu_j}\gamma_{\mu_j}$ *and* $R_{\mu_j}=H(\mu_j)B$*. Let* $\Phi_1\in\mathbb{C}^{m\times p}$ *be a solution to the linear system*

$$\Phi_1 Z_1=\Gamma.$$

Then the feedback matrices F and G defined by

$$F=P_1\Phi_1^T\quad\textit{and}\quad G=Q_1\Phi_1^T$$

are real and solve the PQEAP for the multiple-input control system (1.1)*.*

We observe from Lemma 3.2 and Corollary 3.1 that the solution to the PQEAP is not unique. For practical effectiveness, one may find a robust and minimum norm solution to the PQEAP.

We note that the second-order control system (1.1) can be transformed to the first-order model (2.3). Therefore, the sensitivity of the closed-loop eigenvalues can be reduced if Y is near unitary or orthogonal. This can be measured by the quantity $\mathrm{tr}(I-Y^HY)^2$ (see for instance [12, 36]) instead of the quantity $\|(I-Y^HY)^2\|_F$, which was proposed in [8, 10].

From (2.4) we have

$$\begin{aligned}\|I-Y^HY\|_F^2&=\left\|\begin{bmatrix} I-\widetilde{Y}_1^H\widetilde{Y}_1 & -\widetilde{Y}_1^H\widetilde{X}_2\\ -\widetilde{X}_2^H\widetilde{Y}_1 & I-\widetilde{X}_2^H\widetilde{X}_2\end{bmatrix}\right\|_F^2\\ &=\|I-\widetilde{Y}_1^H\widetilde{Y}_1\|_F^2+2\|\widetilde{Y}_1^H\widetilde{X}_2\|_F^2+\|I-\widetilde{X}_2^H\widetilde{X}_2\|_F^2.\end{aligned}$$

We note that $\widetilde{X}_2$ is unavailable but fixed. It is desired if the quantities $\|I-\widetilde{Y}_1^H\widetilde{Y}_1\|_F$ and $\|\widetilde{Y}_1^H\widetilde{X}_2\|_F$ is minimized. We see that the quantity $\|I-\widetilde{Y}_1^H\widetilde{Y}_1\|_F$ can be minimized if the column vectors of $\widetilde{Y}_1$ are as orthonormal as possible. Despite the unavailability of $\widetilde{X}_2$, by Lemma 3.1, the quantity $\|\widetilde{Y}_1^H\widetilde{X}_2\|_F$ can be reduced if the distance between $\mathcal{R}(\widetilde{Y}_1)$ and $\mathcal{R}([\overline{Q}_1^T,\overline{P}_1^T]^T)$ is minimized. As in [31], we use $\|\sin\Theta(\mathcal{R}(\widetilde{Y}_1),\mathcal{R}([\overline{Q}_1^T,\overline{P}_1^T]^T)\|_F$ to measure the distance between $\mathcal{R}(\widetilde{Y}_1)$ and $\mathcal{R}([\overline{Q}_1^T,\overline{P}_1^T]^T)$. Let

$$K_1=I-\widetilde{Y}_1^H\widetilde{Y}_1=I-Y_1^HY_1-\overline{\Lambda}_cY_1^HY_1\Lambda_c\quad\text{and}\quad\widetilde{H}=\begin{bmatrix}\overline{Q}_1\\ \overline{P}_1\end{bmatrix}T_1^{-1},$$

where $T_1=([Q_1^T,P_1^T][\overline{Q}_1^T,\overline{P}_1^T]^T)^{\frac{1}{2}}$. Hence, $\widetilde{H}^H\widetilde{H}=I$. Then we can use the quantity $\|\widetilde{Y}_1\widetilde{Y}_1^H-\widetilde{H}\widetilde{H}^H\|_F$ as an approximation of $\|\sin\Theta(\mathcal{R}(\widetilde{Y}_1),\mathcal{R}([\overline{Q}_1^T,\overline{P}_1^T]^T)\|_F$.

From the above analysis, we can reduce the feedback norm and the eigenvalue sensitivity simultaneously by solving the following minimization problem:

$$\begin{aligned}\min_{\Gamma\in\mathbb{C}^{m\times p}} J &= \frac{1}{2}\alpha(\|I-\widetilde{Y}_1^H\widetilde{Y}_1\|_F^2+2\|\widetilde{Y}_1\widetilde{Y}_1^H-\widetilde{H}\widetilde{H}^H\|_F^2)+\frac{1}{2}(1-\alpha)\left(\|F\|_F^2+\|G\|_F^2\right)\\ &\equiv \alpha(J_1+2J_2)+(1-\alpha)J_3,\end{aligned} \tag{3.1}$$

where $0\le\alpha\le 1$ is a balance parameter. By using Lemma 3.2 and Corollary 3.1, we know that Λ_c is independent on Γ while Y_1 is dependent on Γ. Hence, J is a function of Γ. In particular, when $\alpha=0$, problem (3.1) is the minimum norm PQEAP [3, 37]. When $\alpha=1$, problem (3.1) is purely a robust PQEAP.

We will propose a gradient-based optimization method for solving problem (3.1). As noted in Lemma 3.2 and Corollary 3.1, the entries of the parameter matrix Γ are not completely independent because of the constraint $\gamma_{\mu_i}=\overline{\gamma_{\mu_j}}$ whenever $\mu_i=\overline{\mu_j}$. To find an independent parameter matrix, we give a real form of problem (3.1) as follows.

By assumption, the set of undesired open-loop eigenvalues $\{\lambda_j\}_{j=1}^p$ and the set of assigned closed-loop eigenvalues $\{\mu_j\}_{j=1}^p$ appear in complex conjugate pairs. Without loss of generality, as in [3], we assume that

$$\begin{aligned}&\mu_{2j-1,2j}=\mu_{jR}\pm\mathrm{i}\mu_{jI}, \quad j=1,\cdots,l, \quad \mu_j\in\mathbb{R}, \quad j=2l+1,\cdots,p \quad (0\le 2l\le p);\\ &\lambda_{2j-1,2j}=\lambda_{jR}\pm\mathrm{i}\lambda_{jI}, \quad j=1,\cdots,t, \quad \lambda_j\in\mathbb{R}, \quad j=2t+1,\cdots,p \quad (0\le 2t\le p);\\ &\mathbf{x}_{2j-1,2j}=\frac{1}{\sqrt{2}}(\mathbf{x}_{jR}\pm\mathrm{i}\mathbf{x}_{jI}), \quad j=1,\cdots,t, \quad \mathbf{x}_j\in\mathbb{R}^n, \quad j=2t+1,\cdots,p,\end{aligned}$$

where $\mu_{jR},\lambda_{jR}\in\mathbb{R}$, $0\neq\mu_{jI},\lambda_{jI}\in\mathbb{R}$ and $\mathbf{x}_{jR}\in\mathbb{R}^n$, $\mathbf{0}\neq\mathbf{x}_{jI}\in\mathbb{R}^n$. Thus the real forms of Λ_c, Λ_1, and X_1 are given by

$$\begin{aligned}\Lambda_{cR}&=\mathrm{diag}(\mu_1^{[2]},\cdots,\mu_l^{[2]},\mu_{2l+1},\cdots,\mu_p)\in\mathbb{R}^{p\times p},\\ \Lambda_{1R}&=\mathrm{diag}(\lambda_1^{[2]},\cdots,\lambda_t^{[2]},\lambda_{2t+1},\cdots,\lambda_p)\in\mathbb{R}^{p\times p},\\ X_{1R}&=[\mathbf{x}_{1R},\mathbf{x}_{1I},\cdots,\mathbf{x}_{tR},\mathbf{x}_{tI},\mathbf{x}_{2t+1},\cdots,\mathbf{x}_p]\in\mathbb{R}^{n\times p},\end{aligned}$$

where

$$\begin{aligned}\mu_j^{[2]}&=\begin{bmatrix}\mu_{jR} & \mu_{jI}\\ -\mu_{jI} & \mu_{jR}\end{bmatrix}\in\mathbb{R}^{2\times 2}, \quad j=1,\cdots,l,\\ \lambda_j^{[2]}&=\begin{bmatrix}\lambda_{jR} & \lambda_{jI}\\ -\lambda_{jI} & \lambda_{jR}\end{bmatrix}\in\mathbb{R}^{2\times 2}, \quad j=1,\cdots,t.\end{aligned}$$

Let

$$P_R=MX_{1R}, \quad Q_R=MX_{1R}\Lambda_{1R}+CX_{1R}, \quad W_R=[Q_R^T,P_R^T]. \tag{3.2}$$

For γ_{μ_j}, R_{μ_j} and $\mathbf{y}_j$ defined in Lemma 3.2, we have

$$\begin{cases} \gamma_{\mu_{2j-1}}=\frac{1}{\sqrt{2}}(\gamma_{jR}+\mathrm{i}\gamma_{jI}), \\ \gamma_{\mu_{2j}}=\frac{1}{\sqrt{2}}(\gamma_{jR}-\mathrm{i}\gamma_{jI}), \end{cases} j=1,\cdots,l, \quad \gamma_{\mu_j}\in\mathbb{R}^m, \quad j=2l+1,\cdots,p;$$
$$\begin{cases} R_{\mu_{2j-1}}=\frac{1}{\sqrt{2}}(R_{jR}+\mathrm{i}R_{jI}), \\ R_{\mu_{2j}}=\frac{1}{\sqrt{2}}(R_{jR}-\mathrm{i}R_{jI}), \end{cases} j=1,\cdots,l, \quad R_{\mu_j}\in\mathbb{R}^n, \quad j=2l+1,\cdots,p; \tag{3.3}$$
$$\mathbf{y}_{2j-1,2j}=\frac{1}{\sqrt{2}}(\mathbf{y}_{jR}\pm\mathrm{i}\mathbf{y}_{jI}), \quad j=1,\cdots,l, \quad \mathbf{y}_j\in\mathbb{R}^n, \quad j=2l+1,\cdots,p,$$

where $\gamma_{jR}\in\mathbb{R}^m$, $0\neq\gamma_{jI}\in\mathbb{R}^m$, $R_{jR}\in\mathbb{R}^{n\times m}$, $0\neq R_{jI}\in\mathbb{R}^{n\times m}$, $\mathbf{y}_{jR}\in\mathbb{R}^n$, $\mathbf{0}\neq\mathbf{y}_{jI}\in\mathbb{R}^n$. Let

$$\Gamma_R=[\gamma_{1R},\gamma_{1I},\cdots,\gamma_{lR},\gamma_{lI},\gamma_{\mu_{2l+1}},\cdots,\gamma_{\mu_p}]\in\mathbb{R}^{m\times p}, \tag{3.4a}$$
$$Y_{1R}=[\mathbf{y}_{1R},\mathbf{y}_{1I},\cdots,\mathbf{y}_{lR},\mathbf{y}_{lI},\mathbf{y}_{2l+1},\cdots,\mathbf{y}_p]\in\mathbb{R}^{n\times p}. \tag{3.4b}$$

By Lemma 3.2 we have

$$\begin{cases} \mathbf{y}_{jR}=\frac{1}{\sqrt{2}}(R_{jR}\gamma_{jR}-R_{jI}\gamma_{jI}), \ \mathbf{y}_{jI}=\frac{1}{\sqrt{2}}(R_{jR}\gamma_{jI}+R_{jI}\gamma_{jR}), & j=1,\cdots,l; \\ \mathbf{y}_j=R_{\mu_j}\gamma_{\mu_j}, & j=2l+1,\cdots,p. \end{cases} \tag{3.5}$$

Define

$$\widetilde{X}_{1R}=\begin{bmatrix} X_{1R} \\ X_{1R}\Lambda_{1R} \end{bmatrix}, \quad \widehat{X}_{1R}=\widetilde{X}_{1R}\Omega_{1R}^{-1}\equiv\begin{bmatrix} \widehat{X}_{11R} \\ \widehat{X}_{12R} \end{bmatrix}, \quad \widetilde{Y}_{1R}=\begin{bmatrix} Y_{1R} \\ Y_{1R}\Lambda_{cR} \end{bmatrix}, \tag{3.6}$$

where $\widehat{X}_{11R}\in\mathbb{C}^{n\times p}$ and $\Omega_{1R}=(\widetilde{X}_{1R}^T\widetilde{X}_{1R})^{\frac{1}{2}}$. We have $\widehat{X}_{1R}^T\widehat{X}_{1R}=I$. Let

$$P_{1R}=M\widehat{X}_{11R}, \quad Q_{1R}=M\widehat{X}_{12R}+C\widehat{X}_{11R}, \quad W_{1R}=[Q_{1R}^T,P_{1R}^T], \quad \widetilde{H}_R=W_{1R}^T T_{1R}^{-1}, \tag{3.7}$$

where $T_{1R}=(W_{1R}W_{1R}^T)^{\frac{1}{2}}$. We have $\widetilde{H}_R^T\widetilde{H}_R=I$. Therefore, problem (3.1) becomes

$$\begin{aligned} \min_{\Gamma_R\in\mathbb{R}^{m\times p}} J&=\frac{1}{2}\alpha(\|I-\widetilde{Y}_{1R}^T\widetilde{Y}_{1R}\|_F^2+2\|\widetilde{Y}_{1R}\widetilde{Y}_{1R}^T-\widetilde{H}_R\widetilde{H}_R^T\|_F^2)+\frac{1}{2}(1-\alpha)\left(\|F\|_F^2+\|G\|_F^2\right) \\ &\equiv\alpha(J_1+2J_2)+(1-\alpha)J_3. \end{aligned} \tag{3.8}$$

In the following, we focus on problem (3.8). We now establish the explicit expression of the gradient of J w.r.t. the parameter matrix Γ_R defined in (3.4a). First, on the real form of a parametric solution to the PQEAP for the multiple-input control system (1.1), we have the following lemma [4, Theorem 3.5].

Lemma 3.3 (Real form)**.** *Suppose Assumption* 2.1 *is satisfied. Let* $\Gamma_R=[\gamma_{1R},\gamma_{1I},\cdots,\gamma_{lR},\gamma_{lI},\gamma_{\mu_{2l+1}},\cdots,\gamma_{\mu_p}]\in\mathbb{R}^{m\times p}$ *be a nontrivial matrix and* $\{H(\mu_j)\}_{j=1}^p$ *be the measured receptances. Define* P_R *and* Q_R *by* (3.2)*. Let* $\{R_{jR}\}_{j=1}^l$, $\{R_{jI}\}_{j=1}^l$, *and* $\{R_{\mu_j}\}_{j=2l+1}^p$ *be defined by* (3.3)*. Set*

$Z_R = P_R^T Y_{1R}\Lambda_{cR} + Q_R^T Y_{1R}$, *where* Y_{1R} *is determined by* (3.4b) *and* (3.5). *Let* $\Phi_R \in \mathbb{R}^{m\times p}$ *be a solution to the linear system*

$$\Phi_R Z_R = \Gamma_R.$$

Then the feedback matrices F *and* G *defined by*

$$F = P_R \Phi_R^T \quad \text{and} \quad G = Q_R \Phi_R^T$$

solves the PQEAP for the multiple-input control system (1.1).

By using (3.2), (3.6), (3.7), we have the following corollary.

Corollary 3.2. *Suppose Assumption* 2.1 *is satisfied. Let* $\Gamma_R = [\gamma_{1R}, \gamma_{1I}, \cdots, \gamma_{lR}, \gamma_{lI}, \gamma_{\mu_{2l+1}}, \cdots, \gamma_{\mu_p}] \in \mathbb{R}^{m\times p}$ *be a nontrivial matrix and* $\{H(\mu_j)\}_{j=1}^p$ *be the measured receptances. Define* P_{1R} *and* Q_{1R} *by* (3.7). *Let* $\{R_{jR}\}_{j=1}^l$, $\{R_{jI}\}_{j=1}^l$, *and* $\{R_{\mu_j}\}_{j=2l+1}^p$ *be defined by* (3.3). *Set* $Z_{1R} = P_{1R}^T Y_{1R}\Lambda_{cR} + Q_{1R}^T Y_{1R}$, *where* Y_{1R} *is determined by* (3.4b) *and* (3.5). *Let* $\Phi_{1R} \in \mathbb{R}^{m\times p}$ *be a solution to the linear system*

$$\Phi_{1R} Z_{1R} = \Gamma_R.$$

Then the feedback matrices F *and* G *defined by*

$$F = P_{1R}\Phi_{1R}^T \quad \text{and} \quad G = Q_{1R}\Phi_{1R}^T$$

solves the PQEAP for the multiple-input control system (1.1).

We see from Lemma 3.3 and Corollary 3.2 that the cost function J defined in (3.8) is a function of Γ_R.

Next, we have the following result on the gradient of $J_3 = \frac{1}{2}(\|F\|_F^2 + \|G\|_F^2)$ w.r.t. Γ_R [3, Theorem 3.2].

Lemma 3.4. *Suppose Assumption* 2.1 *is satisfied. Let* $\Gamma_R = [\gamma_{1R}, \gamma_{1I}, \cdots, \gamma_{lR}, \gamma_{lI}, \gamma_{\mu_{2l+1}}, \cdots, \gamma_{\mu_p}] \in \mathbb{R}^{m\times p}$ *be a nontrivial matrix and* $\{H(\mu_j)\}_{j=1}^p$ *be the measured receptances. Let* $\{R_{jR}\}_{j=1}^l$, $\{R_{jI}\}_{j=1}^l$, *and* $\{R_{\mu_j}\}_{j=2l+1}^p$ *be defined by* (3.3). *Define* P_R, Q_R, *and* W_R *by* (3.2). *Set* $Z_R = P_R^T Y_{1R}\Lambda_{cR} + Q_R^T Y_{1R}$, *where* Y_{1R} *is determined by* (3.4b) *and* (3.5). *Suppose* Z_R *is invertible. Let* $F = P_R\Phi_R^T$ *and* $G = Q_R\Phi_R^T$ *where* Φ_R *is the solution to* $\Phi_R Z_R = \Gamma_R$. *Define* $S = [G^T, F^T]$ *and* $U_R = (Z_R^{-1} W_R S^T \Phi_R)^T = [\mathbf{u}_{1R}, \mathbf{u}_{1I}, \cdots, \mathbf{u}_{lR}, \mathbf{u}_{lI}, \mathbf{u}_{2l+1}, \cdots, \mathbf{u}_p]$. *Then the gradient* $\nabla_{\Gamma_R} J_3$ *of* J_3 *w.r.t.* Γ_R *is given by*

$$\nabla_{\Gamma_R} J_3 = \left(Z_R^{-1} W_R S^T - F_R^T\right)^T,$$

where $F_R = [D_{1R}^T\mathbf{u}_{1R} + D_{1I}^T\mathbf{u}_{1I}, D_{1R}^T\mathbf{u}_{1I} - D_{1I}^T\mathbf{u}_{1R}, \cdots, D_{lR}^T\mathbf{u}_{lR} + D_{lI}^T\mathbf{u}_{lI}, D_{lR}^T\mathbf{u}_{lI} - D_{lI}^T\mathbf{u}_{lR}, D_{2l+1}^T\mathbf{u}_{2l+1}, \cdots, D_p^T\mathbf{u}_p]$ *with*

$$D_{jR} = \frac{1}{\sqrt{2}}(\mu_{jR}P_R^T + Q_R^T)R_{jR} - \frac{1}{\sqrt{2}}\mu_{jI}P_R^T R_{jI}, \quad D_{jI} = \frac{1}{\sqrt{2}}(\mu_{jR}P_R^T + Q_R^T)R_{jI} + \frac{1}{\sqrt{2}}\mu_{jI}P_R^T R_{jR}$$

for $j=1,\cdots,l$ *and*

$$D_j=(\mu_j P_R^T+Q_R^T)R_{\mu_j}$$

for $j=2l+1,\cdots,p$.

From Lemma 3.4 and Corollary 3.2, we have the following corollary.

Corollary 3.3. *Suppose Assumption* 2.1 *is satisfied. Let* $\Gamma_R=[\gamma_{1R},\gamma_{1I},\cdots,\gamma_{lR},\gamma_{lI},\gamma_{\mu_{2l+1}},\cdots,\gamma_{\mu_p}]\in\mathbb{R}^{m\times p}$ *be a nontrivial matrix and* $\{H(\mu_j)\}_{j=1}^p$ *be the measured receptances. Let* $\{R_{jR}\}_{j=1}^l$, $\{R_{jI}\}_{j=1}^l$, *and* $\{R_{\mu_j}\}_{j=2l+1}^p$ *be defined by* (3.3). *Define* P_{1R}, Q_{1R} *and* W_{1R} *by* (3.7). *Set* $Z_{1R}=P_{1R}^TY_{1R}\Lambda_{cR}+Q_{1R}^TY_{1R}$, *where* Y_{1R} *is determined by* (3.4b) *and* (3.5). *Suppose* Z_{1R} *is invertible. Let* $F=P_{1R}\Phi_{1R}^T$ *and* $G=Q_{1R}\Phi_{1R}^T$ *where* Φ_{1R} *is the solution to* $\Phi_{1R}Z_{1R}=\Gamma_R$. *Define* $S=[G^T,F^T]$ *and* $U_{1R}=(Z_{1R}^{-1}W_{1R}S^T\Phi_{1R})^T=[\mathbf{u}_{1R},\mathbf{u}_{1I},\cdots,\mathbf{u}_{lR},\mathbf{u}_{lI},\mathbf{u}_{2l+1},\cdots,\mathbf{u}_p]$. *Then the gradient* $\nabla_{\Gamma_R}J_3$ *of* J_3 *w.r.t.* Γ_R *is given by*

$$\nabla_{\Gamma_R}J_3=\left(Z_{1R}^{-1}W_{1R}S^T-F_{1R}^T\right)^T,$$

where $F_{1R}=[D_{1R}^T\mathbf{u}_{1R}+D_{1I}^T\mathbf{u}_{1I},D_{1R}^T\mathbf{u}_{1I}-D_{1I}^T\mathbf{u}_{1R},\cdots,D_{lR}^T\mathbf{u}_{lR}+D_{lI}^T\mathbf{u}_{lI},D_{lR}^T\mathbf{u}_{lI}-D_{lI}^T\mathbf{u}_{lR},D_{2l+1}^T\mathbf{u}_{2l+1},\cdots,D_p^T\mathbf{u}_p]$ *with*

$$D_{jR}=\frac{1}{\sqrt{2}}(\mu_{jR}P_{1R}^T+Q_{1R}^T)R_{jR}-\frac{1}{\sqrt{2}}\mu_{jI}P_{1R}^TR_{jI},\quad D_{jI}=\frac{1}{\sqrt{2}}(\mu_{jR}P_{1R}^T+Q_{1R}^T)R_{jI}+\frac{1}{\sqrt{2}}\mu_{jI}P_{1R}^TR_{jR}$$

for $j=1,\cdots,l$ *and*

$$D_j=(\mu_jP_{1R}^T+Q_{1R}^T)R_{\mu_j}$$

for $j=2l+1,\cdots,p$.

We now establish the following theorem on the explicit expression of the gradient of $J_1=\frac{1}{2}\|I-Y_{1R}^TY_{1R}-\Lambda_{cR}^TY_{1R}^TY_{1R}\Lambda_{cR}\|_F^2$ w.r.t. Γ_R.

Theorem 3.1. *Suppose Assumption* 2.1 *is satisfied. Let* $\Gamma_R=[\gamma_{1R},\gamma_{1I},\cdots,\gamma_{lR},\gamma_{lI},\gamma_{\mu_{2l+1}},\cdots,\gamma_{\mu_p}]\in\mathbb{R}^{m\times p}$ *be a nontrivial matrix and* $\{H(\mu_j)\}_{j=1}^p$ *be the measured receptances. Let* $\{R_{jR}\}_{j=1}^l$, $\{R_{jI}\}_{j=1}^l$, *and* $\{R_{\mu_j}\}_{j=2l+1}^p$ *be defined by* (3.3). *Let* $K_{1R}=I-Y_{1R}^TY_{1R}-\Lambda_{cR}^TY_{1R}^TY_{1R}\Lambda_{cR}$, *where* Y_{1R} *is defined by* (3.4b) *and* (3.5). *Define* $S_{1R}=K_{1R}+\Lambda_{cR}K_{1R}\Lambda_{cR}^T$ *and* $V_{1R}=Y_{1R}S_{1R}=[\mathbf{v}_{1R},\mathbf{v}_{1I},\cdots,\mathbf{v}_{lR},\mathbf{v}_{lI},\mathbf{v}_{2l+1},\cdots,\mathbf{v}_p]$. *Then the gradient* $\nabla_\Gamma J_1$ *of* J_1 *w.r.t.* Γ_R *is given by*

$$\nabla_{\Gamma_R}J_1=-2G_{1R},$$

where $G_{1R}=[\frac{1}{\sqrt{2}}R_{1R}^T\mathbf{v}_{1R}+\frac{1}{\sqrt{2}}R_{1I}^T\mathbf{v}_{1I},\frac{1}{\sqrt{2}}R_{1R}^T\mathbf{v}_{1I}-\frac{1}{\sqrt{2}}R_{1I}^T\mathbf{v}_{1R},\cdots,\frac{1}{\sqrt{2}}R_{lR}^T\mathbf{v}_{lR}+\frac{1}{\sqrt{2}}R_{lI}^T\mathbf{v}_{lI},\frac{1}{\sqrt{2}}R_{lR}^T\mathbf{v}_{lI}-\frac{1}{\sqrt{2}}R_{lI}^T\mathbf{v}_{lR},R_{\mu_{2l+1}}^T\mathbf{v}_{2l+1},\cdots,R_{\mu_p}^T\mathbf{v}_p]$.

Proof. Since $K_{1R}^T=K_{1R}$, we have $J_1=\frac{1}{2}\|K_{1R}\|_F^2=\frac{1}{2}\mathrm{tr}(K_{1R}^2)$. We now establish the gradient $\nabla_{\Gamma_R}J_1$ of J_1 w.r.t. Γ_R. We note that the first variation of J_1 is given by

$$\Delta J_1(K_{1R},\Delta K_{1R})=\mathrm{tr}(K_{1R}\Delta K_{1R}),$$

where $\Delta K_{1R}=\Delta K_{1R}(Y_{1R},\Delta Y_{1R})$. By using $K_{1R}=I-Y_{1R}^TY_{1R}-\Lambda_{cR}^TY_{1R}^TY_{1R}\Lambda_{cR}$ we have

$$\begin{aligned}\Delta K_{1R}&=\Delta K_{1R}(Y_{1R},\Delta Y_{1R})\\&=-(\Delta Y_{1R}^TY_{1R}+Y_{1R}^T\Delta Y_{1R}+\Lambda_{cR}^T\Delta Y_{1R}^TY_{1R}\Lambda_{cR}+\Lambda_{cR}^TY_{1R}^T\Delta Y_{1R}\Lambda_{cR}).\end{aligned}$$

Thus,

$$\begin{aligned}\Delta J_1(Y_{1R},\Delta Y_{1R})&=-\mathrm{tr}(K_{1R}(\Delta Y_{1R}^TY_{1R}+Y_{1R}^T\Delta Y_{1R}+\Lambda_{cR}^T\Delta Y_{1R}^TY_{1R}\Lambda_{cR}+\Lambda_{cR}^TY_{1R}^T\Delta Y_{1R}\Lambda_{cR}))\\&=-\mathrm{tr}(Z_{1R}Y_{1R}^T\Delta Y_{1R})-\mathrm{tr}(Y_{1R}Z_{1R}\Delta Y_{1R}^T)=-2\mathrm{tr}(Z_{1R}Y_{1R}^T\Delta Y_{1R}),\end{aligned}\tag{3.9}$$

where $\Delta Y_{1R}=\Delta Y_{1R}(\Gamma_R,\Delta\Gamma_R)$.

In the following, we give the first variation of J_1 in terms of $\Delta\Gamma_R$. We now derive the expression of $\mathrm{tr}(Z_{1R}Y_{1R}^T\Delta Y_{1R})$ in terms of $\Delta\Gamma_R$. We have by Lemma 3.3

$$\begin{aligned}Y_{1R}=\Big[&\frac{1}{\sqrt{2}}(R_{1R}\gamma_{1R}-R_{1I}\gamma_{1I}),\frac{1}{\sqrt{2}}(R_{1R}\gamma_{1I}+R_{1I}\gamma_{1R}),\cdots,\frac{1}{\sqrt{2}}(R_{lR}\gamma_{lR}-R_{lI}\gamma_{lI}),\\&\frac{1}{\sqrt{2}}(R_{lR}\gamma_{lI}+R_{lI}\gamma_{lR}),R_{\mu_{2l+1}}\gamma_{\mu_{2l+1}},\cdots,R_{\mu_p}\gamma_{\mu_p}\Big]\end{aligned}$$

and thus

$$\begin{aligned}\Delta Y_{1R}=\Big[&\frac{1}{\sqrt{2}}(R_{1R}\Delta\gamma_{1R}-R_{1I}\Delta\gamma_{1I}),\frac{1}{\sqrt{2}}(R_{1R}\Delta\gamma_{1I}+R_{1I}\Delta\gamma_{1R}),\cdots,\frac{1}{\sqrt{2}}(R_{lR}\Delta\gamma_{lR}-R_{lI}\Delta\gamma_{lI}),\\&\frac{1}{\sqrt{2}}(R_{lR}\Delta\gamma_{lI}+R_{lI}\Delta\gamma_{lR}),R_{\mu_{2l+1}}\Delta\gamma_{\mu_{2l+1}},\cdots,R_{\mu_p}\Delta\gamma_{\mu_p}\Big],\end{aligned}$$

or

$$\mathrm{vec}(\Delta Y_{1R})=B_{1R}\mathrm{vec}(\Delta\Gamma_R),\tag{3.10}$$

where

$$B_{1R}=\begin{bmatrix}\frac{1}{\sqrt{2}}R_{1R} & -\frac{1}{\sqrt{2}}R_{1I} &&&&&\\ \frac{1}{\sqrt{2}}R_{1I} & \frac{1}{\sqrt{2}}R_{1R} &&&&&\\ && \ddots &&&&\\ &&& \frac{1}{\sqrt{2}}R_{lR} & -\frac{1}{\sqrt{2}}R_{lI} &&\\ &&& \frac{1}{\sqrt{2}}R_{lI} & \frac{1}{\sqrt{2}}R_{lR} &&\\ &&&&& R_{\mu_{2l+1}} &&\\ &&&&&& \ddots &\\ &&&&&&& R_{\mu_p}\end{bmatrix}.$$

From (3.10) we have

$$\begin{aligned}\mathrm{tr}(Z_{1R}Y_{1R}^T\Delta Y_{1R})&=\mathrm{tr}(V_R^T\Delta Y_{1R})=\mathrm{vec}(V_R)^T\mathrm{vec}(\Delta Y_{1R})=\mathrm{vec}(V_R)^TB_{1R}\mathrm{vec}(\Delta\Gamma_R)\\&=\Big(B_{1R}^T\mathrm{vec}(V_R)\Big)^T\mathrm{vec}(\Delta\Gamma_R)=\mathrm{vec}(G_{1R})^T\mathrm{vec}(\Delta\Gamma_R)=\mathrm{tr}(G_{1R}^T\Delta\Gamma_R).\end{aligned}\tag{3.11}$$

Substituting (3.11) into (3.9) yields

$$\Delta J_1(\Gamma_R,\Delta\Gamma_R) = -2\mathrm{tr}(G_{1R}^T\Delta\Gamma_R).$$

This implies that

$$\nabla_\Gamma J_1 = -2G_{1R}.$$

This completes the proof. □

By using the similar arguments of Theorem 3.1, we have the following result on the explicit expression of the gradient of $J_2 = \|\widetilde{Y}_{1R}\widetilde{Y}_{1R}^T - \widetilde{H}_R\widetilde{H}_R^T\|_F^2$ w.r.t. Γ_R.

Theorem 3.2. *Suppose Assumption* 2.1 *is satisfied. Let* $\Gamma_R = [\gamma_{1R},\gamma_{1I},\cdots,\gamma_{lR},\gamma_{lI},\gamma_{\mu_{2l+1}},\cdots,\gamma_{\mu_p}] \in \mathbb{R}^{m\times p}$ *be a nontrivial matrix and* $\{H(\mu_j)\}_{j=1}^p$ *be the measured receptances. Let* $\{R_{jR}\}_{j=1}^l$, $\{R_{jI}\}_{j=1}^l$, *and* $\{R_{\mu_j}\}_{j=2l+1}^p$ *be defined by* (3.3). *Define* $\widetilde{H}_R$ *by* (3.7). *Let* $\widetilde{Y}_{1R} = [Y_{1R}^T,(Y_{1R}\Lambda_{cR})^T]^T$, *where* Y_{1R} *is given by* (3.4b) *and* (3.5). *Set* $K_{2R} = \widetilde{H}_R\widetilde{H}_R^T - \widetilde{Y}_{1R}\widetilde{Y}_{1R}^T$, $S_{2R} = K_{2R} + \Lambda_{cR}K_{2R}\Lambda_{cR}^T$, *and* $V_{2R} = Y_{1R}S_{2R} = [\mathbf{v}_{1R},\mathbf{v}_{1I},\cdots,\mathbf{v}_{lR},\mathbf{v}_{lI},\mathbf{v}_{2l+1},\cdots,\mathbf{v}_p]$. *Then the gradient* $\nabla_\Gamma J_2$ *of* J_2 *w.r.t.* Γ_R *is given by*

$$\nabla_{\Gamma_R} J_2 = -2G_{2R},$$

where $G_{2R} = [\frac{1}{\sqrt{2}}R_{1R}^T\mathbf{v}_{1R} + \frac{1}{\sqrt{2}}R_{1I}^T\mathbf{v}_{1I}, \frac{1}{\sqrt{2}}R_{1R}^T\mathbf{v}_{1I} - \frac{1}{\sqrt{2}}R_{1I}^T\mathbf{v}_{1R},\cdots,\frac{1}{\sqrt{2}}R_{lR}^T\mathbf{v}_{lR} + \frac{1}{\sqrt{2}}R_{lI}^T\mathbf{v}_{lI}, \frac{1}{\sqrt{2}}R_{lR}^T\mathbf{v}_{lI} - \frac{1}{\sqrt{2}}R_{lI}^T\mathbf{v}_{lR}, R_{\mu_{2l+1}}^T\mathbf{v}_{2l+1},\cdots,R_{\mu_p}^T\mathbf{v}_p]$.

We see from Lemma 3.4 and Theorems 3.1-3.2 that the gradient of J w.r.t. Γ_R is given by

$$\begin{aligned}\nabla_{\Gamma_R} J &= \alpha(\nabla_{\Gamma_R} J_1 + 2\nabla_{\Gamma_R} J_2) + (1-\alpha)\nabla_{\Gamma_R} J_2\\ &= -2\alpha(G_{1R} + 2G_{2R}) + (1-\alpha)\big(Z_R^{-1}W_RS^T - F_{1R}^T\big)^T.\end{aligned}$$

Based on Corollary 3.2, Corollary 3.3, and Theorems 3.1-3.2, we give a gradient-based optimization algorithm for solving problem (3.8), which is described in Algorithm 1.

Next, we make several remarks on Algorithm 1.

Remark 3.1. In Steps 0–1 of Algorithm 1, one should choose appropriate Γ_R such that the matrix Z_{1R} is nonsingular. The nonsingularity of Z_{1R} can be guaranteed under some conditions on Γ_R as mentioned in [3].

Remark 3.2. We note that the cost function J is nonlinear and nonconvex. Hence, in general, only a local minimum point of J is calculated. A better solution can be obtained by choosing various Γ_R [9].

Algorithm 1 A gradient-based optimization algorithm

Step 0. Choose a nontrivial matrix $\Gamma_R = [\gamma_{1R}, \gamma_{1I}, \cdots, \gamma_{lR}, \gamma_{lI}, \gamma_{\mu_{2l+1}}, \cdots, \gamma_{\mu_p}] \in \mathbb{R}^{m\times p}$ and $\alpha \in [0,1]$. Form the matrices Λ_{1R}, X_{1R}, $\widehat{X}_{11R}$, $\widehat{X}_{12R}$, Λ_{cR}, P_{1R} Q_{1R}, W_{1R}, and $\widetilde{H}_R$. Form $\{R_{jR}\}_{j=1}^{l}$, $\{R_{jI}\}_{j=1}^{l}$, and $\{R_{\mu_j}\}_{j=2l+1}^{p}$ as defined in (3.3). (This needs $\mathcal{O}(n^2p+n^2mp)$ operations).

Step 1. Form the matrix Z_{1R} as defined in Corollary 3.2. (This needs $\mathcal{O}(n^2mp+np^2)$ flops). If the condition number of Z_{1R} is large, then choose another Γ_R.

Step 2. Find a solution Φ_{1R} to the linear system $\Phi_{1R}Z_{1R} = \Gamma_R$. (This needs $\mathcal{O}(p^3+mp^2)$ flops).

Step 3. Compute $S = \Phi_{1R}W_{1R}$, $Y_{1R} = [\frac{1}{\sqrt{2}}(R_{1R}\gamma_{1R} - R_{1I}\gamma_{1I}), \frac{1}{\sqrt{2}}(R_{1R}\gamma_{1I} + R_{1I}\gamma_{1R}), \cdots, \frac{1}{\sqrt{2}}(R_{lR}\gamma_{lR} - R_{lI}\gamma_{lI}), \frac{1}{\sqrt{2}}(R_{lR}\gamma_{lI} + R_{lI}\gamma_{lR}), R_{\mu_{2l+1}}\gamma_{\mu_{2l+1}}, \cdots, R_{\mu_p}\gamma_{\mu_p}]$, $K_{1R} = I - Y_{1R}^TY_{1R} - \Lambda_{cR}^TY_{1R}^TY_{1R}\Lambda_{cR}$, $S_{1R} = K_{1R} + \Lambda_{cR}K_{1R}\Lambda_{cR}^T$, $\widetilde{Y}_{1R} = [Y_{1R}^T, (Y_{1R}\Lambda_{cR})^T]^T$, $K_{2R} = \widetilde{H}_R\widetilde{H}_R^T - \widetilde{Y}_{1R}\widetilde{Y}_{1R}^T$, and $S_{2R} = K_{2R} + \Lambda_{cR}K_{2R}\Lambda_{cR}^T$. (This needs $\mathcal{O}(n^2p+nmp+np^2)$ flops).

Step 4. Form the matrices $U_{1R} = (Z_{1R}^{-1}W_{1R}S^T\Phi_{1R})^T = [\mathbf{u}_{1R}, \mathbf{u}_{1I}, \cdots, \mathbf{u}_{lR}, \mathbf{u}_{lI}, \mathbf{u}_{2l+1}, \cdots, \mathbf{u}_p]$, D_{jR}, D_{jI} for $j=1,\cdots,l$ and D_j for $j=2l+1,\cdots,p$ and F_{1R} as defined in Corollary 3.3, form $V_{1R} = Y_{1R}S_{1R} = [\mathbf{v}_{1R}, \mathbf{v}_{1I}, \cdots, \mathbf{v}_{lR}, \mathbf{v}_{lI}, \mathbf{v}_{2l+1}, \cdots, \mathbf{v}_p]$ and G_{1R} as defined in Theorem 3.1, and form $V_{2R} = Y_{1R}S_{2R} = [\mathbf{v}_{1R}, \mathbf{v}_{1I}, \cdots, \mathbf{v}_{lR}, \mathbf{v}_{lI}, \mathbf{v}_{2l+1}, \cdots, \mathbf{v}_p]$ and G_{2R} as defined in Theorem 3.2. (This needs $\mathcal{O}(nmp^2+np^2)$ operations).

Step 5. Compute the gradient $\nabla_{\Gamma_R}J = \alpha(\nabla_{\Gamma_R}J_1 + 2\nabla_{\Gamma_R}J_2) + (1-\alpha)\nabla_{\Gamma_R}J_3$ as in Corollary 3.3 and Theorems 3.1-3.2. (This needs $\mathcal{O}(nmp+np^2)$ operations).

Step 6. Update Γ_R by using the Broyden-Fletcher-Goldfarb-Shanno (BFGS) method [26, Chap. 8].

Step 7. Using the updated Γ_R we compute F and G as in Corollary 3.2. Stop.

4 Illustrative numerical examples

In this section, we present the performance of Algorithm 1 for solving the minimum norm and robust PQEAP. To show the effectiveness of our method, we compare the proposed algorithms with the methods in [3, 4]. All numerical tests were carried out by using `MATLAB R2019a` on a personal PC of Intel Core i7 of 4.00 GHz CPU and 16GB RAM.

In our numerical experiments, we set $\epsilon = 1.0\times 10^{-6}$, $Max_{iter} = 500$. In what follows, "`tol1`", "`tol2`", and "`tol3`" mean the upper bounds of the errors of the closed-loop

Algorithm 2 Updating Γ_R in Step 6 of Algorithm 1

Step 6.1. Replace Γ_R^{old} by $\widehat{\Gamma}_R=\Gamma_R^{old}+\beta\Psi_{Rk}=[\widehat{\gamma}_{1R},\widehat{\gamma}_{1I},\cdots,\widehat{\gamma}_{lR},\widehat{\gamma}_{lI},\widehat{\gamma}_{\mu_{2l+1}},\cdots,\widehat{\gamma}_{\mu_p}]$, where Γ_R^{old} is the current value of Γ_R and Ψ_{Rk} is a search direction. In the BFGS method, we have $\Psi_{Rk}=-\xi_j\nabla_{\Gamma_R^{old}}J$ where ξ_k is the metric obtained in the BFGS method with $\xi_1=1$.

Step 6.2. Form $\widehat{Y}_{1R}=[\frac{1}{\sqrt{2}}(R_{1R}\widehat{\gamma}_{1R}-R_{1I}\widehat{\gamma}_{1I}),\frac{1}{\sqrt{2}}(R_{1R}\widehat{\gamma}_{1I}+R_{1I}\widehat{\gamma}_{1R}),\cdots,\frac{1}{\sqrt{2}}(R_{lR}\widehat{\gamma}_{lR}-R_{lI}\widehat{\gamma}_{lI}),$ $\frac{1}{\sqrt{2}}(R_{lR}\widehat{\gamma}a_{lI}+R_{lI}\widehat{\gamma}_{lR}),R_{\mu_{2l+1}}\widehat{\gamma}_{\mu_{2l+1}},\cdots,R_{\mu_p}\widehat{\gamma}_{\mu_p}]$ and $\widehat{Z}_{1R}=P_{1R}^T\widehat{Y}_{1R}\Lambda_{cR}+Q_{1R}^T\widehat{Y}_{1R}$. Find a solution $\widehat{\Phi}_{1R}$ to $\widehat{\Phi}_{1R}\widehat{Z}_{1R}=\widehat{\Gamma}_R$ and form $\widehat{S}=\widehat{\Phi}_RW_{1R}$.

Step 6.3. Find $\hat{\beta}=\min_{\beta\in\mathbb{R}}\{\widehat{J}=\frac{1}{2}\alpha(\|I-\widehat{Y}_{1R}^T\widehat{Y}_{1R}-\Lambda_{cR}^T\widehat{Y}_{1R}^T\widehat{Y}_{1R}\Lambda_{cR}\|_F^2+2\|\widetilde{H}_R\widetilde{H}_R^T-[\widehat{Y}_{1R}^T,(\widehat{Y}_{1R}\Lambda_{cR})^T]^T[\widehat{Y}_{1R}^T,(\widehat{Y}_{1R}\Lambda_{cR})^T]\|_F^2)+\frac{1}{2}(1-\alpha)\|\widehat{S}\|_F^2\}$, which is implemented by using the `MATLAB` function `fminsearch`, where the termination tolerance on the function value is set to be $\epsilon>0$ and the maximum number of iterations allowed is set to be Max_{iter}.

Step 6.4. Let $\Gamma_R^{new}=\Gamma_R^{old}+\hat{\beta}\Psi_{Rk}$.

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eigenvalues and eigenvectors accordingly, i.e.,

$$\begin{cases} |\det(\mu_j^2M+\mu_j(C-BF^T)+(K-BG^T))|\le \texttt{tol1}, & 1\le j\le p,\\ \|(\mu_j^2M+\mu_j(C-BF^T)+(K-BG^T))\mathbf{y}_j\|_2\le\texttt{tol2}, & 1\le j\le p,\\ \|(\lambda_j^2M+\lambda_j(C-BF^T)+(K-BG^T))\mathbf{x}_j\|_2\le\texttt{tol3}, & p+1\le j\le 2n \end{cases}$$

and "`CT.`" means the total computing time in seconds. In addition, we use $\kappa=\|\widetilde{Y}\|_2\|\widetilde{Y}^{-1}\|_2$ to denote the condition number of the eigenvector matrix $\widetilde{Y}$ of the model (2.3) at a solution point of the PQEAP.

We consider the following two numerical examples.

Example 4.1. [18] Consider a discrete shear beam in the multiple-input control system (1.1) with $n=10$ and $m=2$, where

$$M=I_n,\quad C=\mathrm{diag}(c_1,\cdots,c_n),\quad K=\begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0\\ -1 & 2 & -1 & \cdots & 0 & 0\\ 0 & -1 & 2 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & \cdots & -1 & 2 & -1\\ 0 & 0 & \cdots & 0 & -1 & 2\end{bmatrix},\quad B=\begin{bmatrix} I_m\\ 0\end{bmatrix},$$

Here, $c_k=2\xi\omega_k$ with the damping factor $\xi=0.1$ and $\omega_k=2\sin(k\pi/(n+1))$ for $k=1,\cdots,n$. There exist 20 open-loop eigenvalues. The four eigenvalues $\{-0.1291\pm1.5063\mathrm{i},-0.1290\pm$

1.3031i} were replaced by $\{-0.8\pm 1.5063\text{i}, -0.4\pm 1.3031\text{i}\}$, and the other eigenpairs were preserved.

By applying Algorithm 1 with $\alpha=1.0$ to Example 4.1, where

$$\Gamma_R=\sqrt{2}\begin{bmatrix} 3 & 4 & 5 & 6 \\ 7 & 8 & 9 & 9 \end{bmatrix}, \tag{4.1}$$

we get the following feedback matrices

$$F=\begin{bmatrix} -1.3986 & -0.1735 \\ 0.0731 & -0.4851 \\ 1.2684 & 0.1098 \\ -0.0271 & 0.9639 \\ -1.0434 & -0.1136 \\ -0.0509 & -1.3619 \\ 0.8209 & 0.1760 \\ 0.0730 & 1.6786 \\ -0.4922 & -0.1825 \\ -0.0479 & -1.7947 \end{bmatrix}, \quad G=\begin{bmatrix} -0.7527 & -0.2041 \\ -0.2287 & 0.1737 \\ 0.9590 & 0.0302 \\ -0.0980 & -0.1596 \\ -0.8750 & -0.1136 \\ 0.5258 & 0.1475 \\ 0.5824 & 0.4594 \\ -0.7311 & -0.3440 \\ -0.2079 & -0.5728 \\ 0.6349 & 0.7136 \end{bmatrix}$$

with $\|F\|_F=3.8423$, $\|G\|_F=2.2825$, $\|K_{1R}\|_F=39.823$, $\|K_{2R}\|_F=40.803$, and $\kappa=30.312$.

Table 1 displays the numerical results for Example 4.1, where Γ_R is chosen as in (4.1). We observe from Table 1 that the method in [3] behaves better than the other methods in reducing feedback norms while Algorithm 1 work more effective than the methods in [4] and [3] in reducing the condition number of the closed-loop eigenvector matrix Y.

Table 1: Numerical results for Example 4.1.

Alg.	$\|F\|_F$	$\|G\|_F$	$\|K_{1R}\|_F$	$\|K_{2R}\|_F$	κ	`tol1.`	`tol2.`	`tol3.`	`CT.`
Theorem 3.5 in [4]	57.238	58.544	861.27	862.31	$1.83\cdot 10^3$	$9.97\cdot 10^{-11}$	$2.63\cdot 10^{-14}$	$1.89\cdot 10^{-13}$	0.0240
Method in [3]	3.4659	2.0935	$1.68\cdot 10^3$	$1.68\cdot 10^3$	165.47	$1.11\cdot 10^{-12}$	$1.29\cdot 10^{-14}$	$1.53\cdot 10^{-14}$	0.1310
Alg. 1 with $\alpha=0.5$	3.7999	2.5007	2.1765	3.2956	20.532	$7.33\cdot 10^{-13}$	$3.04\cdot 10^{-16}$	$1.49\cdot 10^{-14}$	0.1320
Alg. 1 with $\alpha=1.0$	3.8423	2.2825	39.823	40.803	30.312	$6.99\cdot 10^{-13}$	$1.44\cdot 10^{-15}$	$1.47\cdot 10^{-14}$	0.1310

Example 4.2. [2,10] Consider the multiple-input control system (1.1) with $n=10$ and $m=2$, where

$$M=4I_n,\quad C=4I_n,\quad K=\begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0\\ -1 & 2 & -1 & \cdots & 0 & 0\\ 0 & -1 & 2 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & \cdots & -1 & 2 & -1\\ 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix},\quad B=\begin{bmatrix} 1 & 0\\ 0 & 0\\ \vdots & \vdots\\ 0 & 0\\ 0 & 1\\ 2 & -3 \end{bmatrix}.$$

There are 20 open-loop eigenvalues. We reassign the 11-th and 12-th eigenvalues $\{0.0000, -0.0251\}$ to $\{-0.4+0.1\mathrm{i}, -0.4-0.1\mathrm{i}\}$ and keep the remaining eigenpairs unchanged.

The numerical results for Example 4.2 are given in Table 2, where Γ_R is chosen as follows:

$$\Gamma_R=\sqrt{2}\begin{bmatrix} 1 & 2\\ 3 & 4 \end{bmatrix}.$$

We observe from Table 2 that, in terms of feedback norms, the method in [3] work much more effective than the other methods while Algorithm 1 perform much better than the methods in [3,4] in terms of the condition number of Y.

Table 2: Numerical results for Example 4.2.

Alg.	$\|F\|_F$	$\|G\|_F$	$\|K_{1R}\|_F$	$\|K_{2R}\|_F$	κ	tol1.	tol2.	tol3.	CT.
Theorem 3.5 in [4]	5.1751	5.1695	$1.70\cdot10^3$	$1.70\cdot10^3$	$3.25\cdot10^3$	$1.65\cdot10^{-15}$	$1.06\cdot10^{-14}$	$1.59\cdot10^{-14}$	0.0230
Method in [3]	2.8800	2.8451	$2.07\cdot10^3$	$2.08\cdot10^3$	$1.72\cdot10^3$	$1.71\cdot10^{-15}$	$2.07\cdot10^{-14}$	$1.56\cdot10^{-14}$	0.1170
Alg. 1 with $\alpha=0.5$	2.9562	2.9207	1.1553	1.4585	145.387	$2.07\cdot10^{-16}$	$1.13\cdot10^{-16}$	$1.56\cdot10^{-14}$	0.1170
Alg. 1 with $\alpha=1.0$	3.3772	3.3622	0.9329	1.4890	70.490	$5.14\cdot10^{-16}$	$2.58\cdot10^{-16}$	$1.56\cdot10^{-14}$	0.1140

To further illustrate the effectiveness of our algorithms, in Fig. 1, we give the final values of the cost function J_1 and J_2 computed by Algorithm 1 with $\alpha=1.0$ for Examples 4.1-4.2 for different choices of Γ_R.

We observe from Fig. 1 that, for Example 4.1, Algorithm 1 with $\alpha=1.0$ gives different local minima of J_1 and J_2 for different choices of Γ_R while, for Example 4.2, Algorithm 1 with $\alpha=1.0$ gives almost the same values of J_1 and J_2 for different choices of Γ_R.

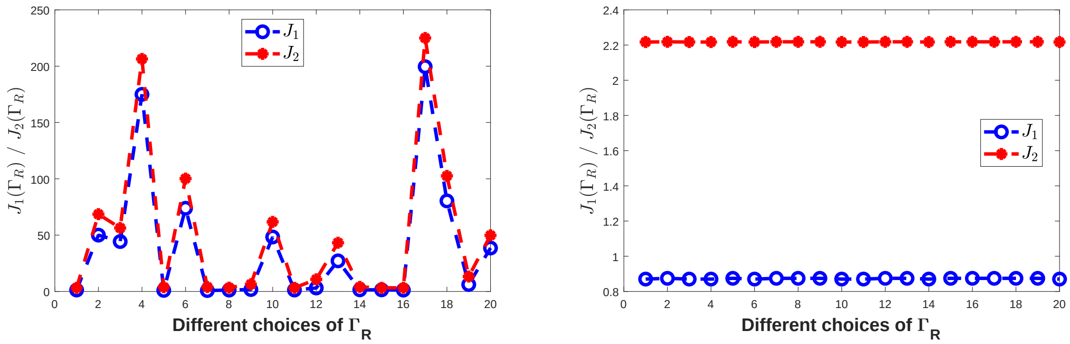

Figure 1: Values of J_1 and J_2 for Example 4.1 (left) and Example 4.2 (right) with different choices of Γ_R.

5 Conclusions

In this paper, we propose a receptance-based optimization approach for solving the minimum norm and robust partial quadratic eigenvalue assignment problem for multiple-input vibration control systems. We provide a new cost function for measuring the robustness and the feedback norms simultaneously. In particular, we measure the robustness by orthogonalizing the closed-loop eigenvector matrix as much as possible. By using the measured receptances, the system matrices, and a few undesired open-loop eigenpairs, we establish the explicit expression of the gradient of the cost function. To further improve the effectiveness, we also present the real form of our method. Numerical results show that the proposed method is effective. One may measure the robustness by using the condition number of Y as in [11] or the sensitivity measurement as in [25]. This needs further study.

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