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Radial Transonic Shock Solutions to Euler-Poisson System with Varying Background Charge in an Annulus

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Abstract. This paper concerns both the structural and dynamical stabilities of radially symmetric transonic shock solutions for two-dimensional Euler-Poisson system in an annulus. The density of fixed, positively charged background ions is allowed to be different constants in supersonic and subsonic regimes. First, the existence and structural stability of a steady transonic shock solution are obtained by the monotonicity between the shock location and the density on the outer circle. Second, any radially symmetric transonic shock solution with respect to small perturbations of the initial data is shown to be dynamically stable. The proof relies on the decay estimates and coupled effects from electric field and geometry of the annulus, together with the methods from [18]. These results generalize previous stability results on transonic shock solutions for constant background charge.

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1 Introduction and main results

The propagation of electrons in submicron semiconductor devices and plasma is governed by the Euler-Poisson equations [22]. In this paper, we focus on the two-dimensional Euler-Poisson equations in an annulus

$$\Omega=\left\{(x,y): 0<r_1<r=\sqrt{x^2+y^2}<r_2<+\infty\right\}$$

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as follows:

$$\begin{cases} \rho_t+(\rho u_1)_x+(\rho u_2)_y=0, \\ (\rho u_1)_t+(\rho u_1^2)_x+(\rho u_1 u_2)_y+p_x=\rho E_1, \\ (\rho u_2)_t+(\rho u_1 u_2)_x+(\rho u_2^2)_y+p_y=\rho E_2, \\ (E_1)_x+(E_2)_y=\rho-b, \end{cases} \tag{1.1}$$

where $\boldsymbol{u}=(u_1,u_2)$ and $\rho>0$ represent the macroscopic particle velocity field and electron density, respectively. $\boldsymbol{E}=(E_1,E_2)$ is the electric field generated by the Coulomb force of particles, and $b>0$ stands for the density of fixed, positively charged background ions. The pressure p is given by $p=A\rho^\gamma (A>0,\gamma>1)$ and thus satisfies

$$p(0)=0, \quad p'>0, \quad p''>0 \quad \text{for} \quad \rho>0, \quad p(+\infty)=+\infty.$$

Moreover, the local sound speed c and Mach number M are defined by $c(\rho)=\sqrt{p'(\rho)}$ and $M=|\boldsymbol{u}|/c$, respectively. The flow is called supersonic if $|\boldsymbol{u}|>c(\rho)$; subsonic if $|\boldsymbol{u}|<c(\rho)$; sonic if $|\boldsymbol{u}|=c(\rho)$. The pure supersonic and subsonic flows have been studied by many people (see [2,3,5,11,21,30] and references therein). On the other hand, the discontinuous transonic flow, that is, transonic shock contains a free boundary (shock) on the left of the subsonic region. This leads to some essential difficulties for mathematical analysis of transonic shock solutions.

For Euler system, Courant and Friedrichs [10] described the transonic shock phenomena in a de Laval nozzle. According to the above phenomena, there are numerous significant results about the existence and stability of steady transonic shock solutions to Euler system in a nozzle (see [7–9, 14, 17, 29, 31, 32] and references therein). The global in time stability of transonic shock solutions was investigated by Liu [16] and Xin and Yin [32]. In [16], the author used a wave front tracking variant of Glimm's scheme to prove that, for quasi-one-dimensional system, a weak transonic shock solution is dynamically stable in divergent nozzle and dynamically unstable in convergent nozzle. These results were improved by Rauch *et al.* [27]. Other related results about the transonic flows can be found [6, 28, 33].

Concerning Euler-Poisson system, there have been only a few results for the transonic shock solutions. In one-dimensional case, a transonic shock problem with a linear pressure $p(\rho)=k\rho$ and special boundary conditions was discussed in [1]. For more general case, Gamba [13] constructed a transonic shock solution, which may contain boundary layers due to the technical limit. A thorough study of the transonic shock solutions for one-dimensional Euler-Poisson equations with a constant background charge $b=b_0$ in flat nozzles was given by Luo and Xin [19], where the existence, non-existence, uniqueness and non-uniqueness of solutions with transonic shock were established. Bae and Park [4] established the well-posedness of radial transonic shock problem for Euler-Poisson equations in a two-dimensional convergent nozzle under a strong effect of self-generated electric field. Luo *et al.* [18] proved that a steady transonic shock solution with supersonic background charge, obtained in [19], is structurally stable under small perturbations of the background charge and is dynamically stable with respect to small perturbation of the initial data. Later on, Duan *et al.* [12] extended the one-dimensional results [18] to the quasi-one-dimensional case, and showed that there indeed exists a dynamically stable transonic shock solution for the Euler-Poisson system in convergent nozzles, which is not true for the Euler system. These one-dimensional or quasi-one-dimensional results [12, 18, 19] are closely related to the assumptions on the background charge. Physically, the background charge b represents the propagation state of particles in semiconductor devices. b may be different constants in supersonic region and subsonic region. Thus, it would be interesting to study transonic flows with jumping background charge. Motivated by the pioneer work [18], in this paper, our purpose is to establish stability results for radial symmetric transonic shock solutions to Euler-Poisson equations in a two-dimensional annulus with jumping background charge. These results generalize previous stability results on transonic shock solutions for constant background charge [19], or background charge around a constant [12, 18].

In particular, we consider the two-dimensional Euler-Poisson equations (1.1) in an annulus

$$\Omega=\left\{(x,y): 0<r_1<r=\sqrt{x^2+y^2}<r_2<+\infty\right\},$$

where the background charge b is assumed to be

$$b=\begin{cases} b_1, & \text{if} \quad |\boldsymbol{u}|>c, \\ b_2, & \text{if} \quad |\boldsymbol{u}|<c. \end{cases} \tag{1.2}$$

Here b_1,b_2 are positive constants. Since we focus on the radical symmetric flows, in the polar coordinate (r,θ), let the density, velocity field and electric field be $\rho(r)$, $\boldsymbol{u}=u(r)\boldsymbol{e_r}$ and $\boldsymbol{E}=E(r)\boldsymbol{e_r}$, respectively, where

$$r=\sqrt{x^2+y^2}, \quad \boldsymbol{e_r}=\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}.$$

(We assume that the velocity is always positive.)
Therefore, (1.1) can be written as

$$\begin{cases} (r\rho)_t+(r\rho u)_r=0, \\ (r\rho u)_t+(r\rho u^2)_r+rp_r=r\rho E, \\ (rE)_r=r(\rho-b). \end{cases} \tag{1.3}$$

The corresponding time-independent equations read,

$$\begin{cases} (r\rho u)_r=0, & (1.4\text{a}) \\ (r\rho u^2)_r+rp_r=r\rho E, & (1.4\text{b}) \\ (rE)_r=r(\rho-b). & (1.4\text{c}) \end{cases}$$

We aim to investigate two distinct stability results. One result concerns the existence and stability of steady transonic shock solutions to (1.4). The other concerns the global in time stability for solutions whose initial data is a small perturbation of a stationary transonic shock solution to (1.4).

First, consider the boundary value problem for (1.4) in $0<r_1<r<r_2$ with the boundary conditions

$$\begin{aligned}&(\rho,u,E)(r_1)=(\rho_l,u_l,E_l) \quad \text{for} \quad E_l>0,\\&\rho(r_2)=\rho_e.\end{aligned} \tag{1.5}$$

The flow velocity is assumed to be supersonic on the inner circle and subsonic on the outer circle of the annulus, i.e

$$\rho_l>0, \quad u_l>\sqrt{p'(\rho_l)}, \quad u_e<\sqrt{p'(\rho_e)}.$$

By Eq. (1.4a), $r\rho u(r)=r_1\rho_l u_l:=J\,(r_1\le r\le r_2)$ and the velocity is given by $u=J/(r\rho)$. Thus the boundary value problem (1.4)-(1.5) can be reduced to

$$\begin{cases}\left(\dfrac{J^2}{r^2\rho}+p\right)_r=\rho E-\dfrac{J^2}{r^3\rho}, & (1.6a)\\ (rE)_r=r(\rho-b), & (1.6b)\\ (\rho,E)(r_1)=(\rho_l,E_l),\ \rho(r_2)=\rho_e. & (1.6c)\end{cases}$$

Let ρ_s be the unique solution of the equation

$$\frac{J^2}{r^2\rho^2}=p'(\rho),$$

which is the density for the sonic state.

The steady transonic shock solution to (1.4)-(1.5) is defined as follows.

Definition 1.1. *The piecewise smooth function*

$$(\rho,E)=\begin{cases}(\rho_-,E_-), & r_1\le r<r_s,\\ (\rho_+,E_+), & r_s<r\le r_2\end{cases}$$

is said to be a steady transonic shock solution with shock located at $r_s\in(r_1,r_2)$ to the boundary value problem (1.4)-(1.5) *in Ω provided*

(i) (ρ_-,E_-) and (ρ_+,E_+) satisfy (1.4)-(1.5) *in (r_1,r_s) and (r_s,r_2) piecewisely, with $M_->1$ and $M_+<1$, where M is the Mach number.*

(ii) The Rankine-Hugoniot conditions hold acrossing the shock at $r=r_s$,

$$\begin{aligned}&\rho_-u_-(r_s)=\rho_+u_+(r_s),\\&\rho_-u_-^2(r_s)+p(\rho_-(r_s))=\rho_+u_+^2(r_s)+p(\rho_+(r_s)),\\&E_-(r_s)=E_+(r_s).\end{aligned} \tag{1.7}$$

(iii) The Lax's entropy condition holds at $r=r_s$,

$$\rho_+(r_s)>\rho_s>\rho_-(r_s). \tag{1.8}$$

Remark 1.1. The Rankine-Hugoniot conditions (1.7) are equivalent to

$$p(\rho_-(r_s))+\frac{J^2}{r_s^2\rho_-(r_s)}=p(\rho_+(r_s))+\frac{J^2}{r_s^2\rho_+(r_s)}, \tag{1.9a}$$

$$E_-(r_s)=E_+(r_s). \tag{1.9b}$$

Furthermore, Eq. (1.4b) becomes

$$\left(\frac{1}{2}u^2+\frac{\gamma p}{(\gamma-1)\rho}\right)_r=E$$

since $p=A\rho^\gamma$. Therefore, we have

$$\frac{1}{2}u_-^2(r_s)+\frac{\gamma p(\rho_-(r_s))}{(\gamma-1)\rho_-(r_s)}=\frac{1}{2}u_+^2(r_s)+\frac{\gamma p(\rho_+(r_s))}{(\gamma-1)\rho_+(r_s)} \tag{1.10}$$

in the Rankine-Hugoniot conditions as well.

The structural stability of the steady transonic shock solution is analyzed in the following theorem.

Theorem 1.1. *For given positive constants J,r_1,r_2 and background charge $0<b<\rho_s$ as in* (1.2)*, there exists an interval $I=(\overline{\rho},\underline{\rho})$. Then for any density on the outer circle $\rho_e\in I$, the boundary value problem* (1.4)-(1.5) *admits a unique transonic shock solution (ρ,u,E) as in Definition* 1.1*. Moreover, the position of the transonic shock is automatically adjusted according to ρ_e.*

Second, we investigate the dynamical stability for the steady transonic shock solutions. For the background charge $0<b<\rho_s$ as in (1.2) and a given constant $\bar{J}>0$, let $(\bar{\rho},\bar{u},\bar{E})(r)$ be the steady transonic shock solution to the boundary value problem (1.4)-(1.5) with shock location r_s. In the following, we consider the initial boundary value problem for (1.3) with the initial data

$$(\rho,u,E)(0,r)=(\rho_0,u_0,E_0)(r),\quad r_1\le r\le r_2 \tag{1.11}$$

and the boundary conditions

$$(\rho,u,E)(t,r_1)=(\rho_l,\bar{J}/\rho_l,E_l),\quad \rho(t,r_2)=\rho_e,\quad t>0, \tag{1.12}$$

where ρ_l,E_l,ρ_e are the same as in (1.5). The initial data is of the form

$$(\rho_0,u_0)(r)=\begin{cases}(\rho_{0-},u_{0-})(r), & r_1\le r\le\tilde{r}_0,\\ (\rho_{0+},u_{0+})(r), & \tilde{r}_0\le r\le r_2,\end{cases} \tag{1.13}$$

and

$$E_0(r)=\frac{r_1}{r}E_l+\frac{1}{r}\int_{r_1}^{r}s\big(\rho_0(s)-b_1\big)\,\mathrm{d}s, \qquad \text{if} \quad r\in(r_1,\tilde r_0), \tag{1.14}$$

$$E_0(r)=\frac{r_1}{r}E_l+\frac{1}{r}\int_{r_1}^{\tilde r_0}s\big(\rho_0(s)-b_1\big)\,\mathrm{d}s+\frac{1}{r}\int_{\tilde r_0}^{r}s\big(\rho_0(s)-b_2\big)\,\mathrm{d}s, \quad \text{if} \quad r\in(\tilde r_0,r_2). \tag{1.15}$$

At $r=\tilde r_0$, the Rankine-Hogoniot conditions hold

$$\begin{aligned}&\big(p(\rho_{0+})+\rho_{0+}u_{0+}^2-p(\rho_{0-})-\rho_{0-}u_{0-}^2\big)\cdot\big(\rho_{0+}-\rho_{0-}\big)(\tilde r_0)\\ =&\big(\rho_{0+}u_{0+}-\rho_{0-}u_{0-}\big)^2(\tilde r_0).\end{aligned} \tag{1.16}$$

Furthermore, we assume that the initial data is a small perturbation of $(\bar\rho,\bar u,\bar E)$ in the sense that

$$\begin{aligned}&|\tilde r_0-r_s|+\|(\rho_{0-},u_{0-})-(\bar\rho_-,\bar u_-)\|_{H^{k+2}([r_1,\hat r_0])}\\ &\qquad+\|(\rho_{0+},u_{0+})-(\bar\rho_+,\bar u_+)\|_{H^{k+2}([\check r_0,r_2])}<\epsilon\end{aligned} \tag{1.17}$$

for some small $\epsilon>0$ and some integer $k\geq 15$, where $\hat r_0=\max\{r_s,\tilde r_0\}$ and $\check r_0=\min\{r_s,\tilde r_0\}$. A discussion on the regularity assumption can be found in [20,23].

We give the definition of piecewise smooth entropy solutions to (1.3) as follows.

Definition 1.2. *A piecewise smooth entropy solution of* (1.3) *at $r=r(t)$ to Euler-Poisson equations* (1.3) *in Ω can be formulated as*

$$(\rho,u,E)=\begin{cases}(\rho_-,u_-,E_-), & r_1\leq r\leq r(t),\\ (\rho_+,u_+,E_+), & r(t)\leq r\leq r_2,\end{cases}$$

where

(i) (ρ_-,u_-,E_-) and (ρ_+,u_+,E_+) are C^1 smooth solutions of (1.3), (1.11)-(1.12) *in the regions $\{(t,r)\,|\,t\geq 0, r_1\leq r\leq r(t)\}$ and $\{(t,r)\,|\,t\geq 0, r(t)\leq r\leq r_2\}$ respectively, with $M_->1$ and $M_+<1$, where M is the Mach number.*

(ii) The Rankine-Hugoniot conditions hold at $r=r(t)$,

$$\rho u\big(t,r(t)+\big)-\rho u\big(t,r(t)-\big)=\big(\rho(t,r(t)+)-\rho(t,r(t)-)\big)r'(t), \tag{1.18a}$$

$$\begin{aligned}&\big(p(\rho)+\rho u^2\big)\big(t,r(t)+\big)-\big(p(\rho)+\rho u^2\big)\big(t,r(t)-\big)\\ =&\big(\rho u(t,r(t)+)-\rho u(t,r(t)-)\big)r'(t),\end{aligned} \tag{1.18b}$$

$$E\big(t,r(t)+\big)=E\big(t,r(t)-\big). \tag{1.18c}$$

(iii) The Lax geometric entropy conditions hold at $r=r(t)$,

$$\begin{aligned}&\Big(u-\sqrt{p'(\rho)}\Big)\big(t,r(t)-\big)>r'(t)>\Big(u-\sqrt{p'(\rho)}\Big)\big(t,r(t)+\big),\\ &\Big(u+\sqrt{p'(\rho)}\Big)\big(t,r(t)+\big)>r'(t).\end{aligned}$$

The dynamical stability of transonic shock solutions to (1.3) is given in the following theorem.

Theorem 1.2. *Suppose* $(\bar{\rho},\bar{u},\bar{E})(r)$ *obtained in Theorem* 1.1, *is a steady transonic shock solution to problem* (1.4)-(1.5) *with shock location* r_s *and* $\bar{E}_+(r_s)>0$. *Then there exists* $\epsilon_0>0$ *such that for any* $\epsilon\le\epsilon_0$, *if the initial data* (ρ_0,u_0,E_0) *satisfy* (1.13)-(1.17) *and the k+2-th order compatibility conditions at* $r=r_1,r=r_s$ *and* $r=r_2$, *then the initial boundary value problem* (1.3), (1.11)-(1.12) *admits a unique piecewise smooth entropy solution* $(\rho,u,E)(t,r)$ *for* $(t,r)\in[0,\infty)\times[r_1,r_2]$ *containing a single transonic shock at* $r=r(t)$ *with* $r(0)=\tilde{r}_0$. *Moreover, there exist* $T_*>0$ *and* $\lambda>0$ *such that*

$$(\rho_-,u_-,E_-)(t,r)=(\bar{\rho}_-,\bar{u}_-,\bar{E}_-)(r) \quad \textit{for} \quad r_1\le r<r(t), \quad t>T_*$$

and

$$\begin{aligned}
&\|(\rho_+,u_+)(\cdot,t)-(\bar{\rho}_+,\bar{u}_+)(\cdot)\|_{\mathbf{W}^{k-7,\infty}(r(t),r_2)}+\|E_+(\cdot,t)-\bar{E}_+(\cdot)\|_{\mathbf{W}^{k-6,\infty}(r(t),r_2)}\le C\epsilon e^{-\lambda t},\\
&\sum_{m=0}^{k-6}\left|\partial_t^m\big(r(t)-r_s\big)\right|\le C\epsilon e^{-\lambda t} \quad \textit{for} \quad t\ge 0.
\end{aligned}$$

Remark 1.2. The compatibility conditions for the initial boundary value problems for hyperbolic equations were discussed in detail in [20,23,25].

In this paper, the background charge b is assumed to be two different constants in supersonic and subsonic regions respectively. Here, the discontinuity of b is actually determined by shock fronts. Thus the function b depends on transonic shock solutions, and the equations change across shocks. Accordingly, the solution trajectories have more plentiful phenomena than the ones of constant background charge. Compared with previous studies, one of the main difficulties is the uncertainty of the jumping of b which depends only on shock locations. In addition, the shock is a free boundary connecting supersonic and subsonic regions. The key approach is that we formulate the dynamical stability problem into analyzing the second order quasi-linear hyperbolic equation by using the difference of shock locations $r(t)-r_s$ and the difference of background charges b_2-b_1. Applying the idea in [18], we obtain additional a priori estimate involving the difference of background charges and shock locations, also the dynamical stability is established in Theorem 1.2 with decay estimates and coupled effects from electric field and geometry.

The rest of this paper is organized as follows. In Section 2, the structural stability of radially symmetric steady transonic shock solutions to (1.4) is established by the stability analysis for subsonic/supersonic solutions, and the monotonicity between the shock position and the density on the outer circle. In Subsection 3.1, the dynamical stability problem is transformed to the global well-posedness of a free boundary problem. Then, after analyzing the associated linearized problem in Subsection 3.2, we state our main proposition (Proposition 3.1), which will lead to Theorem 1.2. Subsection 3.3 is devoted to prove Proposition 3.1 by deriving uniform a priori estimates on the solutions, and thus, Theorem 1.2 holds.

2 Structural stability of steady transonic shock solutions

In this section, we investigate the steady transonic shock solutions to (1.4)-(1.5) under jumping background charge $b\in(0,\rho_s)$ as in (1.2).

Lemma 2.1 concerns the existence of a supersonic solution in the annulus.

Lemma 2.1 (Existence of supersonic solutions)**.** *For any given interval $[r_*,r^*]\subseteq[r_1,r_2]$, suppose (ρ_*,u_*,E_*) is any supersonic data at $r=r_*$, then Eq.* (1.4) *with the boundary condition $(\rho,u,E)(r_*)=(\rho_*,u_*,E_*)$ admits a C^1 supersonic solution $(\rho,u,E)(r)$ on $[r_*,r^*]$.*

Proof. A direct computation from (1.4) shows that a C^1 non-sonic solution (ρ,u,E) satisfies

$$\begin{cases} \dfrac{\mathrm{d}\rho}{\mathrm{d}r}=\dfrac{\rho E}{p'(\rho)-u^2}+\dfrac{\rho u^2}{r\big(p'(\rho)-u^2\big)}, \\ \dfrac{\mathrm{d}u}{\mathrm{d}r}=-\dfrac{uE}{p'(\rho)-u^2}-\dfrac{uc^2}{r\big(p'(\rho)-u^2\big)}, \\ \dfrac{\mathrm{d}E}{\mathrm{d}r}=-\dfrac{E}{r}+\rho-b. \end{cases} \tag{2.1}$$

Eqs. (2.1) describe the variation of states of subsonic and supersonic flows in the annulus and its integral curve is called subsonic curve or supersonic curve. By the local existence theory of ODE system, one can define L_{r_*} to be the lifespan of the supersonic solution (ρ,u,E) to (1.4) with $(\rho,u,E)(r_*)=(\rho_*,u_*,E_*)$.

For any $\tau\in(0,\rho_s)$, set

$$r(\tau)=\int_{\rho_*}^{\tau}\frac{p'(\rho)-J^2/(r^2\rho^2)}{\rho E+J^2/(r^3\rho)}\,\mathrm{d}\rho.$$

Note that in supersonic region, $0<\rho<\rho_s$. Thus

$$r'(\tau)=\frac{p'(\tau)-J^2/(r^2\tau^2)}{\tau E_-(r)+J^2/(r^3\tau)}<0$$

for any $0<\tau<\rho_s$ and $E(\tau)>0$. The monotonicity of $r(\tau)$ in $(0,\rho_s)$ leads to $L_{r_*}\geq r^*$. That is, Eq. (1.4) admits a C^1 supersonic solution on $[r_*,r^*]$. □

Now, for any left supersonic state (ρ_-,u_-,E_-) satisfying $\rho_-<\rho_s$, via a shock at $r=s$, one can connect it to a unique right state (ρ_+,u_+,E_+) at $r=s$, where

$$\begin{cases} u_+=\dfrac{(\rho_-u_-^3)/2+[\gamma/(\gamma-1)]p_-u_-}{[(\gamma+1)/(2(\gamma-1))]\rho_-u_-}\cdot\dfrac{1}{u_-}=\dfrac{\gamma-1}{\gamma+1}u_-+\dfrac{2\gamma}{(\gamma+1)J}sp_-, \\ \rho_+=\dfrac{J}{su_+}=\dfrac{1}{s}\cdot\dfrac{J^2(\gamma+1)}{(\gamma-1)u_-J+2\gamma sp_-}, \\ p_+=\dfrac{Ju_-}{s}+p_--\dfrac{J}{s}\left(\dfrac{\gamma-1}{\gamma+1}u_-+\dfrac{2\gamma}{(\gamma+1)J}sp_-\right), \\ E_+=E_- \end{cases} \tag{2.2}$$

are determined by the Rankine-Hugoniot conditions (1.7) and (1.10). Moreover, the entropy condition (1.8) implies that

$$u_+{}^2 - c_+{}^2 = u_+{}^2 - \gamma \frac{p_+}{\rho_+} = u_+^2 - \frac{\gamma s p_+ u_+}{J} = u_+ u_- \left(\frac{c_-^2}{u_-^2} - 1 \right) < 0,$$

which means (ρ_+, u_+, E_+) is a subsonic state.

Remark 2.1. For any given supersonic initial data (ρ_l, E_l) in (1.5), by Lemma 2.1 one can always solve the boundary value problem (1.4)-(1.5) to get a C^1 supersonic solution $(\rho_-, u_-, E_-)(r)$ on $[r_1, s)$ for any $s \in (r_1, r_2)$. Then one obtains the right subsonic state $(\rho_+, u_+, E_+)(s)$ according to (2.2). Now fix $s \in (r_1, r_2)$, the same argument as in Lemma 2.1 shows that the problem (1.4) with the initial data $(\rho_+, u_+, E_+)(s)$ at $r = s$ admits a C^1 subsonic solution $(\rho_+, u_+, E_+)(r; s)$ on $(s, r_2]$. Consequently, a solution with transonic shock located at $r = s$ to the boundary value problem (1.4)-(1.5) on $[r_1, r_2]$ is obtained.

The monotone relation between the shock location and the density on the outer circle is given by the following lemma.

Lemma 2.2 (Monotone Dependence). *For any $s \in (r_1, r_2)$ and any given supersonic initial data (ρ_l, u_l, E_l) in* (1.5)*, let $(\rho, u, E)(r)$ be the transonic shock solution located at $r = s$ to the boundary value problem* (1.4)-(1.5) *on $[r_1, r_2]$. Assume that $\rho_+(r_2; s)$ is the density of the transonic shock solution $(\rho, u, E)(r)$ at $r = r_2$. Then $\rho_+(r_2; s)$ is a decreasing continuous function of s in (r_1, r_2).*

Proof. For any $s \in (r_1, r_2)$ and any given supersonic initial data (ρ_l, u_l, E_l) in (1.5), the transonic shock solution $(\rho, u, E)(r)$ located at s can be represented as

$$(\rho, u, E) = \begin{cases} (\rho_-, u_-, E_-)(r), & r_1 \le r < s, \\ (\rho_+, u_+, E_+)(r; s), & s < r \le r_2. \end{cases}$$

Then (2.2) and the continuous dependence of initial data for solutions to ODE system yield that $\rho_+(r_2; s)$ is a continuous function of s. The right subsonic state $\rho_+(s)$ at $r = s$ is also a continuous function of s and its image is called the $R - H$ curve of ρ.

In order to obtain the monotone relation, one needs to compare the subsonic curve $\rho_+(r; s)$ and the R-H curve $\rho_+(s)$. Differentiating Eq. (1.9a) in Rankine-Hugoniot conditions (1.9) with respect to s, we have

$$\begin{aligned} &\left(p'(\rho_-(s)) - \frac{J^2}{s^2 \rho_-(s)^2} \right) \frac{d\rho_-(s)}{ds} - \frac{2J^2}{s^3 \rho_-(s)} \\ = &\left(p'(\rho_+(s)) - \frac{J^2}{s^2 \rho_+(s)^2} \right) \frac{d\rho_+(s)}{ds} - \frac{2J^2}{s^3 \rho_+(s)}. \end{aligned} \tag{2.3}$$

Note that

$$s = \int_{\rho_l}^{\rho_-(s)} \frac{p'(\rho_-(r)) - J^2/(r^2 \rho_-(r)^2)}{\rho(r) E(r) + J^2/(r^3 \rho_-(r))} \mathrm{d}\rho. \tag{2.4}$$

Differentiating (2.4) with respect to s leads to

$$\frac{\mathrm{d}\rho_-(s)}{\mathrm{d}s}=\frac{\rho_-(s)E_-(s)+J^2/\left(s^3\rho_-(s)\right)}{p'\left(\rho_-(s)\right)-J^2/\left(s^2\rho_-(s)^2\right)},$$

which together with (2.3) implies

$$\frac{\mathrm{d}\rho_+(s)}{\mathrm{d}s}=\frac{\rho_-(s)E_-(s)-J^2/\left(s^3\rho_-(s)\right)+2J^2/\left(s^3\rho_+(s)\right)}{p'\left(\rho_+(s)\right)-J^2/\left(s^2\rho_+(s)^2\right)}.$$

On the other hand, it follows from (2.1) and the fact $r\rho u(r)=J$ that the subsonic curve issuing from $(s,\rho_+(s))$ satisfies

$$\left.\frac{\mathrm{d}\rho_+(r;s)}{\mathrm{d}r}\right|_{r=s}=\frac{\rho_+(s)E_+(s)+J^2/\left(s^3\rho_+(s)\right)}{p'\left(\rho_+(s)\right)-J^2/\left(s^2\rho_+(s)^2\right)}.$$

Thus, at $r=s$, one has

$$\left.\frac{\mathrm{d}\rho_+(r,s)}{\mathrm{d}r}\right|_{r=s}-\frac{\mathrm{d}\rho_+(s)}{\mathrm{d}s}=\frac{\left(\rho_+(s)-\rho_-(s)\right)E_-(s)}{p'\left(\rho_+(s)\right)-J^2/\left(s^2\rho_+(s)^2\right)}+\frac{J^2/\left(s^3\rho_-(s)\right)-J^2/\left(s^3\rho_+(s)\right)}{p'\left(\rho_+(s)\right)-J^2/\left(s^2\rho_+(s)^2\right)}>0,$$

which implies that the ρ-subsonic curve issuing from $(s,\rho_+(s))$ lies always above the R-H curve of ρ for $r>s$. Therefore, $\rho_+(r_2;r_1)>\rho_+(r_2)=\rho_+(r_2;r_2)$ and $\rho_+(r_2;s)$ is decreasing on s according to the uniqueness of solution to Eqs. (2.1). □

Based on the monotonicity between the shock location and the density on the outer circle in Lemma 2.2, we finish the proof of Theorem 1.1.

Proof of Theorem 1.1. Since $\rho_+(r_2;s)$ is a decreasing continuous function of shock locations, one has $\mathrm{d}\rho_+(r_2)/\mathrm{d}s<0$. Set

$$\underline{\rho}=\rho_+(r_2;r_2),\quad \overline{\rho}=\rho_+(r_2;r_1),\quad I=(\underline{\rho},\overline{\rho}).$$

For any $\rho_e\in I$, the shock location s can be uniquely determined by $\rho_+(r_2;s)=\rho_e$, and consequently, there exists a unique transonic shock solution $(\rho,u,E)(r)$ located at $r=s$ to the boundary value problem (1.4)-(1.5) as in Definition 1.1. □

3 Dynamical stability of transonic shock solutions

This section is devoted to establish the dynamical stability of transonic shock solutions for the system in (1.3).

First, it follows from the argument in [15] that there exists a local (in time) piecewise smooth solution to Euler-Poisson equations (1.3).

Lemma 3.1 ([15, 18], Local existence). *Let $(\bar\rho,\bar u,\bar E)$ be the steady transonic shock solution to* (1.4)-(1.5). *Suppose that the initial data (ρ_0,u_0,E_0) satisfies* (1.13), (1.16), (1.17) *and the $k+2$-th order compatibility conditions at $r=r_1$, $r=r_s$ and $r=r_2$. Then there exists a time $T^*>0$, and a piecewise smooth solution (ρ,u,E) containing a single shock $r=r(t)$ (with $(r(0)=\tilde r_0)$ for the Euler-Poisson equations* (1.3) *in the form of Definition* 1.2 *on $[0,T^*]$. Furthermore, if ϵ in* (1.17) *is sufficiently small, then for some $0<T_*<T^*$,*

$$(\rho_-,u_-,E_-)(t,r)=(\bar\rho_-,\bar u_-,\bar E_-)(r)\quad in\ \ (T_*,+\infty)\times\big[r_1,r(t)\big). \tag{3.1}$$

Without loss of generality, we can assume that $T^*=0$ for simplicity. We intend to extend the local (in time) solution in Lemma 3.1 to all time $t>0$. In view of (3.1), we focus on the region $r(t)<r\le r_2, t>0$ and formulate a free boundary value problem in Subsection 3.1. Some necessary uniform estimates will be stated in Proposition 3.1 and then Theorem 1.2 follows consequently.

3.1 Formulation of the problem

In view of the Eqs. (1.3) and Rankine-Hugoniot conditions (1.18), it holds that

$$\begin{aligned}
&rE_+(t,r)=r_1E_l+\int_{r_1}^{r(t)}y\big(\rho_-(t,y)-b_1\big)\mathrm{d}y+\int_{r(t)}^{r}y\big(\rho_+(t,y)-b_2\big)\mathrm{d}y,\\
&\partial_t\big(rE_+(t,r)\big)=-J_+(t,r)+\bar J+(b_2-b_1)r(t)r'(t).
\end{aligned}$$

Set

$$Y=r\big(E_+(t,r)-\bar E_+(r)\big)-\frac12(b_2-b_1)\big(r(t)^2-r_s^2\big).$$

We readily check that

$$Y_t=\bar J-J_+(t,r),\quad Y_r=r\big(\rho_+(t,r)-\bar\rho_+(r)\big). \tag{3.2}$$

According to (1.3), we have

$$(J_+-\bar J)_t+\partial_r\left(rp(\rho_+)+\frac{J_+^2}{r\rho_+}-rp(\bar\rho_+)-\frac{\bar J^2}{r\bar\rho_+}\right)-r\rho_+E_++r\bar\rho_+\bar E_+-p(\rho_+)+p(\bar\rho_+)=0.$$

Therefore, Y satisfies the following equation:

$$\begin{aligned}
&Y_{tt}+\partial_r\left(\frac{\bar J^2}{r\bar\rho_+}-\frac{(\bar J-Y_t)^2}{r\bar\rho_++Y_r}\right)+r\partial_r\left(p(\bar\rho_+)-p\left(\bar\rho_++\frac{Y_r}{r}\right)\right)+\bar E_+Y_r\\
&\quad+\frac{Y_r}{2r}(b_2-b_1)\big(r(t)^2-r_s^2\big)+\bar\rho_+Y+\frac{YY_r}{r}+\frac12(b_2-b_1)\big(r(t)^2-r_s^2\big)\bar\rho_+=0,
\end{aligned}\tag{3.3}$$

where $\partial_r=\partial/\partial_r$. Let

$$(\eta_0,\eta_1):=(t,r),\quad \sigma(t):=r(t)-r_s,\quad \partial_0:=\frac{\partial}{\partial_t},\quad \partial_1:=\frac{\partial}{\partial_r},\quad \partial_{ij}:=\frac{\partial^2}{\partial_i\partial_j},\quad i,j=0,1.$$

We can rewrite (3.3) as the following hyperbolic equation:

$$\begin{aligned}L(r,Y_t,Y_r,\sigma)Y:=&\sum_{0\le i,j\le 1}\bar{a}_{ij}(r,Y_t,Y_r,\sigma)\partial_{ij}Y+\sum_{0\le i\le 1}\bar{b}_i(r,Y_t,Y_r,\sigma)\partial_i Y\\&+\bar{c}(r,Y_t,Y_r,\sigma)Y+\bar{d}(r,Y_t,Y_r,\sigma)=0,\end{aligned}$$

where

$$\begin{aligned}
&\bar{a}_{00}(r,Y_t,Y_r,\sigma)=1,\quad \bar{a}_{01}(r,Y_t,Y_r,\sigma)=\bar{a}_{10}(r,Y_t,Y_r,\sigma)=\frac{\bar{J}-Y_t}{r\bar{\rho}_++Y_r},\\
&\bar{a}_{11}(r,Y_t,Y_r,\sigma)=-\left(p'\left(\bar{\rho}_++\frac{1}{r}Y_r\right)-\frac{(\bar{J}-Y_t)^2}{(r\bar{\rho}_++Y_r)^2}\right),\\
&\bar{b}_0(r,Y_t,Y_r,\sigma)=-\frac{2\bar{J}-Y_t}{(r\bar{\rho}_++Y_r)^2}\cdot\frac{\partial(r\bar{\rho}_+)}{\partial r},\\
&\bar{b}_1(r,Y_t,Y_r,\sigma)=-\int_0^1\left(p''\left(\bar{\rho}_++\theta\frac{Y_r}{r}\right)\right)\mathrm{d}\theta\cdot\frac{\partial\bar{\rho}_+}{\partial r}-\int_0^1\left(\frac{2\bar{J}^2}{(r\bar{\rho}_++\theta Y_r)^3}\right)\mathrm{d}\theta\cdot\frac{\partial(r\bar{\rho}_+)}{\partial r}\\
&\qquad\qquad+\bar{E}_++\frac{1}{r}\cdot p'(\bar{\rho}_++\frac{1}{r}Y_r)+\frac{1}{2r}(b_2-b_1)\big(\sigma(t)^2+2r_s\sigma(t)\big),\\
&\bar{c}(r,Y_t,Y_r,\sigma)=\bar{\rho}_++\frac{1}{r}Y_r,\quad \bar{d}(r,Y_t,Y_r,\sigma)=\frac{1}{2}(b_2-b_1)\big(\sigma(t)^2+2r_s\sigma(t)\big)\bar{\rho}_+.
\end{aligned}$$

Moreover,

$$\begin{aligned}L(r,0,0,0)Y=&Y_{tt}-\partial_r\left(\left(p'(\bar{\rho}_+)-\frac{\bar{J}^2}{(r\bar{\rho}_+)^2}\right)Y_r\right)+\partial_r\left(\frac{2\bar{J}}{r\bar{\rho}_+}Y_t\right)\\&+\frac{1}{r}\cdot p'(\bar{\rho}_+)Y_r+\bar{E}_+Y_r+\bar{\rho}_+Y=0.\end{aligned}$$

By the Rankine-Hugoniot conditions (1.18), one has

$$\begin{aligned}&\left(\frac{J_+\big(t,r(t)\big)-J_-\big(t,r(t)\big)}{r}\right)^2\\
=&\left(p(\rho_+)\big(t,r(t)\big)+\frac{J_+^2}{r^2\rho_+}\big(t,r(t)\big)-p(\rho_-)\big(t,r(t)\big)-\frac{J_-^2}{r^2\rho_-}\big(t,r(t)\big)\right)\\
&\times\big(\rho_+\big(t,r(t)\big)-\rho_-\big(t,r(t)\big)\big),\end{aligned}$$

where $J_-(t,r(t))=\bar{J}=r_1\rho_l u_l$. Then, it follows from Taylor expansions that,

$$\begin{aligned}&\Bigg(\left(p'(\bar{\rho}_+)-\frac{\bar{J}^2}{(r\bar{\rho}_+)^2}\right)\big(r(t)\big)\cdot\big(\rho_+\big(t,r(t)\big)-\bar{\rho}_+\big(r(t)\big)\big)+\frac{2\bar{J}}{r^2\bar{\rho}_+}\big(r(t)\big)\cdot\big(J_+\big(t,r(t)\big)-\bar{J}\big(r(t)\big)\big)\\
&\quad+\partial_r\left(p(\bar{\rho}_+)+\frac{\bar{J}^2}{r^2\bar{\rho}_+}-p(\bar{\rho}_-)-\frac{\bar{J}^2}{r^2\bar{\rho}_-}\right)(r_s)\cdot\big(r(t)-r_s\big)+R_1\Bigg)\end{aligned}$$

$$\times\big(\bar\rho_+(r_s)-\bar\rho_-(r_s)+R_2\big)-\left(\frac{J_+\big(t,r(t)\big)-\bar J\big(r(t)\big)}{r}\right)^2=0.$$

Here

$$\begin{aligned}
R_1=&\int_{\bar\rho_+(r(t))}^{\rho_+(t,r(t))}\left(p''(\theta)+\frac{2J_+^2}{r^2\rho_+^3}(t,\theta)\right)\cdot\big(\rho_+\big(t,r(t)\big)-\theta\big)\,\mathrm{d}\theta\\
&+\int_{\bar J}^{J_+(t,r(t))}\frac{2}{r^2\rho_+}(t,\theta)\cdot\big(J_+\big(t,r(t)\big)-\theta\big)\,\mathrm{d}\theta\\
&+\int_{r_s}^{r(t)}\left(\partial_{rr}\left(p(\bar\rho_+)+\frac{\bar J^2}{r^2\bar\rho_+}\right)-\partial_{rr}\left(p(\bar\rho_-)+\frac{\bar J^2}{r^2\bar\rho_-}\right)\right)(\theta)\cdot\big(r(t)-\theta\big)\mathrm{d}\theta,\\
R_2=&\left(\frac{\partial\rho_+}{\partial r}-\frac{\partial\bar\rho_-}{\partial r}\right)(r_s)\big(r(t)-r_s\big)+\int_{r_s}^{r(t)}(\partial_{rr}\rho_+-\partial_{rr}\bar\rho_-)(t,\theta)\cdot\big(r(t)-\theta\big)\mathrm{d}\theta.
\end{aligned}$$

Implicit function theorem implies

$$(J_+-\bar J)\big(t,r(t)\big)=\mathcal{T}_1\big((\rho_+-\bar\rho_+)\big(t,r(t)\big),r(t)-r_s\big) \tag{3.4}$$

with $\mathcal{T}_1(0,0)=0$ satisfying

$$\begin{aligned}
\frac{\partial\mathcal{T}_1}{\partial(\rho_+-\bar\rho_+)}&=-\frac{r\big(p'(\bar\rho_+)-\bar u_+^2\big)}{2\bar u_+}(r_s),\\
\frac{\partial\mathcal{T}_1}{\partial\big(r(t)-r_s\big)}&=-(\bar\rho_+-\bar\rho_-)\cdot\left(\frac{\bar u_-}{2}+\frac{r\bar E_+}{2\bar u_+}\right)(r_s).
\end{aligned}$$

Substituting $\mathcal{T}_1$ into Eq. (1.18a) of Rankine-Hugoniot conditions (1.18) yields

$$r'(t)=\mathcal{T}_2\big(\rho_+-\bar\rho_+,r(t)-r_s\big), \tag{3.5}$$

where $\mathcal{T}_2(0,0)=0$ satisfies

$$\begin{aligned}
\frac{\partial\mathcal{T}_2}{\partial(\rho_+-\bar\rho_+)}&=-\frac{p'(\bar\rho_+)-\bar u_+^2}{2\bar u_+(\bar\rho_+-\bar\rho_-)}(r_s),\\
\frac{\partial\mathcal{T}_2}{\partial(r(t)-r_s)}&=-\left(\frac{\bar u_-}{2r}+\frac{\bar E_+}{2\bar u_+}\right)(r_s).
\end{aligned}$$

Combing (3.2) and (3.4)-(3.5) leads to

$$Y_t=-\mathcal{T}_1\left(\frac{Y_r}{r},\sigma(t)\right),\quad \sigma'(t)=\mathcal{T}_2\left(\frac{Y_r}{r},\sigma(t)\right). \tag{3.6}$$

Moreover,

$$Y\big(t,r(t)\big)=r(t)\big(E_+\big(t,r(t)\big)-\bar E_+\big(r(t)\big)\big)-\frac{1}{2}(b_2-b_1)\big(r(t)^2-r_s^2\big)$$

$$
\begin{aligned}
&=\partial_r\big(r(t)\bar{E}_-\big(r(t)\big)-r(t)\bar{E}_+\big(r(t)\big)\big)(r_s)\cdot\big(r(t)-r_s\big)\\
&\quad-\frac{1}{2}(b_2-b_1)\big(r(t)^2-r_s^2\big)+\mathrm{O}\big((r(t)-r_s)^2\big)\\
&=\left(r_s\bar{\rho}_-(r_s)-r_s\bar{\rho}_+(r_s)-\frac{1}{2}\big(r(t)-r_s\big)(b_2-b_1)\right)\big(r(t)-r_s\big)+\mathrm{O}\big((r(t)-r_s)^2\big).
\end{aligned}
$$

By using the implicit function theorem again, we derive the equation for the shock front as follows:

$$
r(t)-r_s=\sigma(t)=\mathcal{T}_3\big(Y(t,r(t))\big)\quad\text{at}\ \ r=r(t), \tag{3.7}
$$

where

$$
\mathcal{T}_3(0)=0,\quad \frac{\partial\mathcal{T}_3}{\partial Y}=\frac{1}{r_s\big(\bar{\rho}_-(r_s)-\bar{\rho}_+(r_s)\big)}.
$$

It follows from (3.6)-(3.7) that

$$
Y_t=\mathcal{T}_4(Y_r,Y)\quad\text{at}\ \ r=r(t) \tag{3.8}
$$

with

$$
\mathcal{T}_4(0,0)=0,\quad \frac{\partial\mathcal{T}_4}{\partial Y_r}=\frac{p'(\bar{\rho}_+)-\bar{u}_+^2}{2\bar{u}_+}(r_s),\quad \frac{\partial\mathcal{T}_4}{\partial Y}=-\left(\frac{\bar{u}_-}{2r}+\frac{\bar{E}_+}{2\bar{u}_+}\right)(r_s).
$$

Note that on the right boundary, Y satisfies

$$
\partial_r Y=0\quad\text{at}\ \ r=r_2.
$$

Our purpose is to derive the uniform estimates on the solution Y to the free boundary value problem (3.3), (3.7)-(3.8) in the region $\{(t,r)\,|\,t\geq 0, r_s\leq r\leq r_2\}$.

In order to transform the problem to the fixed domain $[r_s,r_2]$, we introduce the transformation

$$
\tilde{t}=t,\quad \tilde{r}=(r_2-r_s)\frac{r-r(t)}{r_2-r(t)}+r_s,\quad \sigma(\tilde{t})=r(t)-r_s.
$$

Let

$$
q_1(\tilde{r},\sigma)=\frac{r_2-\tilde{r}}{r_2-r_s-\sigma(\tilde{t})},\quad q_2(\sigma)=\frac{r_2-r_s}{r_2-r_s-\sigma(\tilde{t})}.
$$

Then (3.3) becomes

$$
\begin{aligned}
&q_1\sigma''(\tilde{t})\partial_{\tilde{r}}Y-\frac{1}{2}(b_2-b_1)\big(\sigma(\tilde{t})^2+2r_s\sigma(\tilde{t})\big)\bar{\rho}_+\\
=&\partial_{\tilde{t}\tilde{t}}Y+\big(q_1\sigma'(\tilde{t})\big)^2\partial_{\tilde{r}\tilde{r}}Y-2q_1\sigma'(\tilde{t})\partial_{\tilde{t}\tilde{r}}Y+q_2\partial_{\tilde{r}}\left(\frac{\bar{J}^2}{m(\tilde{t},\tilde{r})\bar{\rho}_+}-\frac{\big(\bar{J}-Y_{\tilde{t}}+q_1\sigma'(\tilde{t})Y_{\tilde{r}}\big)^2}{m(\tilde{t},\tilde{r})\bar{\rho}_++q_2Y_{\tilde{r}}}\right)\\
&+m(\tilde{t},\tilde{r})q_2\partial_{\tilde{r}}\left(p(\bar{\rho}_+)-p\left(\bar{\rho}_++\frac{q_2Y_{\tilde{r}}}{m(\tilde{t},\tilde{r})}\right)\right)-\frac{2q_1\sigma'(\tilde{t})^2}{r_2-r_s-\sigma(\tilde{t})}\partial_{\tilde{r}}Y+q_2\bar{E}_+\partial_{\tilde{r}}Y\\
&+\frac{q_2Y}{m(\tilde{t},\tilde{r})}\partial_{\tilde{r}}Y+\frac{b_2-b_1}{2m(\tilde{t},\tilde{r})}\big(\sigma(\tilde{t})^2+2r_s\sigma(\tilde{t})\big)q_2\partial_{\tilde{r}}Y+\bar{\rho}_+Y
\end{aligned}
$$

$$
\begin{aligned}
&=\sum_{0\le i,j\le 1} a_{ij}(\tilde r,Y,\nabla Y,\sigma,\sigma')\partial_{ij}Y+\sum_{0\le i\le 1} b_i(\tilde r,Y,\nabla Y,\sigma,\sigma')\partial_i Y+c(\tilde r,Y,\nabla Y,\sigma,\sigma')Y\\
&=:\mathcal{L}(\tilde r,Y,\sigma)Y
\end{aligned}
$$

with

$$
m(\tilde t,\tilde r)=\frac{\sigma(\tilde t)(r_2-\tilde r)}{r_2-r_s}+\tilde r.
$$

Furthermore, $\mathcal{L}(\tilde r,0,0)Y=L(r,0,0,0)Y$ and

$$
\begin{aligned}
&a_{00}(\tilde r,Y,\nabla Y,\sigma,\sigma')=1,\\
&a_{01}(\tilde r,0,0,0,0)=a_{10}(\tilde r,0,0,0,0)=\frac{\bar J}{\tilde r\bar\rho_+}=\bar u_+,\\
&a_{11}(\tilde r,0,0,0,0)=-\left(p'(\bar\rho_+)-\frac{\bar J^2}{(\tilde r\bar\rho_+)^2}\right)=-\left(p'(\bar\rho_+)-\bar u_+^2\right),\\
&b_0(\tilde r,0,0,0,0)=\partial_{\tilde r}\left(\frac{2\bar J}{\tilde r\bar\rho_+}\right)=\partial_{\tilde r}(2\bar u_+),\\
&b_1(\tilde r,0,0,0,0)=-\partial_{\tilde r}\left(p'(\bar\rho_+)-\frac{\bar J^2}{(\tilde r\bar\rho_+)^2}\right)+\bar E_++\frac{1}{\tilde r}p'(\bar\rho_+)\\
&\qquad\qquad\qquad\;=-\partial_{\tilde r}\left(p'(\bar\rho_+)-\bar u_+^2\right)+\bar E_++\frac{1}{\tilde r}p'(\bar\rho_+),\\
&\bar c(\tilde r,0,0,0,0)=\bar\rho_+.
\end{aligned}
$$

Next, (3.5) and (3.7) are transformed into

$$
\sigma'(\tilde t)=\mathcal{T}_2\left(\frac{q_2(\sigma)}{m(\tilde t,\tilde r)}Y_{\tilde r},\sigma(\tilde t)\right),\quad \sigma(\tilde t)=\mathcal{T}_3\big(Y(t,\tilde r=r_s)\big). \tag{3.9}
$$

At $\tilde r=r_s$, it holds that

$$
\sigma'(\tilde t)+\left(\frac{\bar u_-}{2m(\tilde t,\tilde r)}+\frac{\bar E_+}{2\bar u_+}\right)(r_s)\sigma=\mathcal{C}_2(Y_{\tilde r},Y), \tag{3.10}
$$

where

$$
\mathcal{C}_2(Y_{\tilde r},Y)=-\frac{p'(\bar\rho_+)-\bar u_+^2}{2\tilde r(\bar\rho_+-\bar\rho_-)\bar u_+}(r_s)+\mathrm{O}\big(Y_{\tilde r}^2+Y^2\big)
$$

by using Taylor expansions. Therefore,

$$
\left|\mathcal{C}_2(Y_{\tilde r},Y)+\frac{p'(\bar\rho_+)-\bar u_+^2}{2\tilde r(\bar\rho_+-\bar\rho_-)\bar u_+}(r_s)Y_{\tilde r}\right|\le C\big(Y_{\tilde r}^2+Y^2\big).
$$

In view of (3.9)-(3.10) and the implicit function theorem, σ and σ' are functions of Y and its derivatives at $\tilde r=r_s$. Moreover, combining (3.8)-(3.10) together gives

$$
Y_{\tilde t}=\frac{p'(\bar\rho_+)-\bar u_+^2}{2\bar u_+}(r_s)Y_{\tilde r}-\left(\frac{\bar u_-}{2r}+\frac{\bar E_+}{2\bar u_+}\right)Y+\mathrm{O}\big(Y_{\tilde r}^2+Y^2\big)\quad\text{at }\ \tilde r=r_s,
$$

which implies

$$Y_{\tilde{r}}=\mathcal{C}_3(Y_{\tilde{t}},Y)\quad \text{at } \tilde{r}=r_s$$

by using implicit function theorem again. Here $\mathcal{C}_3$ satisfies

$$\left|\mathcal{C}_3(Y_{\tilde{t}},Y)-\frac{2\bar{u}_+}{p'(\bar{\rho}_+)-\bar{u}_+^2}(r_s)Y_{\tilde{t}}-\left(\frac{\bar{E}_+}{p'(\bar{\rho}_+)-\bar{u}_+^2}+\frac{\bar{u}_+\bar{u}_-}{r\big(p'(\bar{\rho}_+)-\bar{u}_+^2\big)}\right)(r_s)Y\right|\le C\big(Y_{\tilde{t}}^2+Y^2\big).$$

In the rest of the paper, we use r and t instead of $\tilde{r}$ and $\tilde{t}$ for simplicity. The free boundary value problem (3.3), (3.7)-(3.8) is transformed to the problem in the region $\{(t,r)|t\ge 0,r_s\le r\le r_2\}$ with the compact form

$$\begin{cases}\mathcal{L}(r,Y,\sigma)Y=\sigma''(\tilde{t})q_1\partial_r Y-\frac{1}{2}(b_2-b_1)\big(\sigma(t)^2+2r_s\sigma(t)\big)\bar{\rho}_+, & (t,r)\in\Omega, \quad (3.11a)\\ \partial_r Y=e(Y_t,Y)Y_t+f(Y_t,Y)Y & \text{at } r=r_s, \quad (3.11b)\\ \partial_r Y=0 & \text{at } r=r_2, \quad (3.11c)\\ \sigma(t)=\mathcal{T}_3\big(Y(t,r_s)\big). & (3.11d)\end{cases}$$

Here

$$\begin{aligned}&e(Y_t,Y)=\int_0^1\frac{\partial\mathcal{C}_3}{\partial Y_t}(\theta Y_t,\theta Y)\,\mathrm{d}\theta,\quad e(0,0)=\frac{2\bar{u}_+}{p'(\bar{\rho}_+)-\bar{u}_+^2}(r_s),\\ &f(Y_t,Y)=\int_0^1\frac{\partial\mathcal{C}_3}{\partial Y}(\theta Y_t,\theta Y)\,\mathrm{d}\theta,\quad f(0,0)=\left(\frac{\bar{E}_+}{p'(\bar{\rho}_+)-\bar{u}_+^2}+\frac{\bar{u}_+\bar{u}_-}{r\big(p'(\bar{\rho}_+)-\bar{u}_+^2\big)}\right)(r_s).\end{aligned}$$

The initial conditions are given by

$$Y(0,r)=h_1(r),\quad Y_t(0,r)=h_2(r),\quad r_s<r<r_2,\quad \sigma(0)=\sigma_0. \tag{3.12}$$

3.2 A priori estimates and proof of Theorem 1.2

In this subsection, after analyzing an associated linearized problem, some uniform a priori estimate on the solutions to problem (3.11)-(3.12) will be stated in Proposition 3.1. Then Theorem 1.2 is a consequence of Lemma 3.1 and Proposition 3.1.

We consider the following linearized problem:

$$\begin{cases}\mathcal{L}(r,0,0)Y=0,\quad (t,r)\in[0,\infty)\times[r_s,r_2], & (3.13a)\\ \partial_r Y=\frac{2\bar{u}_+}{p'(\bar{\rho}_+)-\bar{u}_+^2}(r_s)Y_t+\left(\frac{\bar{E}_+}{p'(\bar{\rho}_+)-\bar{u}_+^2}+\frac{\bar{u}_+\bar{u}_-}{r(p'(\bar{\rho}_+)-\bar{u}_+^2)}\right)(r_s)Y\quad \text{at } r=r_s, & (3.13b)\\ \partial_r Y=0\quad \text{at } r=r_2, & (3.13c)\\ Y(0,r)=h_1(r),\quad Y_t(0,r)=h_2(r),\quad r_s<r<r_2. & (3.13d)\end{cases}$$

Multiplying Eq. (3.13a) by $Y_t/(r\bar{\rho}_+(r))$ and integrating the resulting equation over $[r_s,r_2]$, we deduce that

$$\begin{aligned}
&\frac{1}{2}\partial_t\int_{r_s}^{r_2}\frac{1}{r\bar{\rho}_+}\left(|\partial_t Y|^2+\left(p'(\bar{\rho}_+)-\frac{\bar{J}^2}{(r\bar{\rho}_+)^2}\right)|\partial_r Y|^2+\bar{\rho}_+Y^2\right)\mathrm{d}r\\
&\quad+\int_{r_s}^{r_2}\left(-\partial_r\left(\frac{1}{r\bar{\rho}_+}\right)\cdot\frac{\bar{J}}{r\bar{\rho}_+}+\frac{1}{r\bar{\rho}_+}\cdot\partial_r\left(\frac{\bar{J}}{r\bar{\rho}_+}\right)\right)|\partial_t Y|^2\mathrm{d}r\\
&\quad+\int_{r_s}^{r_2}\left(\partial_r\left(\frac{1}{r\bar{\rho}_+}\right)\cdot\left(p'(\bar{\rho}_+)-\frac{\bar{J}^2}{(r\bar{\rho}_+)^2}\right)+\frac{1}{r^2\bar{\rho}_+}p'(\bar{\rho}_+)+\frac{1}{r\bar{\rho}_+}\bar{E}_+\right)\partial_t Y\partial_r Y\,\mathrm{d}r\\
&\quad+\frac{\bar{J}}{(r\bar{\rho}_+)^2}|\partial_t Y|^2\Big|_{r_s}^{r_2}-\frac{1}{r\bar{\rho}_+}\cdot\left(p'(\bar{\rho}_+)-\frac{\bar{J}^2}{(r\bar{\rho}_+)^2}\right)\partial_t Y\partial_r Y\Big|_{r_s}^{r_2}\\
=&\partial_t\int_{r_s}^{r_2}\frac{1}{r\bar{\rho}_+}\left(|\partial_t Y|^2+\left(p'(\bar{\rho}_+)-\frac{\bar{J}^2}{(r\bar{\rho}_+)^2}\right)|\partial_t Y|^2+\bar{\rho}_+Y^2\right)\mathrm{d}r\\
&\quad+\partial_t\left(\left(\frac{\bar{E}_+}{r\bar{\rho}_+}+\frac{\bar{J}^2}{r^4\bar{\rho}_+^2\bar{\rho}_-}\right)Y^2\right)\Big|_{r_s}+2\frac{\bar{J}}{(r\bar{\rho}_+)^2}|\partial_t Y|^2\Big|_{r_2}+2\frac{\bar{J}}{(r\bar{\rho}_+)^2}|\partial_t Y|^2\Big|_{r_s}=0,
\end{aligned}\tag{3.14}$$

where Eq. (1.6a) and the boundary conditions in (3.13) have been used. Integrating (3.14) with respect to t gives

$$\varphi_0(Y,0)=\varphi_0(Y,t)+D_0(Y,t)\tag{3.15}$$

with

$$\begin{aligned}
\varphi_0(Y,t)=&\left(\frac{\bar{E}_+}{r\bar{\rho}_+}+\frac{\bar{J}^2}{r^4\bar{\rho}_+^2\bar{\rho}_-}\right)(r_s)Y^2(t,r_s)\\
&+\int_{r_s}^{r_2}\frac{1}{r\bar{\rho}_+}\left(|\partial_t Y|^2+\left(p'(\bar{\rho}_+)-\frac{\bar{J}^2}{(r\bar{\rho}_+)^2}\right)|\partial_r Y|^2+\bar{\rho}_+Y^2\right)(t,r)\,\mathrm{d}r,\\
D_0(Y,t)=&2\left(\int_0^t\frac{\bar{J}}{(r\bar{\rho}_+)^2}|\partial_t Y|^2(s,r_2)\,\mathrm{d}s+\int_0^t\frac{\bar{J}}{(r\bar{\rho}_+)^2}|\partial_t Y|^2(s,r_s)\,\mathrm{d}s\right).
\end{aligned}$$

Lemma 3.2. *Assume that $\bar{E}_+(r_s)>0$ and Y is a smooth solution to the linearized problem* (3.13), *then there exist constants $\lambda_0>0$ and $c>0$ such that for any solution Y to* (3.13), *it holds that*

$$\varphi_0(Y,t)\le ce^{-\lambda_0 t}\varphi_0(Y,0)$$

and

$$\int_0^\infty e^{\frac{\lambda_0 t}{4}}\left(|Y_t|^2(t,r_s)+|Y_t|^2(t,r_2)\right)\mathrm{d}t\le c\varphi_0(Y,0).$$

Proof. Define the solution operator $S_t:\mathbf{X}\mapsto\mathbf{X}$ to the problem (3.13) as

$$(h_1,h_2)\ \mapsto\ \big(Y(t,\cdot),Y_t(t,\cdot)\big),$$

where $h=(h_1,h_2)\in\mathbf{H}^1\times\mathbf{L}^2([r_s,r_2])$ is the initial data and $\mathbf{X}$ is the associated complex Hilbert space with the norm

$$\begin{aligned}\|h\|_{\mathbf{X}}^2=\varphi_0(Y,0)=&\left(\frac{\bar{E}_+}{r\bar{\rho}_+}+\frac{\bar{J}^2}{r^4\bar{\rho}_+^2\bar{\rho}_-}\right)(r_s)|h_1|^2(r_s)\\&+\int_{r_s}^{r_2}\frac{1}{r\bar{\rho}_+}\left(|h_2(r)|^2+\left(p'(\bar{\rho}_+)-\frac{\bar{J}^2}{(r\bar{\rho}_+)^2}\right)|h_1'(r)|^2+\bar{\rho}_+|h_1(r)|^2\right)\mathrm{d}r.\end{aligned}$$

By (3.15), $\|S_t(h)\|_{\mathbf{X}}\le\|h\|_{\mathbf{X}}$, that is, S_t is bounded and $\|S_t\|\le 1$. It follows from the spectrum radius theorem that $|\sigma(S_t)|\le 1$. We also define a compact map $K:\mathbf{X}\mapsto\mathbf{L}^2([0,T])$ by

$$K(h)=Y(t,r_s).$$

According to the Rauch-Taylor type estimates [26], one has for some $T>0$,

$$(1+C_1)\varphi_0(Y,T)\le\varphi_0(Y,0)+C_2\int_0^T Y^2(t,r_s)\,\mathrm{d}t, \tag{3.16}$$

which can also be written as

$$(1+C_1)\|S_T(h)\|_{\mathbf{X}}\le\|h\|_{\mathbf{X}}+C_2\|K(h)\|_{\mathbf{L}^2([0,T])}.$$

Moreover, [18, Proposition 8] and [24] yield

$$\sigma(S_T)\subset\{|z|\le\sqrt{\alpha_0}\}\quad\text{for}\quad 0<\alpha_0<1. \tag{3.17}$$

Then

$$\|S_T(h)\|_{\mathbf{X}}=\varphi_0(Y,T)\le\alpha_0\varphi_0(Y,0)=\alpha_0\|h\|_{\mathbf{X}}.$$

Noting that $\varphi_0(Y,t)$ is decreasing with respect to t, thus for any $t\in[nT,(n+1)T)$ and $n\in\mathbf{N}$, it holds that

$$\varphi_0(Y,t)\le\varphi_0(Y,nT)\le\alpha_0^n\varphi_0(Y,0)\le\alpha_0^{\frac{t}{T}-1}\varphi_0(Y,0)=e^{-\lambda_0 t}\alpha_0^{-1}\varphi_0(Y,0) \tag{3.18}$$

with $\lambda_0=-\ln\alpha_0/T$. By virtue of (3.18), one gets

$$\begin{aligned}&\int_{2^iT}^{2^{i+1}T}e^{\frac{\lambda_0 t}{4}}\left(|Y_t|^2(t,r_s)+|Y_t|^2(t,r_2)\right)\mathrm{d}t\\\le& Ce^{\lambda_0 2^{i-1}T}\left(\varphi_0(Y,2^iT)-\varphi_0(Y,2^{i+1}T)\right)\le C\alpha_0^{-1}e^{-\lambda_0 2^{i-1}T}\varphi_0(Y,0).\end{aligned}$$

Therefore,

$$\int_0^\infty e^{\frac{\lambda_0 t}{4}}\left(|Y_t|^2(t,r_s)+|Y_t|^2(t,r_2)\right)\mathrm{d}t\le C\sum_{i=0}^{\infty}\alpha_0^{-1}e^{-\lambda_0 2^{i-1}T}\varphi_0(Y,0)=C\varphi_0(Y,0).$$

The proof is complete. □

Remark 3.1. Differentiating (3.13) m times with respect to t, one deduces that

$$\varphi_m(Y,t)=\left(\frac{\bar{E}_+}{r\bar{\rho}_+}+\frac{\bar{J}^2}{r^4\bar{\rho}_+^2\bar{\rho}_-}\right)(r_s)|\partial_t^m Y|^2(t,r_s)$$
$$+\int_{r_s}^{r_2}\frac{1}{r\bar{\rho}_+}\left(|\partial_t^{m+1}Y|^2+\left(p'(\bar{\rho}_+)-\frac{\bar{J}^2}{(r\bar{\rho}_+)^2}\right)|\partial_r\partial_t^m Y|^2+\bar{\rho}_+|\partial_t^m Y|^2\right)(t,r)\mathrm{d}r.$$

Summing these estimates from 0 to k, one gets

$$\sum_{m=0}^{k}\varphi_m(Y,t)\leq Ce^{-\lambda_0 t}\sum_{m=0}^{k}\varphi_m(Y,0),$$

and

$$\int_0^\infty e^{\frac{\lambda_0 t}{4}}\sum_{m=1}^{k+1}\left(|\partial_t^m Y|^2(t,r_s)+|\partial_t^m Y|^2(t,r_2)\right)\mathrm{d}t\leq C\sum_{m=0}^{k}\varphi_m(Y,0).$$

Now, due to decay for the linearized problem in Lemma 3.2, some uniform a priori estimate of the solutions to problem (3.11)-(3.12) comes out by quasilinear technical argument. We introduce the notations for $k\geq 15$ and $t>T$ with T in (3.16),

$$|||(Y,\sigma)|||:=\rceil(Y,\sigma)\rceil+\|(Y,\sigma)\|,$$

where

$$\rceil(Y,\sigma)\rceil:=\sup_{\tau\in[0,t)}\sum_{0\leq m\leq k-6}\left(\sum_{0\leq l\leq m}e^{\frac{\lambda\tau}{16}}\|\partial_t^l\partial_r^{m-l}Y(\tau,\cdot)\|_{\mathbf{L}^\infty([r_s,r_2])}+e^{\frac{\lambda\tau}{16}}\left|\frac{\mathrm{d}^m\sigma}{\mathrm{d}t^m}\right|(\tau)\right),$$

$$\|(Y,\sigma)\|:=\sup_{\tau\in[0,t]}\left(\sum_{\substack{0\leq l\leq m\\0\leq m\leq k}}\|\partial_t^l\partial_r^{m-l}Y(\tau,\cdot)\|_{\mathbf{L}^2([r_s,r_2])}+\sum_{0\leq l\leq k}\|\partial_t^l\partial_r^{k+1-l}Y(\tau,\cdot)\|_{\mathbf{L}^2([r_s,r_2])}\right)$$
$$+\sup_{\tau\in[0,t]}\left\|\partial_t^{k+1}Y(\tau,\cdot)-\frac{\mathrm{d}^{k+1}\sigma}{\mathrm{d}t^{k+1}}q_1(\cdot,\sigma)\partial_r Y(\tau,\cdot)\right\|_{\mathbf{L}^2([r_s,r_2])}$$
$$+\sum_{\substack{0\leq l\leq m\\0\leq m\leq k+1}}\left(\|\partial_t^l\partial_r^{m-l}Y(\cdot,r_s)\|_{\mathbf{L}^2([0,t])}+\|\partial_t^l\partial_r^{m-l}Y(\cdot,r_2)\|_{\mathbf{L}^2([0,t])}\right)$$
$$+\sum_{0\leq m\leq k+1}\left\|\frac{\mathrm{d}^m\sigma}{\mathrm{d}t^m}\right\|_{\mathbf{L}^2([0,t])}.$$

The uniform estimates are stated in the following proposition.

Proposition 3.1. *Assume that $\bar{E}_+(r_s)>0$ and (Y,σ) is a smooth solution to the problem* (3.11)-(3.12) *in $\{(t,r)|t\geq 0, r_s\leq r\leq r_2\}$, if*

$$|\sigma_0|+\|h_1\|_{\mathbf{H}^{k+2}}+\|h_2\|_{\mathbf{H}^{k+1}}\leq\epsilon^2\leq\epsilon_0^2$$

for some $\epsilon_0>0$ and $|||(Y,\sigma)|||\leq\epsilon$ for some $t>T$ with T as in (3.16)*, then*

$$|||(Y,\sigma)|||\leq\frac{\epsilon}{2}.$$

Consequently, Theorem 1.2 holds.

Proof of Theorem 1.2. In view of Proposition 3.1, one may apply the standard continuation argument to extend the local (in time) solution in Lemma 3.1 to all time $t>0$. ☐

It remains to prove Proposition 3.1, which will be shown in Subsection 3.3.

3.3 Proof of Proposition 3.1

In this subsection, we will establish some necessary global a priori estimates for solutions to the problem (3.11)-(3.12) in $\{(t,r)|t\geq 0, r_s\leq r\leq r_2\}$, and then finish the proof of Proposition 3.1.

For any $l\in\mathbf{N}$ and $|||(Y,\sigma)|||<\infty$, we define

$$\hat{\Phi}_l(Z,t;Y,\sigma)=\tilde{\Phi}_{l-1}(Z,t;Y,\sigma)+\Phi_0\left(\partial_t^l Z-q_1(r,\sigma)Y_r\frac{\mathrm{d}^l\sigma}{\mathrm{d}t^l},t;Y,\sigma\right)$$

with

$$\tilde{\Phi}_l(Z,t;Y,\sigma)=\sum_{m=0}^{l}\Phi_m(Z,t;Y,\sigma),$$

and

$$\Phi_m(Z,t;Y,\sigma)=\frac{a_{11}f(\partial_t^m Z)^2}{r\bar{\rho}_+}(t,r_s)+\int_{r_s}^{r_2}\frac{1}{r\bar{\rho}_+}\left((\partial_t^{m+1}Z)^2-a_{11}(\partial_r\partial_t^m Z)^2+c(\partial_t^m Z)^2\right)(t,r)\mathrm{d}r.$$

Similarly, we define

$$\hat{\mathcal{D}}_l(Z,t;Y,\sigma)=\tilde{\mathcal{D}}_{l-1}(Z,t;Y,\sigma)+\mathcal{D}_0\left(\partial_t^l Z-q_1(r,\sigma)Y_r\frac{\mathrm{d}^l\sigma}{\mathrm{d}t^l},t;Y,\sigma\right)$$

with

$$\tilde{\mathcal{D}}_m(Z,t;Y,\sigma)=\sum_{l=0}^{m}\mathcal{D}_l(Z,t;Y,\sigma),$$

and

$$\mathcal{D}_l(Z,t;Y,\sigma)=\int_0^t\frac{-(a_{11}e+a_{01})}{r\bar{\rho}_+}\left(\partial_t^{l+1}Z\right)^2(\tau,r_s)\,\mathrm{d}\tau+\int_0^t\frac{a_{01}}{r\bar{\rho}_+}\left(\partial_t^{l+1}Z\right)^2(\tau,r_2)\,\mathrm{d}\tau.$$

By the coefficients in (3.11) and (3.13), it holds that

$$\hat{\Phi}_k(Z,t;0,0)=\sum_{m=0}^{k}\varphi_m(Z,t),$$

where $\varphi_m(Z,t)$ is defined as in Remark 3.1. Moreover,

$$\Phi_m(Z,t;Y,\sigma)(t)\geq C\int_{r_s}^{r_2}\left(\left(\partial_t^{m+1}Z\right)^2+\left(\partial_r\partial_t^m Z\right)^2+\left(\partial_t^m Z\right)^2\right)(t,r)\,\mathrm{d}r$$

holds if $|||(Y,\sigma)|||\leq\epsilon$ for some $\epsilon>0$, where $C>0$ is a constant independent of t. We start with the following lower order energy estimate.

Lemma 3.3. *Assume that $\bar{E}_+(r_s)>0$ and (Y,σ) is a smooth solution to the problem* (3.11)-(3.12) *satisfying the assumptions in Proposition* 3.1*, then*

$$
\begin{aligned}
&\tilde{\Phi}_m(Y,t;Y,\sigma)+\tilde{D}_m(Y,\tau;Y,\sigma) \qquad (3.19)\\
\leq&\tilde{\Phi}_m(Y,0;Y,\sigma)+C|||(Y,\sigma)|||\left(\int_0^t e^{-\frac{\lambda\tau}{64}}\left(\tilde{\Phi}_m(Y,\tau;Y,\sigma)+\sum_{l=0}^{m+2}\left|\frac{\mathrm{d}^l\sigma}{\mathrm{d}t^l}\right|^2\right)\mathrm{d}\tau+\tilde{D}_m(Y,t;Y,\sigma)\right).
\end{aligned}
$$

Proof. Taking the m-th $(0\leq m\leq k-1)$ order derivative of Eq. (3.11) with respect to t, one has

$$\mathcal{L}(r,Y,\sigma)\partial_t^m Y=\mathcal{F}_m(r,Y,\sigma)+\hat{\mathcal{F}}_m(r,Y,\sigma)+\check{\mathcal{F}}_m(r,\sigma), \tag{3.20}$$

where

$$
\begin{aligned}
&\mathcal{F}_m(r,Y,\sigma)=\sum_{1\leq l\leq m}C_m^l\left(-\sum_{i,j=0}^{1}\partial_t^l a_{ij}\partial_{ij}\partial_t^{m-l}Y-\sum_{i=0}^{1}\partial_t^l b_i\partial_i\partial_t^{m-l}Y-\partial_t^l c\partial_t^{m-l}Y\right),\\
&\hat{\mathcal{F}}_m(r,Y,\sigma)=\sum_{0\leq l\leq m}C_m^l\frac{\mathrm{d}^{l+2}\sigma}{\mathrm{d}t^{l+2}}\partial_t^{m-l}\big(q_1(r,\sigma)Y_r\big),\\
&\check{\mathcal{F}}_m(r,\sigma)=-\frac{1}{2}\bar{\rho}_+(b_2-b_1)\left(\sum_{0\leq l\leq m}C_m^l\frac{\mathrm{d}^l\sigma}{\mathrm{d}t^l}\cdot\frac{\mathrm{d}^{m-l}\sigma}{\mathrm{d}t^{m-l}}+2r_s\frac{\mathrm{d}^m\sigma}{\mathrm{d}t^m}\right).
\end{aligned}
$$

Multiplying both sides of (3.20) by $\partial_t^{m+1}Y/(r\bar{\rho}_+(r))$ and integrating over $\Omega'=[0,t]\times[r_s,r_2]$ lead to

$$
\begin{aligned}
&\iint_{\Omega'}\mathcal{L}(r,Z,\sigma)\partial_t^m Y\frac{1}{r\bar{\rho}_+(r)}\partial_t^{m+1}Y(\tau,r)\mathrm{d}\tau\mathrm{d}r\\
=&\int_{r_s}^{r_2}\frac{1}{2r\bar{\rho}_+(r)}\left(\big(\partial_t^{m+1}Y\big)^2-a_{11}\big(\partial_r\partial_t^m Y\big)^2+c\big(\partial_t^m Y\big)^2\right)(\tau,r)\mathrm{d}r\\
&+\iint_{\Omega'}\left(\frac{b_0}{r\bar{\rho}_+(r)}-\partial_r\left(\frac{a_{01}}{r\bar{\rho}_+(r)}\right)\right)\big(\partial_t^{m+1}Y\big)^2(\tau,r)\mathrm{d}\tau\mathrm{d}r\\
&+\iint_{\Omega'}\left(\frac{b_1}{r\bar{\rho}_+(r)}-\partial_r\left(\frac{a_{11}}{r\bar{\rho}_+(r)}\right)\right)\partial_t^m\partial_r Y\partial_t^{m+1}Y(\tau,r)\mathrm{d}\tau\mathrm{d}r\\
&+\iint_{\Omega'}\frac{\partial_t a_{11}}{r\bar{\rho}_+(r)}\cdot\frac{(\partial_r\partial_t^m Y)^2}{2}(\tau,r)-\frac{\partial_t c}{r\bar{\rho}_+(r)}\cdot\frac{(\partial_t^m Y)^2}{2}(\tau,r)\mathrm{d}\tau\mathrm{d}r\\
&+\int_0^t\left(\frac{a_{11}\partial_r\partial_t^m Y\partial_t^{m+1}Y}{r\bar{\rho}_+(r)}+\frac{a_{01}}{r\bar{\rho}_+(r)}\big(\partial_t^{m+1}Y\big)^2\right)(\tau,r_2)\mathrm{d}\tau\\
&-\int_0^t\left(\frac{a_{11}\partial_r\partial_t^m Y\partial_t^{m+1}Y}{r\bar{\rho}_+(r)}+\frac{a_{01}}{r\bar{\rho}_+(r)}\big(\partial_t^{m+1}Y\big)^2\right)(\tau,r_s)\mathrm{d}\tau\\
&-\int_{r_s}^{r_2}\frac{1}{2r\bar{\rho}_+(r)}\left(\big(\partial_t^{m+1}Y\big)^2-a_{11}\big(\partial_r\partial_t^m Y\big)^2+c\big(\partial_t^m Y\big)^2\right)(0,r)\mathrm{d}r
\end{aligned}
$$

$$
\begin{aligned}
&\geq \Phi_m(Y,t;Y,\sigma)-\Phi_m(Y,0;Y,\sigma)+\mathcal{D}_m(Y,t;Y,\sigma)\\
&\qquad -C|||(Y,\sigma)|||\left(\int_0^t e^{-\frac{\lambda\tau}{64}}\tilde{\Phi}_m(Y,\tau;Y,\sigma)\mathrm{d}\tau+\tilde{\mathcal{D}}_m(Y,t;Y,\sigma)\right).
\end{aligned}
\tag{3.21}
$$

On the other hand, it holds that

$$
\begin{aligned}
&\iint_{\Omega'}\mathcal{F}_m\frac{1}{r\bar{\rho}_+(r)}\partial_t^{m+1}Y\,\mathrm{d}t\,\mathrm{d}r\\
=&-\Bigg(\iint_{\Omega'}\sum_{\substack{1\leq l\leq 6\\ 0\leq i,j\leq 1}}C_m^l\left(\frac{\partial_t^l a_{ij}}{r\bar{\rho}_+(r)}\partial_{ij}\partial_t^{m-l}Y+\frac{\partial_t^l b_i}{r\bar{\rho}_+(r)}\partial_i\partial_t^{m-l}Y\right)\partial_t^{m+1}Y\\
&\qquad+\iint_{\Omega'}\sum_{1\leq l\leq 6}C_m^l\frac{\partial_t^l c}{r\bar{\rho}_+(r)}\partial_t^{m-l}Y\partial_t^{m+1}Y\\
&\qquad+\iint_{\Omega'}\sum_{\substack{7\leq l\leq m\\ 0\leq i,j\leq 1}}C_m^l\left(\frac{\partial_t^l a_{ij}}{r\bar{\rho}_+(r)}\partial_{ij}\partial_t^{m-l}Y+\frac{\partial_t^l b_i}{r\bar{\rho}_+(r)}\partial_i\partial_t^{m-l}Y\right)\partial_t^{m+1}Y\\
&\qquad+\iint_{\Omega'}\sum_{7\leq l\leq m}\frac{\partial_t^l c}{r\bar{\rho}_+(r)}\partial_t^{m-l}Y\partial_t^{m+1}Y\Bigg)\\
\leq& C|||(Y,\sigma)|||\int_0^t e^{-\frac{\lambda\tau}{64}}\left(\tilde{\Phi}_m(Y,\tau;Y,\sigma)+|||(Y,\sigma)|||\sum_{l=0}^{m+1}\left|\frac{\mathrm{d}^l\sigma}{\mathrm{d}t^l}\right|^2\right)\mathrm{d}\tau,
\end{aligned}
\tag{3.22}
$$

and

$$
\begin{aligned}
&\iint_{\Omega'}\hat{\mathcal{F}}_m\frac{1}{r\bar{\rho}_+(r)}\partial_t^{m+1}Y\,\mathrm{d}t\,\mathrm{d}r\\
=&\iint_{\Omega'}\sum_{0\leq l\leq 6}C_m^l\frac{\mathrm{d}^{l+2}\sigma}{\mathrm{d}t^{l+2}}\partial_t^{m-l}\big(q_1(r,\sigma)Y_r\big)\frac{1}{r\bar{\rho}_+(r)}\partial_t^{m+1}Y\mathrm{d}\tau\mathrm{d}r\\
&+\iint_{\Omega'}\sum_{7\leq l\leq m}C_m^l\frac{\mathrm{d}^{l+2}\sigma}{\mathrm{d}t^{l+2}}\partial_t^{m-l}\big(q_1(r,\sigma)Y_r\big)\frac{1}{r\bar{\rho}_+(r)}\partial_t^{m+1}Y\mathrm{d}\tau\mathrm{d}r\\
\leq& C\int_0^t\sum_{0\leq l\leq 6}\left|\frac{\mathrm{d}^{l+2}\sigma}{\mathrm{d}t^{l+2}}\right|\cdot\left\|\partial_t^{m-l}(q_1(r,\sigma)Y_r)\right\|_{\mathbf{L}^2([r_s,r_2])}\left\|\frac{\partial_t^{m+1}Y}{r\bar{\rho}_+(r)}\right\|_{\mathbf{L}^2([r_s,r_2])}\mathrm{d}\tau\\
&+C\int_0^t\sum_{7\leq l\leq m}\left|\frac{\mathrm{d}^{l+2}\sigma}{\mathrm{d}t^{l+2}}\right|\cdot\left\|\partial_t^{m-l}(q_1(r,\sigma)Y_r)\right\|_{\mathbf{L}^\infty([r_s,r_2])}\left\|\frac{\partial_t^{m+1}Y}{r\bar{\rho}_+(r)}\right\|_{\mathbf{L}^2([r_s,r_2])}\mathrm{d}\tau\\
\leq& C|||(Y,\sigma)|||\int_0^t e^{-\frac{\lambda\tau}{64}}\left(\tilde{\Phi}_m(Y,\tau;Y,\sigma)+\sum_{l=0}^{m+2}\left|\frac{\mathrm{d}^l\sigma}{\mathrm{d}t^l}\right|^2\right)\mathrm{d}\tau.
\end{aligned}
\tag{3.23}
$$

Noting that

$$
\left\|\frac{\mathrm{d}^{l+2}\sigma}{\mathrm{d}t^{l+2}}\right\|_{\mathbf{L}^2([r_s,r_2])}\leq C\left|\frac{\mathrm{d}^{l+2}\sigma}{\mathrm{d}t^{l+2}}\right|,
$$

one has

$$\begin{aligned}
&\iint_{\Omega'}\check{\mathcal{F}}_m\frac{1}{r\bar{\rho}_+(r)}\partial_t^{m+1}Y\,\mathrm{d}t\,\mathrm{d}r\\
=&-\iint_{\Omega'}\bar{\rho}_+(b_2-b_1)r_s\frac{\mathrm{d}^m\sigma}{\mathrm{d}t^m}\frac{1}{r\bar{\rho}_+(r)}\partial_t^{m+1}Y\,\mathrm{d}\tau\,\mathrm{d}r\\
&-\iint_{\Omega'}\frac{1}{2}\bar{\rho}_+(b_2-b_1)\sum_{0\le l\le m}C_m^l\frac{\mathrm{d}^l\sigma}{\mathrm{d}t^l}\cdot\frac{\mathrm{d}^{m-l}\sigma}{\mathrm{d}t^{m-l}}\frac{1}{r\bar{\rho}_+(r)}\partial_t^{m+1}Y\,\mathrm{d}\tau\,\mathrm{d}r\\
\le& C\int_0^t\left|\frac{\mathrm{d}^m\sigma}{\mathrm{d}t^m}\right|\cdot\left\|\frac{1}{r\bar{\rho}_+(r)}\partial_t^{m+1}Y\right\|_{\mathbf{L}^2([r_s,r_2])}\mathrm{d}\tau\\
&+C\int_0^t\sum_{0\le l\le m}\left|\frac{\mathrm{d}^l\sigma}{\mathrm{d}t^l}\right|\cdot\left|\frac{\mathrm{d}^{m-l}\sigma}{\mathrm{d}t^{m-l}}\right|\cdot\left\|\frac{1}{r\bar{\rho}_+(r)}\partial_t^{m+1}Y\right\|_{\mathbf{L}^2([r_s,r_2])}\mathrm{d}\tau\\
\le& C|||(Y,\sigma)|||\int_0^t e^{-\frac{\lambda\tau}{64}}\left(\tilde{\Phi}_m(Y,\tau;Y,\sigma)+\sum_{l=0}^{m+2}\left|\frac{\mathrm{d}^l\sigma}{\mathrm{d}t^l}\right|^2\right)\mathrm{d}\tau. \qquad (3.24)
\end{aligned}$$

Combining the estimates (3.21)-(3.24) implies the lower order estimate (3.19). □

Next, the highest order energy estimate will be given in the following lemma.

Lemma 3.4. *Assume that $\bar{E}_+(r_s)>0$ and (Y,σ) is a smooth solution to the problem* (3.11)-(3.12) *satisfying the assumptions in Proposition* 3.1*, then*

$$\begin{aligned}
&\hat{\Phi}_k(Y,t;Y,\sigma)+\hat{\mathcal{D}}_k(Y,t;Y,\sigma) \qquad (3.25)\\
\le&\hat{\Phi}_k(Y,0;Y,\sigma)+C|||(Y,\sigma)|||\left(\int_0^t e^{-\frac{\lambda\tau}{64}}\left(\hat{\Phi}_k(Y,\tau;Y,\sigma)+\sum_{l=0}^{k+1}\left|\frac{\mathrm{d}^l\sigma}{\mathrm{d}t^l}\right|^2\right)\mathrm{d}\tau+\hat{\mathcal{D}}_k(Y,t;Y,\sigma)\right).
\end{aligned}$$

Proof. Taking the k-th order derivative to (3.11) with respect to t gives

$$\begin{aligned}
\mathcal{L}(r,Y,\sigma)\partial_t^kY=&\mathcal{F}_k(r,Y,\sigma)+\frac{\mathrm{d}^{k+2}\sigma}{\mathrm{d}t^{k+2}}q_1(r,\sigma)Y_r+\sum_{0\le l\le k-1}C_k^l\frac{\mathrm{d}^{l+2}\sigma}{\mathrm{d}t^{l+2}}\partial_t^{k-l}\big(q_1(r,\sigma)Y_r\big)\\
&-\frac{1}{2}\bar{\rho}_+(b_2-b_1)\left(\sum_{0\le l\le k}C_k^l\frac{\mathrm{d}^l\sigma}{\mathrm{d}t^l}\cdot\frac{\mathrm{d}^{k-l}\sigma}{\mathrm{d}t^{k-l}}+2r_s\frac{\mathrm{d}^k\sigma}{\mathrm{d}t^k}\right).
\end{aligned}$$

In order to handle the term $\mathrm{d}^{k+2}\sigma/\mathrm{d}t^{k+2}$, we rewrite the above equation as

$$\mathcal{L}(r,Y,\nabla Y,\sigma,\sigma')\check{Y}=\mathcal{F}_k(r,Y,\sigma)+\tilde{\mathcal{F}}_k(r,Y,\sigma), \qquad (3.26)$$

where

$$\check{Y}=\partial_t^kY-q_1(r,\sigma)Y_r\frac{\mathrm{d}^k\sigma}{\mathrm{d}t^k},$$

and

$$\begin{aligned}\tilde{\mathcal{F}}_k(r,Y,\sigma)=&-2\frac{\mathrm{d}^{k+1}\sigma}{\mathrm{d}t^{k+1}}\partial_t\big(q_1(r,\sigma)Y_r\big)-\frac{\mathrm{d}^k\sigma}{\mathrm{d}t^k}\partial_t^2\big(q_1(r,\sigma)Y_r\big)\\&-\left(2a_{01}\partial_t\partial_r+a_{11}\partial_r^2+\sum_{i=0}^{1}b_i\partial_i+c\right)\left(q_1(r,\sigma)Y_r\frac{\mathrm{d}^k\sigma}{\mathrm{d}t^k}\right)\\&+\sum_{0\le l\le k-1}C_k^l\frac{\mathrm{d}^{l+2}\sigma}{\mathrm{d}t^{l+2}}\partial_t^{k-l}\big(q_1(r,\sigma)Y_r\big)\\&-\frac{1}{2}\bar{\rho}_+(b_2-b_1)\left(\sum_{0\le l\le k}C_k^l\frac{\mathrm{d}^l\sigma}{\mathrm{d}t^l}\cdot\frac{\mathrm{d}^{k-l}\sigma}{\mathrm{d}t^{k-l}}+2r_s\frac{\mathrm{d}^k\sigma}{\mathrm{d}t^k}\right).\end{aligned}$$

Multiplying both sides of (3.26) by $\partial_t\check{Y}/(r\bar{\rho}_+)$ and integrating over $\Omega'=[0,t]\times[r_s,r_2]$ lead to

$$\begin{aligned}&\Phi_0(\check{Y},t;Y,\sigma)+\mathcal{D}_0(\check{Y},t;Y,\sigma)\\&\le\Phi_0(\check{Y},0;Y,\sigma)+C|||(Y,\sigma)|||\left(\int_0^t e^{-\frac{\lambda\tau}{64}}\left(\tilde{\Phi}_k(Y,\tau;Y,\sigma)+\sum_{l=0}^{k+1}\left|\frac{\mathrm{d}^l\sigma}{\mathrm{d}t^l}\right|^2\right)\mathrm{d}\tau+\hat{\mathcal{D}}_k(Y,t;Y,\sigma)\right),\end{aligned}\tag{3.27}$$

where the associated boundary condition for $\check{Y}$ at $r=r_s$ and $r=r_2$, and the estimate

$$\sum_{l=0}^{k+1}\left\|\partial_t^l\partial_r^{k-l}Y(\tau,\cdot)\right\|^2_{\mathbf{L}^2([r_s,r_2])}\le C\left(\hat{\Phi}_k(Y,\tau;Y,\sigma)+|||(Y,\sigma)|||^2\sum_{l=0}^{k+1}\left|\frac{\mathrm{d}^l\sigma}{\mathrm{d}t^l}\right|^2\right)$$

have been used. Combining (3.19) and (3.27) gives (3.25). □

Finally, base on the estimates in Lemmas 3.3-3.4, the proof of Proposition 3.1 is analogous to the discussion in [12, 18]; we only sketch the proof for completeness.

Proof of Proposition 3.1. Differentiating Eq. (3.11d) with respect to t gives

$$\sum_{l=0}^{k+1}\left|\frac{\mathrm{d}^l\sigma}{\mathrm{d}t^l}\right|^2(\tau)\le C\left(|\sigma(\tau)|^2+\sum_{l=0}^{k+1}|\partial_t^lY(\tau,r_s)|^2\right).\tag{3.28}$$

Thus, one has

$$\sum_{l=0}^{k+1}\left|\frac{\mathrm{d}^l\sigma}{\mathrm{d}t^l}\right|^2\le C\sum_{l=1}^{k}\left|\partial_t^lY(\tau,r_s)\right|^2+|\partial_t\check{Y}|^2,$$

which together with (3.28) implies

$$\int_0^t e^{-\frac{\lambda\tau}{64}}\sum_{l=0}^{k+1}\left|\frac{\mathrm{d}^l\sigma}{\mathrm{d}t^l}\right|^2\mathrm{d}\tau\le C\left(\int_0^t e^{-\frac{\lambda\tau}{64}}\Phi_0(Y,\tau;Y,\sigma)\mathrm{d}\tau+\hat{\mathcal{D}}_k(Y,t;Y,\sigma)\right).$$

Then (3.25) becomes

$$\begin{aligned}&\hat{\Phi}_k(Y,t;Y,\sigma)+\hat{\mathcal{D}}_k(Y,t;Y,\sigma)\\ \le&\hat{\Phi}_k(Y,0;Y,\sigma)+C|||(Y,\sigma)|||\left(\hat{\mathcal{D}}_k(Y,t;Y,\sigma)+\int_0^t e^{-\frac{\lambda\tau}{64}}\hat{\Phi}_k(Y,\tau;Y,\sigma)\mathrm{d}\tau\right).\end{aligned}\tag{3.29}$$

Therefore, for $|||(Y,\sigma)|||\le\epsilon$ small enough, one deduces that

$$\tilde{|||}(Y,\sigma)\tilde{|||}\le C\left(\sup_{0\le\tau\le t}\hat{\Phi}_k^{\frac{1}{2}}(Y,\tau;Y,\sigma)+\hat{\mathcal{D}}_k^{\frac{1}{2}}(Y,t;Y,\sigma)\right)\le C\epsilon^2\le\frac{\epsilon}{4}.$$

In the following, the decay of the lower order energy and the shock position is obtained by controlling the deviation of the solution Y to the nonlinear problem (3.11) from the solution $\bar{Y}$ to the linear problem (3.13). The contraction of the energy for $\bar{Y}$ yields the contraction of the energy for Y. At time $\tau=t_0$, choose $\bar{h}_1\in\mathbf{H}^k$ and $\bar{h}_2\in\mathbf{H}^{k-1}$ such that there exists a solution $\bar{Y}\in\mathbf{C}^{k-1-i}([t_0,\infty);\mathbf{H}^i([r_s,r_2]))$ of the linear problem (3.13) satisfying

$$\bar{Y}(t_0,\cdot)=\bar{h}_1,\quad \bar{Y}_t(t_0,\cdot)=\bar{h}_2$$

and

$$\sum_{l=0}^{k-1}\sum_{i=0}^{l}\left\|\partial_t^i\partial_r^{l-i}\bar{Y}(t_0,\cdot)\right\|_{\mathbf{L}^2([r_s,r_2])}\le C|||(Y,\sigma)|||.$$

Here C is a uniform constant, and

$$\hat{\Phi}_{k-4}(Y-\bar{Y},t_0;Y,\sigma)\le C|||(Y,\sigma)|||\hat{\Phi}_{k-4}(Y,t_0;Y,\sigma).$$

Furthermore, $Y-\bar{Y}$ satisfies the following equation:

$$\begin{aligned}&\sum_{i,j=0}^{1}a_{ij}(r,Y,\sigma)\partial_{ij}(Y-\bar{Y})+\sum_{i=0}^{1}b_i(r,Y,\sigma)\partial_i(Y-\bar{Y})+c(r,Y,\sigma)(Y-\bar{Y})\\ =&\sum_{i,j=0}^{1}\big(a_{ij}(r,Y,\sigma)-a_{ij}(r,0,0)\big)\partial_{ij}\bar{Y}+\sum_{i=0}^{1}\big(b_i(r,Y,\sigma)-b_i(r,0,0)\big)\partial_i\bar{Y}\\ &+\big(c(r,Y,\sigma)-c(r,0,0)\big)\bar{Y}+\sigma''(t)q_1(r,\sigma)\partial_rY-\frac{1}{2}(b_2-b_1)\bar{\rho}_+\big(\sigma(t)^2+2r_s\sigma(t)\big)\end{aligned}$$

with the boundary conditions

$$\begin{aligned}\partial_r(Y-\bar{Y})=&e(Y_t,Y)\partial_t(Y-\bar{Y})+f(Y_t,Y)(Y-\bar{Y})\\ &+\big(e(Y_t,Y)-e(0,0)\big)\partial_t\bar{Y}\\ &+\big(f(Y_t,Y)-f(0,0)\big)\partial_t\bar{Y}\quad\text{at } r=r_s,\end{aligned}$$

and

$$\partial_r(Y-\bar{Y})=0\quad\text{at } r=r_2.$$

Then, we verify that

$$\begin{aligned}&\hat{\Phi}_{k-4}(Y-\bar{Y},t_0+T;Y,\sigma)+\hat{\mathcal{D}}_{k-4}(Y-\bar{Y},t_0+T;Y,\sigma)-\hat{\mathcal{D}}_{k-4}(Y-\bar{Y},t_0;Y,\sigma)\\ \le&\hat{\Phi}_{k-4}(Y-\bar{Y},t_0;Y,\sigma)+C|||(Y,\sigma)|||\\ &\times\Bigg(\int_{t_0}^{t_0+T}\sum_{l=0}^{k-3}\left(\left|\frac{\mathrm{d}^l\sigma}{dt^l}\right|^2+\|\partial_t^i\partial_r^{k-3-i}(Y-\bar{Y})\|_{\mathbf{L}^2}^2\right)\mathrm{d}\tau\\ &\quad+\int_{t_0}^{t_0+T}\left(\hat{\Phi}_{k-4}^{\frac{1}{2}}(Y,\tau;Y,\sigma)\hat{\Phi}_{k-4}^{\frac{1}{2}}(Y-\bar{Y},\tau;Y,\sigma)+\hat{\Phi}_{k-4}(Y-\bar{Y},\tau;Y,\sigma)\right)\mathrm{d}\tau\\ &\quad+\big(\hat{\mathcal{D}}_{k-4}(Y-\bar{Y},t_0+T;Y,\sigma)-\hat{\mathcal{D}}_{k-4}(Y-\bar{Y},t_0;Y,\sigma)\big)\\ &\quad+\big(\hat{\mathcal{D}}_{k-4}(Y,t_0+T;Y,\sigma)-\hat{\mathcal{D}}_{k-4}(Y,t_0;Y,\sigma)\big)\Bigg),\end{aligned}$$

where $T>0$ is the one as in (3.16). According to Lemma 3.2, there exists $\alpha\in(\alpha_0,1)$ such that

$$\hat{\Phi}_{k-4}(Y,t_0+T;Y,\sigma)\le\alpha\hat{\Phi}_{k-4}(Y,t_0;Y,\sigma), \tag{3.30}$$

where $\alpha_0\in(0,1)$ is the constant in (3.17). By (3.30) and the shock front equation in (3.11), the same argument in Lemma 3.2 leads to

$$\hat{\Phi}_{k-4}(Y_t,t;Y,\sigma)+\sigma^2(t)\le C\big(\tilde{\Phi}_{k-4}(Y,0;Y,\sigma)+\sigma^2(0)\big)e^{-2\lambda t},$$

where $\lambda=-\ln((1+\alpha)/2)/(2T)$. Now, if $\hat{\Phi}_{k-4}(Y,0;Y,\sigma)\le\epsilon^4$, then

$$\sum_{l=0}^{k-6}\|Y(t,\cdot)\|_{\mathbf{L}^\infty([r_s,r_2])}\le C\epsilon^2e^{-\lambda t},$$

which yields

$$\sum_{l=0}^{k-6}\left|\frac{\mathrm{d}^l\sigma}{\mathrm{d}t^l}\right|\le C\epsilon^2e^{-\lambda t}.$$

This together with (3.29) gives $|||(Y,\sigma)|||\le\epsilon/2$. This finishes the proof. □

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