CSIAM Trans. Appl. Math.
doi: 10.4208/csiam-am.SO-2023-0034

Vol. **5**, No. 1, pp. 142-181
March 2024

Dynamic Pricing with Surging Demand

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Received 8 July 2023; Accepted 30 November 2023

Abstract. This paper considers the case of a firm's dynamic pricing problem for a non-perishable product experiencing surging demand caused by rare events modelled by a marked point process. The firm aims to maximize its running revenue by selecting an optimal price process for the product until its inventory is depleted. Using the dynamic program and inspired by the viscosity solution technique, we solve the resulting integro-differential Hamilton-Jacobi-Bellman (HJB) equation and prove that the value function is its unique classical solution. We also establish structural properties for our problem and find that the optimal price always decreases with initial inventory level in the absence of surging demand. However, with surging demand, we find that the optimal price could increase rather than decrease at the initial inventory level.

AMS subject classifications: 90B05, 49L20, 60G55

Key words: Dynamic pricing, surging demand, HJB equation, viscosity solution, linear demand.

1 Introduction

In many settings, the occurrence of unexpected rare events may result in surging demand. This has been highlighted by the COVID-19 outbreak, which imposed an unexpected demand surge for a range of consumer goods, and such a surge often has a major impact on commodity prices. In April 2020, the price index for meats, poultry, and fish as well as the index for cereal and bakery products both rose from the month preceding, the first month of the global quarantine [21]. During this time, demand rose to an unexpected level in numerous sectors from disposable medical supplies to groceries, thereby placing upward pressure on prices. In a vivid example of this, products like face masks

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saw markups as high as 582% during the pandemic [24]. In this paper, we aim to formally understand how surging demand affects a firm's pricing decisions. To this end, we model two types of demands: continuous and surging. We first introduce the continuous demand model (see, e.g. [18, 32]) and then use a marked point process to describe the surging demand. From this, we model the cumulative demand process as a drift-jump process, with the price process as a control variable in which the forecast demand rate is a deterministic known function.

From there, we investigate the impact of the key factor – surging demand – on the firm's price policies. To the best of our knowledge, the existing literature on the dynamic price-based revenue management problem in the absence of surging demand has always showcased how optimal price decreases with respect to initial inventory level ([11, 18, 38]). In our paper, however, we can prove that for the linear demand rate function with surging demand it may occur that the optimal price can in fact increase with the initial inventory level. Intuitively speaking, if one considers a surge in demand, the firm with more inventory may need to adopt the opposite price strategy and adjust prices in order to make the maximal expected profit.

We then extend the demand model by introducing Brownian random fluctuations to assist in describing the demand variability. Several studies in the traditional literature have focused on the Poisson demand model, for which the uncertainty in demand is in the form of a discrete shock. In contrast, our model incorporates uncertainty via Brownian motion, letting us capture the variability right around the expected level [27]. There has also recently been an increased interest in the Brownian-demand model in the context of dynamic pricing, such as how [12] consider a diffusion demand model with the Brownian demand-forecast variability. For their part, [35] studies a stochastic production/inventory system with finite production capacity and random demand, where the cumulative production and demand are jointly described as a two-dimensional Brownian motion, while [34] explores a case where the consumer's cumulative utility for a new product is characterized by a diffusion process. We follow suit to incorporate a Brownian motion in a surging demand model.

Under the above model settings with and without Brownian motion, we seek an optimal pricing policy that maximizes expected profit by solving the associated Hamilton-Jacobi-Bellman (HJB) equation (see [26]) and arrive at a class of integro-differential HJB equations by using the dynamic program. The works [9, 10] prove that a kind of linear ordinary integro-differential equations (OIDEs) can be transformed into a homogeneous linear high-order ordinary differential equation (ODE), thereby having an explicit solution. Integro-differential HJB equations, however, are fully nonlinear for which no analytical solution can usually be expected, in which case the value function is typically verified as the unique solution to the HJB equation in terms of the viscosity solution (see, e.g. [4, 29]). Inspired by the viscosity solution technique (see, e.g. [15, 16]), we first establish the existence of the viscosity solutions of our integro-differential HJB equation via the study of a few important structural properties of the value function. Next, by applying the theory of degenerate and non-degenerate quasilinear differential equations [31], we further improve the smoothness of the viscosity solution by treating the (non-local) integral part of the HJB equation as nonhomogeneous in terms of the value function [17]. One of the main difficulties in studying this stems from the continuity of the value function with a stochastic exit time at the zero-initial inventory level, particularly under the Brownian motion demand model. To overcome this difficulty, we develop a stochastic integral argument to prove that the value function is indeed continuous at the zero-initial inventory level. In turn, the unique viscosity solution indicates that our value function is continuous at all initial inventory levels and satisfies an important regular property that serves as an improvement in the smoothness of the viscosity solution to our integro-differential HJB equation in the subsequent stage. Then we identify the optimal (feedback) price implied by the HJB equation and prove that the viscosity solution (or equivalent value function) analyzed above is, in fact, the unique classical solution of our integro-differential HJB equation. We refer the reader interested in stochastic exit times for control problems to [5,6,8].

Our paper has three main contributions. The first one lies in the formulation of the problem and modeling of the surging demand. The second is that we prove the existence and uniqueness of smooth solutions to associated integro-differential HJB equation, which helps us establish structural properties of the value function and optimal price policy. The third is, interestingly, for the linear demand rate function, we prove that the optimal price could increase rather than decrease with respect to the initial inventory level under a surging demand model.

The remainder of the paper is organized in the following manner. Section 2 introduces the continuous demand model, studies the well-posedness of the HJB equation, and characterizes the structural property of the optimal price. Section 3 extends the continuous model by incorporating the surging demand, and establishes the well-posedness of the resulting (integro-differential) HJB equation as well as the structural property for optimal price with the linear demand rate function. Section 4 provides the extension to the continuous/surging model by taking into account Brownian random fluctuations. Section 5 presents the concluding remarks. All proofs are presented in the Appendices A and B.

2 Continuous demand

Consider a firm that sells a single nonperishable product to a stream of price-sensitive customers over the sales horizon until the item has sold out. Following the classical literature [18], we consider a continuous time framework. At any time $t \geq 0$, the cumulative demand quantity up to time t is described in the following manner:

$$D_t = \int_0^t \lambda(p_s) ds, \tag{2.1}$$

where we assumed that there is no demand at the initial time, that is, $D_0=0$. In (2.1), p_t is the price charged by the firm at time t, and so the demand rate for continuous demand is given by $\lambda(p_t)$ at time t, with the demand rate function $p \to \lambda(p)$ being known to the firm.

Assumption 2.1. There is a one-to-one correspondence between the price p and the demand rate $\lambda(p)$ so that $\lambda(p)$ has an inverse, denoted by $p(\lambda)$. Moreover, $p\to\lambda(p)$ is continuous and non-increasing.

Let $K>0$ be a price ceiling, the maximum price that can be charged for the product. Then, the only price interval at which the product can be traded is $I:=[0,K]$. In view of Assumption 2.1, the demand rate $\lambda(p)\in I_K:=[\lambda(K),\lambda(0)]$ for all $p\in I$. Throughout this paper, it is assumed that $\lambda(K)>0$. The firm is initially endowed with $X_0\in\mathbb{R}_+:=(0,\infty)$ product units and so its inventory level at time $t\geq 0$ becomes

$$X_t=X_0-D_t. \tag{2.2}$$

Then, the time taken for all items to sell out is defined as

$$\tau_o:=\inf\{t\geq 0;X_t\leq 0\}=\inf\{t\geq 0;D_t\geq X_0\}, \tag{2.3}$$

where we define $\tau_o=+\infty$ by convention if the time set $\{t\geq 0;X_t\leq 0\}=\emptyset$.

Let

$$\mathcal{P}_K:=\{p=(p_t)_{t\geq 0};\ p \text{ is measurable, } p_t\in I,\ \forall t\geq 0\}$$

be the set of all allowable pricing policies. With an initial inventory of $x\in\mathbb{R}_+$ units (i.e. $X_0=x$) and a price function $p\in\mathcal{P}_K$, the cumulative revenue generated until all items are sold out (i.e. before τ_o) is

$$J(x,p):=\int_0^{\tau_o}e^{-\rho s}p_s dD_s, \tag{2.4}$$

where $\rho>0$ is the discount rate. In contrast to the traditional revenue management literature (see, e.g. [18, 38]), as the product is assumed to be nonperishable, we consider an infinite-horizon stochastic control problem with the exit time τ_o (see, e.g. [3, 22, 37]) rather than a fixed finite horizon. The stopping time τ_o implies that the firm would not stop selling the product until it is completely sold out.

It must also be noted that if the firm has an initial inventory of zero units, then it obviously cannot sell anything and so its revenue is zero, formulated thusly

$$J(0,p)=0,\quad \forall p\in\mathcal{P}_K. \tag{2.5}$$

The firm's optimal pricing problem is to find an allowable pricing policy $p^*\in\mathcal{P}_K$ (if it exists) that maximizes the total revenue $J(x,p)$ over $p\in\mathcal{P}_K$. In other words, this p^* satisfies that, for the initial inventory units of $x\in\mathbb{R}_+$,

$$V(x)=J(x,p^*)=\sup_{p\in\mathcal{P}_K}J(x,p)=\sup_{p\in\mathcal{P}_K}\int_0^{\tau_o}e^{-\rho s}p_s dD_s. \tag{2.6}$$

The boundary condition of the value function from (2.5) is given as $V(0)=0$.

Using the dynamic programming principle, we formally obtain the HJB equation satisfied by the value function V

$$\sup_{\lambda\in I_K}[-\lambda V'(x)+r(\lambda)]=\rho V(x),\quad x\in\mathbb{R}_+ \tag{2.7}$$

with the boundary condition $V(0)=0$. Here, $r(\lambda):=\lambda p(\lambda)$ is defined as revenue rate.

The next lemma can help us characterize the value function and optimal policy.

Lemma 2.1. *Under Assumption* 2.1*, the value function* $V\in C^1(\mathbb{R}_+)\cap C(\overline{\mathbb{R}}_+)$ *is a unique classical solution of the HJB equation* (2.7)*, where* $\overline{\mathbb{R}}_+:=[0,\infty)$*. There exists a measurable function* $\lambda^*:\mathbb{R}_+\to I_K$ *such that*

$$\mathcal{H}_1\big(x;\lambda^*(x)\big)=\sup_{\lambda\in I_K}\mathcal{H}(x;\lambda):=\sup_{\lambda\in I_K}[-\lambda V'(x)+r(\lambda)],\quad x\in\mathbb{R}_+.$$

Moreover, if $\lambda\to\mathcal{H}_1(x;\lambda)$ *is concave, then* λ^* *is unique.*

To the extent of our reading, the traditional literature (e.g. [7,18,38]) focuses on the Poisson-demand model where optimal price drops as initial inventory increases. The following proposition shows that under some minimal assumption regarding the revenue rate function $\lambda\to r(\lambda)$, our continuous demand model yields a similar result.

Proposition 2.1. *Let Assumption* 2.1 *hold. Assume that the revenue rate function* $r(\cdot)\in C^1(I_K)$ *is strictly concave. Then, under the continuous demand model* (2.1)*, the value function* V *is concave on* $\overline{\mathbb{R}}_+$*, and the optimal price* $p^*(x)$ *decreases in the initial inventory level* x*.*

The linear demand rate function is rather common in the literature on dynamic pricing (see, e.g. [1,13,20]). For our purposes, it will help us to obtain qualitative insights without too much analytical complexity. The linear demand rate function considered in this section assumes the following form:

$$\lambda(p)=M-mp,\quad p\in[0,K(\epsilon)],\tag{2.8}$$

where $M>0$ denotes the demand rate when the price p is zero, with a higher value of M representing a higher overall potential for demand, while $m>0$ represents the price sensitivity of the demand rate ([28]). Here, $K(\epsilon):=(M-\epsilon)/m$ is the price ceiling with $\epsilon\in[0,M]$.

For the continuous linear demand model, the resulting HJB equation from (2.7) becomes that

$$\sup_{\lambda\in[\epsilon,M]}\left[-\lambda V'(x)+\frac{(M-\lambda)\lambda}{m}\right]=\rho V(x).\tag{2.9}$$

It follows from Lemma 2.1 that the corresponding Hamiltonian is read in the following manner:

$$\mathcal{H}_1(x;\lambda)=-\lambda V'(x)+\frac{(M-\lambda)\lambda}{m},\quad (x,\lambda)\in\mathbb{R}_+\times I_K.$$

It is clear here that $\lambda\to\mathcal{H}_1(x;\lambda)$ is concave and this yields the unique optimal (feedback) demand rate

$$\lambda^*(x)=\frac{1}{2}\big(M-mV'(x)\big)\vee\epsilon,\quad x\in\mathbb{R}_+.\tag{2.10}$$

From Assumption 2.1, the optimal (feedback) price is accordingly given as

$$p^*(x)=\frac{1}{2m}\big(M+mV'(x)\big)\wedge K(\epsilon),\quad x\in\mathbb{R}_+.\tag{2.11}$$

Note that both of the optimal price and demand depend on the solution of Eq. (2.9). By solving Eq. (2.9) in terms of (2.10) (or (2.11)), we obtain the following result.

Lemma 2.2. *Under the continuous demand model with the linear demand rate* (2.8)*, the optimal price and demand rate are given by, respectively*

$$p^*(x)=\begin{cases}\dfrac{M-\epsilon}{m}, & 0<x<x^*,\\ \dfrac{M}{m}-\sqrt{\dfrac{\rho V(x)}{m}}, & x\geq x^*,\end{cases}$$
$$\lambda^*(x)=\begin{cases}\epsilon, & 0<x<x^*,\\ \sqrt{\rho m V(x)}, & x\geq x^*,\end{cases} \tag{2.12}$$

where the critical inventory level x^* is given as

$$x^*=\begin{cases}\dfrac{\epsilon}{\rho}\log\dfrac{M-\epsilon}{M-2\epsilon}, & \epsilon\in(0,M/2),\\ \infty, & \epsilon\in[M/2,M].\end{cases} \tag{2.13}$$

The value function satisfies

$$V(x)=\frac{K(\epsilon)\epsilon}{\rho}\left(1-e^{-\frac{\rho}{\epsilon}x}\right), \quad x\in(0,x^*],$$
$$V'(x)=\frac{M}{m}-\sqrt{\frac{4\rho V(x)}{m}}, \quad x>x^*.$$

Moreover, for any $\epsilon\in(0,M/2)$, the optimal price $p^*(x)\downarrow M/(2m)$, as the initial inventory level $x\to\infty$, and the corresponding demand rate $\lambda^*(x)\uparrow M/2$.

Lemma 2.2 provides a closed-form representation (via the value function V) of the product's optimal price and demand rate. Lemma 2.2 also indicates that a static price is asymptotically optimal [30] when the initial inventory level increases to infinity (see Fig. 1 for an illustration).

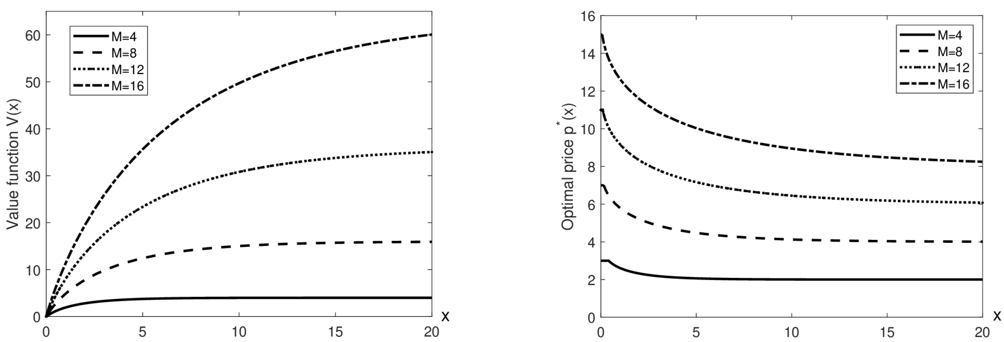

Figure 1: Left panel: Value function $x\to V(x)$. Right panel: Optimal price $x\to p^*(x)$. Parameters are set to be $m=1,\rho=1$, and $\epsilon=1$.

We also note that for the case with $\epsilon \in [M/2, M]$, the firm always sets the ceiling price $K(\epsilon)$ as the optimal price and has no need to adjust its price strategy during the selling season.

3 Continuous-surging demand

In this section, we extend the continuous demand model by describing the cumulative demand process $D=(D_t)_{t\geq 0}$ as the following controlled jump process:

$$D_t = \underbrace{\int_0^t \lambda(p_s)ds}_{\text{continuous demand}} + \underbrace{\int_0^t \int_\Gamma \alpha(p_s,\gamma,X_{s-})N(d\gamma,ds)}_{\text{surging demand}}. \tag{3.1}$$

The second term in (3.1) represents the surging demand, whose arrival process is a Poisson point process $N=(N(t,\gamma);\gamma\in\Gamma)_{t\geq 0}$ with an intensity measure μ on a Borel set Γ with $\mu(\Gamma)<\infty$. The set Γ is called mark space, which records the locations where the rare event occurs. This notion is very versatile, as it can represent the longitude and latitude of a location [36] when it is continuous (e.g. $\Gamma=\mathbb{R}^3$), or even represent different locations when it is discrete (e.g. $\Gamma=\{\gamma_1,\ldots,\gamma_n\}$). Given a location $\gamma\in\Gamma$, the function $\alpha(p,\gamma,x)\in[0,x]$ represents the magnitude of surging demand for the product price $p\in I=[0,K]$ and inventory level $x>0$.

Assumption 3.1. For $(p,\gamma)\in I\times\Gamma$, $x\to\alpha(p,\gamma,x)$ is 1-Lipschitz continuous on $\mathbb{R}_+$, that is

$$|\alpha(p,\gamma,x)-\alpha(p,\gamma,y)|\leq|x-y|, \quad \forall x,y\in\mathbb{R}_+.$$

Fig. 2 displays an arbitrary sample path of the cumulative demand process $t\to D_t$ and the corresponding inventory process $t\to X_t$, when $N=(N_t)_{t\geq 0}$ is a Poisson process with

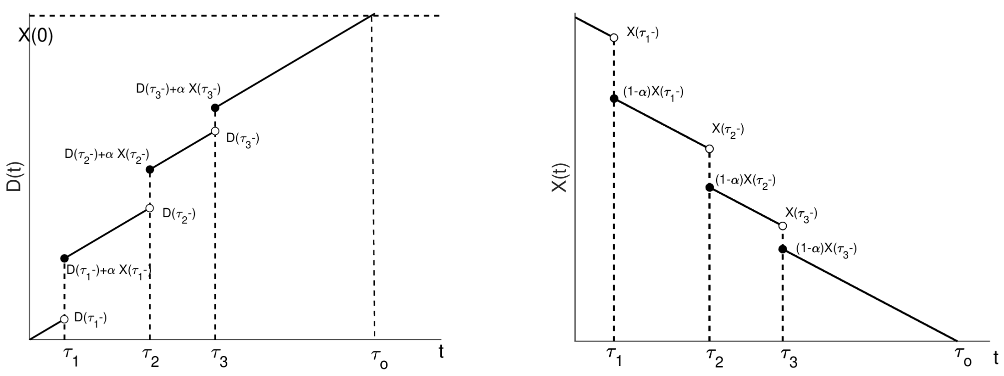

Figure 2: Left panel: Sample path of the cumulative demand process $t\to D_t$ with a linear surging demand $\alpha(p,\gamma,x)=\alpha_0 x$ ($\alpha_0\in[0,1]$). Right panel: Sample path of the inventory level $t\to X_t$. Parameters are set to be $\mu(\Gamma)=0.15, \alpha_0=0.2, \lambda(p)=10-p, p_t=10-0.3t$, and $X_0=80$.

a constant intensity $\mu > 0$ and $\alpha(x) = \alpha_0 x$ (with $\alpha_0 \in [0,1]$) is proportionate to the inventory level via the Monte-Carlo simulation. This can be seen as a special case of the continuous-surging demand model (3.1), which may help us illustrate the model intuitively.

Under the surging demand model (3.1), we rewrite the firm's pricing problem as

$$V(x) = \sup_{p \in \mathcal{P}_K} J(x,p) = \sup_{p \in \mathcal{P}_K} \mathbb{E}\left[\int_0^{\tau_o} e^{-\rho s} p_s dD_s \middle| X(0) = x\right], \tag{3.2}$$

where $\mathcal{P}_K$ denotes the set of all allowable pricing policies. In other words

$$\mathcal{P}_K = \{p = (p_t)_{t \geq 0}; p \text{ is a predictable process such that } p_t \in I, \forall t \geq 0, \text{a.s.}\}. \tag{3.3}$$

Next, we formulate the HJB equation based on the dynamic pricing problem (3.2) using the dynamic programming principle. In terms of the surging demand model (3.1), the value function $V(x)$ formally satisfies the following HJB equation:

$$\sup_{\lambda \in I_K} \left[\mathcal{L}^\lambda V(x) + r(\lambda) + \phi\big(p(\lambda), x\big)\right] = \rho V(x), \quad x \in \mathbb{R}_+ \tag{3.4}$$

with boundary condition $V(0) = 0$. Here, $\mathcal{L}^\lambda$ is an integro-differential operator defined by, for all $f \in C^1(\mathbb{R}_+)$,

$$\mathcal{L}^\lambda f(x) := -\lambda f'(x) + \int_\Gamma \left\{f\big(x - \alpha\big(p(\lambda), \gamma, x\big)\big) - f(x)\right\} \mu(d\gamma), \quad (\lambda, x) \in I_K \times \mathbb{R}_+. \tag{3.5}$$

The function $\phi(p,x)$ for $(p,x) \in I \times \mathbb{R}_+$ measures surging revenue over all the locations where rare events occur, that is

$$\phi(p,x) := p \int_\Gamma \alpha(p,\gamma,x) \mu(d\gamma). \tag{3.6}$$

Note that if we take $\alpha(\cdot) \equiv 0$ in the HJB equation (3.4), then it returns to the case of the continuous demand model.

3.1 The optimal price and well-posedness of the HJB equation

In this section, we focus on the well-posedness (i.e. the existence and uniqueness) of the HJB equation (3.4) in the classical sense, which is crucial for the following discussion on structural properties of optimal price. In the traditional literature on dynamic pricing, particularly studies that consider the Poisson-demand model where inventory level takes only integral values, the proof of well-posedness of their HJB equations is straightforward (see, e.g. [3, 18]). Our model, however, is continuous not only in terms of time, but also in terms of initial inventory level. Taking into account that the surging demand yields an integro-differential HJB equation to which it is difficult to apply a unique classical solution, we overcome this challenge by utilizing the viscosity solution technique.

3.1.1 The optimal price

In this section, we propose a characterization on the optimal (feedback) price implied by the HJB equation.

Lemma 3.1. *Let Assumptions* 2.1 *and* 3.1 *hold. If value function V under demand model* (3.1) *is a classical solution of the integro-differential HJB equation* (3.4)*, there exists a measurable function $\lambda^*:\mathbb{R}_+\to I_K$ such that*

$$\mathcal{H}_2\big(x;\lambda^*(x)\big)=\sup_{\lambda\in I_K}\mathcal{H}(x;\lambda):=\sup_{\lambda\in I_K}\big[\mathcal{L}^\lambda V(x)+\lambda p(\lambda)+\phi\big(p(\lambda),x\big)\big],\quad x\in\mathbb{R}_+, \tag{3.7}$$

where the operator $\mathcal{L}^\lambda$ is defined by (3.5)*. Moreover, if $\lambda\to\mathcal{H}_2(x;\lambda)$ is concave, then λ^* is unique.*

We can apply Lemma 3.1 to characterize the optimal demand rate, and with it the optimal price, if the prior result of the value function is indeed a classical solution to the HJB equation. We then study the viscosity solution of the HJB equation (3.4), whose well-posedness under the continuous demand model can be followed simply. To achieve this, we prove certain structural properties satisfied by the value function under the continuous-surging demand model (3.1).

Lemma 3.2. *Under Assumptions* 2.1 *and* 3.1*, the value function V defined by* (3.2) *under the continuous-surging demand model satisfies that*

(i) $x\to V(x)$ *increases on* $\mathbb{R}_+$,

(ii) $x\to V(x)$ *satisfies the linear growth condition on* $\mathbb{R}_+$,

(iii) $x\to V(x)$ *is right continuous at* $x=0$.

Lemma 3.2(i) indicates that the firm's optimal revenue increases as the initial inventory level also does. More specifically, when the firm has more inventory units at the initial time, then it can sell more item units and earn more profit. Lemma 3.2(ii) proves that the optimal cumulative revenue can be dominated by a nonnegative linear profit function in the initial inventory level, while Lemma 3.2(iii) establishes the continuity of the value function at the vanishing (initial) inventory level. This property plays an important role in the proof of the existence and uniqueness of viscosity solutions to the integro-differential HJB equation (3.4).

3.1.2 The well-posedness of the HJB equation (3.4)

In this section, we investigate the existence and uniqueness of classical solutions to the integro-differential HJB equation (3.4). To do so, we rewrite the HJB equation (3.4) in the following equivalent abstract form:

$$\begin{cases} F(x,V,V')=0, & x>0,\\ V(x)=0, & x=0, \end{cases} \tag{3.8}$$

where the functional F, for any $u \in C^1(\mathbb{R}_+)$, is defined as

$$F(x,u,u') := \rho u(x) - \sup_{\lambda \in I_K} [-\lambda u'(x) + d^u(x,\lambda)], \quad x \in \mathbb{R}_+. \tag{3.9}$$

In (3.9), the non-local term $d^u(x,\lambda)$ for any measurable function u on $\mathbb{R}_+$ is defined as, for $(x,\lambda) \in \mathbb{R}_+ \times I_K$,

$$d^u(x,\lambda) := \int_\Gamma \{u(x - \alpha(p(\lambda),\gamma,x)) - u(x)\} \mu(d\gamma) + \lambda p(\lambda) + \phi(p(\lambda),x). \tag{3.10}$$

Our main strategy for studying classical solutions to the integro-differential HJB equation (3.4) consists of two key steps:

(i) We prove that the value function $V(x)$ for $x \in \mathbb{R}_+$ defined as (3.2) is indeed the viscosity solution to the abstract equation (3.8) (for the definition of viscosity solutions, one may refer to [14–16]).

(ii) Given value function V, we formulate Eq. (3.8) as a degenerate elliptic equation and prove that this has a unique classical solution.

For the first step, we obtain the following main result.

Lemma 3.3. *Let Assumptions* 2.1 *and* 3.1 *hold. Then, for a sufficiently large discount factor* $\rho > 0$*, the value function* V *defined as* (3.2) *under the continuous-demand model* (3.1) *is a viscosity solution that satisfies the linear growth to the abstract equation* (3.8).

In the sequel, we will improve the smoothness of the viscosity solution (i.e. the value function V defined by (3.2) under the demand model (3.1) equipped with Lemma 3.3) of the HJB equation (3.8). More precisely, given value function V defined by (3.2) under the demand model (3.1), we define the following functional, for any $u \in C^1(\mathbb{R}_+)$:

$$F^V(x,u,u') := \rho u(x) - \sup_{\lambda \in I_K} \left[-\lambda u'(x) + d^V(x,\lambda)\right], \quad x \in \mathbb{R}_+, \tag{3.11}$$

where $d^V(x,\lambda)$ is given as (3.10) with u replaced by value function V. We consider the following abstract equation given as

$$\begin{cases} F^V(x,u,u') = 0, & x > 0, \\ u = 0, & x = 0. \end{cases} \tag{3.12}$$

Then, we have lemma.

Lemma 3.4. *Let assumptions of Lemma* 3.3 *hold. The value function* V *defined by* (3.2) *under demand model* (3.1) *is a unique viscosity solution that satisfies the linear growth to the abstract equation* (3.12). *Moreover, it is continuous, that is,* $V \in C(\overline{\mathbb{R}}_+)$.

We then transform the abstract equation (3.12) into a standard elliptic one by utilizing the assumption $\lambda(K)>0$. We define the following function as

$$H^V(x,m):=\sup_{\lambda\in I_K}\frac{1}{\lambda}\left[-\rho m+d^V(x,\lambda)\right],\quad (x,m)\in\mathbb{R}_+\times\mathbb{R}. \tag{3.13}$$

Then, Eq. (3.12) above is equivalent to the following standard elliptic equation:

$$\begin{cases} u'(x)-H^V\big(x,u(x)\big)=0, & x>0,\\ u(x)=0, & x=0. \end{cases} \tag{3.14}$$

Proposition 3.1. *Let assumptions of Lemma* 3.3 *hold. Given value function V defined by* (3.2) *under demand model* (3.1)*, the elliptic equation* (3.14) *has a solution $u\in C^1(\mathbb{R}_+)\cap C(\overline{\mathbb{R}}_+)$ that satisfies the linear growth condition.*

For value function V defined by (3.2) under demand model (3.1), let $u\in C^1(\mathbb{R}_+)\cap C(\overline{\mathbb{R}}_+)$ be the classical solution of Eq. (3.14) in lieu of Proposition 3.1. Then, it is obvious that this u is also a viscosity solution of Eq. (3.14) (or equivalently Eq. (3.12)). Hence, it follows from Lemma 3.4 that

$$V=u\quad \text{on } \overline{\mathbb{R}}_+. \tag{3.15}$$

This leads to the main result of this section on the well-posedness of classical solutions to the abstract equation (3.8).

Theorem 3.1. *Let assumptions of Lemma* 3.3 *hold. Then, the value function $V\in C^1(\mathbb{R}_+)\cap C(\overline{\mathbb{R}}_+)$ is a unique classical solution to the abstract equation* (3.8).

3.2 Example: Optimal price with linear demand

This section analyzes the HJB equation (3.4) with linear demand rate (2.8). We provide interesting insights on the optimal (feedback) price implied by the HJB equation.

In order to analyze the case with the continuous-surging demand model (3.1), we consider the following surging demand function:

$$\alpha(x)=\begin{cases} x, & x\le\Lambda,\\ \Lambda, & x>\Lambda, \end{cases} \tag{3.16}$$

where the constant $\Lambda>0$ represents the potential demand caused by the occurrence of rare events. In this case, the actual surging demand increases with respect to the initial inventory when $x\le\Lambda$, yet when initial inventory level x exceeds Λ, the surging demand has reached its saturation point and will remain constant at Λ.

Remark 3.1. For the generality of the model, the magnitude of surging demand $\alpha(p,\gamma,x)$ is assumed to be dependent on the product price p, the location γ and the inventory level x in the continuous/surging demand model (3.1). However, in reality, sometimes the occurrence of rare event leads to a surge in demand and in turn the strain supply on many kinds of products. In such a case, the price factor often has little impact on demand. For example, due to the shortage of masks at the beginning of the COVID-19 epidemic, although the price of some masks is much higher than usual (see [24]), people would still buy them without hesitation. Furthermore, in this section we focus on the situation of product sales in just one location (e.g. a state or a city). Then the mark space is a single point and the location variable γ can also be ignored. Hence, one can see that we study the surging demand function (3.16) from not only the tractability but also the practice.

The resulting HJB equation from (3.4) then becomes that

$$\begin{aligned}&\sup_{\lambda\in[\epsilon,M]}\left[-\lambda V'(x)+\frac{(M-\lambda)\lambda}{m}-\frac{\lambda\mu\alpha(x)}{m}\right]+\frac{\mu M\alpha(x)}{m}\\&\quad+\mu V\big(x-\alpha(x)\big)-(\rho+\mu)V(x)=0.\end{aligned}\tag{3.17}$$

In view of Theorem 3.1, the corresponding Hamiltonian is read as

$$\mathcal{H}_2(x;\lambda)=-\lambda V'(x)+\frac{(M-\lambda)\lambda}{m}-\frac{\lambda\mu\alpha(x)}{m},\quad (x,\lambda)\in\mathbb{R}_+\times I_K.$$

Note that $\lambda\to\mathcal{H}_2(x;\lambda)$ is concave. It follows from Lemma 3.1 that the unique optimal (feedback) demand rate is

$$\lambda^*(x)=\frac{1}{2}\big\{M-mV'(x)-\mu\alpha(x)\big\}\vee\epsilon,\quad x\in\mathbb{R}_+.\tag{3.18}$$

The optimal (feedback) price from Assumption 2.1 is given by

$$p^*(x)=\frac{1}{2m}\big\{M+mV'(x)+\mu\alpha(x)\big\}\wedge K,\quad x\in\mathbb{R}_+.\tag{3.19}$$

Note that both the optimal price and demand depend on the solution of Eq. (3.17). By solving Eq. (3.17) in terms of (3.18) (or (3.19)), we obtain the following lemma.

Lemma 3.5. *Under the continuous surging demand model* (3.1)*, with linear demand rate function* (2.8)*, the optimal price and demand rate are given respectively as*

$$\begin{aligned}&p^*(x)=\begin{cases}\dfrac{M-\epsilon}{m}, & x\in\mathcal{O},\\ \dfrac{M+\alpha(x)\mu+mV'(x)}{2m}, & x\in\mathcal{O}^c,\end{cases}\\&\lambda^*(x)=\begin{cases}\epsilon, & x\in\mathcal{O},\\ \dfrac{M-\alpha(x)\mu-mV'(x)}{2}, & x\in\mathcal{O}^c,\end{cases}\end{aligned}\tag{3.20}$$

where the set $\mathcal{O}$ is defined as

$$\mathcal{O}:=\left\{x\in\mathbb{R}_+;\frac{1}{2}\big(M-\mu\alpha(x)-mV'(x)\big)\le\epsilon\right\}.\tag{3.21}$$

The value function $V(x)$ for $x\in\mathbb{R}_+$ solves the following equation:

- *on $\mathcal{O}$,*

$$
\begin{aligned}
-\epsilon V'(x)+\frac{(M-\epsilon)\epsilon}{m}-\frac{\epsilon\mu\alpha(x)}{m}+\frac{\mu M\alpha(x)}{m}& \\
+\mu V[x-\alpha(x)]-(\rho+\mu)V(x)=0,& \qquad (3.22)
\end{aligned}
$$

- *on $\mathcal{O}^c$,*

$$
\begin{aligned}
\frac{m}{4}V'(x)^2+\frac{1}{2}\big(\alpha(x)\mu-M\big)V'(x)+\mu V[x-\alpha(x)]& \\
-(\rho+\mu)V(x)+\frac{\alpha(x)^2\mu^2}{4m}+\frac{\alpha(x)\mu M}{2m}+\frac{M^2}{4m}=0.& \qquad (3.23)
\end{aligned}
$$

The boundary condition is $V(0)=0$.

We do not expect to obtain an analytical solution to Eq. (3.17), so we instead solve it numerically for analyzing structural properties satisfied by the optimal price strategy. By solving Eq. (3.17) numerically, both the resulting value function and optimal price with respect to initial inventory level under different values of surging demand size Λ are displayed in Figs. 3 and 4, respectively.

Our first observation is that under certain cases, the static price strategy is optimal. It follows from Lemma 3.2(i) that the value function $V(x)$ is increasing in $x\in\mathbb{R}_+$. In view of (3.20), if the maximal normal demand rate is smaller than the expected surging demand size (i.e. $M<\mu\Lambda$), then optimal demand rate is given by ϵ. The optimal price, therefore, is always the ceiling price $K(\epsilon)=(M-\epsilon)/m$ when the initial inventory level $x\geq\Lambda$ (see Figs. 3 and 4 for an illustration). This indicates that under such conditions, there is no need for the firm to adjust its price when initial inventory level is sufficiently large, since the static price strategy is always optimal. Here, the firm with a sizable initial inventory can save significantly on administrative costs associated with price changes. In contrast, for the firm with a low initial inventory level, the flexibility to change the price becomes crucial.

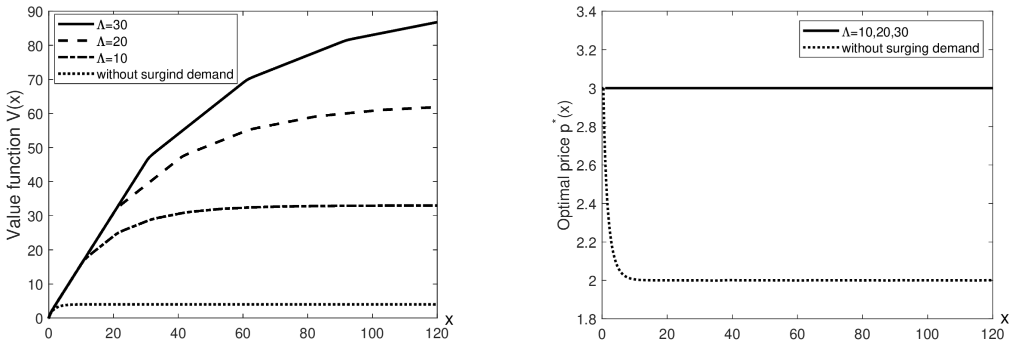

Figure 3: Left panel: Value function $x\to V(x)$. Right panel: Optimal price $x\to p^*(x)$. Parameters are set to be $M=4, m=1, \rho=1, \mu=1$, and $\epsilon=1$.

In another contrast, the optimal price may fail to be monotone in the initial inventory level (Fig. 4), indicating that the optimal price first decreases, then increases to the ceiling price as the initial inventory also increases. Our reading of the existing literature on the dynamic price-based revenue management problem in the absence surging demand has always shown decreasing optimal price with respect to initial inventory level (see, e.g. [11,18,38]). To fill this gap in the literature, we find in our case of surging demand caused by the occurrence of rare events that if the maximal normal demand rate is smaller than the expected surging demand size, the optimal price indeed increases at a certain interval belonging to which the initial inventory level belongs. This finding is summarized in the following proposition.

Proposition 3.2. *For the sufficiently small floor of the demand rate $\epsilon>0$, if the maximal demand rate is smaller than the expected surging demand size (i.e. $M<\mu\Lambda$), then there exists a nonempty interval $J\subset\mathbb{R}_+$, such that the optimal price $x\mapsto p^*(x)$ is strictly increasing for $x\in J$.*

Fig. 4 displays the existence of the abnormal interval J for variously surging demand size Λ. As we analyzed in the previous subsection, there does not exist an abnormal interval J in cases without surging demand (i.e. $\Lambda=0$), meaning that the optimal price function $x\mapsto p^*(x)$ always decreases. In an exception, however, when Λ is away from zero, there appears at least one interval J in which $x\mapsto p^*(x)$ shows increase, because when rare events occur, initial inventory level plays a crucial role in helping improve sales. In fact, when initial inventory level is $x\leq\Lambda$, the surging demand function $\alpha(x)=x$ increases with respect to initial inventory level x, implying that higher initial inventory level may lead to higher demand rate. Sometimes the revenue generated by the surging demand exceeds that generated through price reduction and the corresponding increase in demand rate. In such a case as is indicated by Proposition 3.2, the firm must adopt an opposite price adjustment strategy compared to the one followed in cases of normal demand. We call this strategy as "sell slow to sell fast" strategy, i.e. using a high price to sell slow at regular time but sell fast during demand surges.

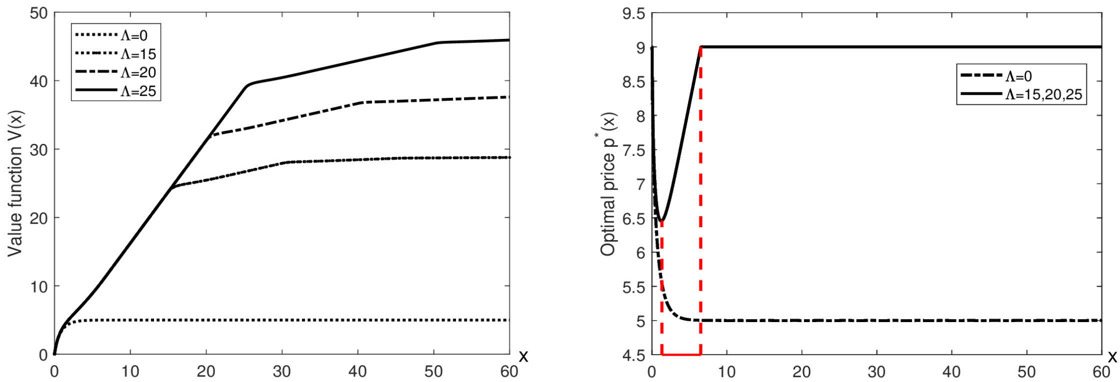

Figure 4: Left panel: Value function $x\to V(x)$. Right panel: Optimal price $x\to p^*(x)$. Parameters are set to be $M=10, m=1, \rho=5, \mu=1$, and $\epsilon=1$.

4 Model extension with random fluctuations

Thus far, we have introduced the continuous demand model and used a marked point process to describe the surge in demand. A more realistic dynamic demand model is needed, however, to consider the random fluctuations that are not captured by the model (e.g. [12]).

The primary focus in this section is to extend the continuous/surging demand model (3.1) by taking into account Brownian random fluctuations, which leads to the following generalized surging demand model as a controlled jump-diffusion process:

$$D_t = \overbrace{\underbrace{\int_0^t \lambda(p_s)ds}_{\text{continuous demand}} + \underbrace{\int_0^t \int_\Gamma \alpha(p_s,\gamma,X_{s-})N(d\gamma,ds)}_{\text{surging demand}}}^{\text{surging demand model (3.1)}} + \underbrace{\int_0^t \sigma\big(\lambda(p_s)\big)dW_s}_{\text{random fluctuations}}. \tag{4.1}$$

Here, $W=(W_t)_{t\geq 0}$ is a Brownian motion that is independent of the marked point process $N(d\gamma,dt)$, while $\sigma(\lambda)$ measures the variability of demand (or demand random fluctuations) when the demand rate is λ. In order to highlight the path behavior of the surging demand process with Brownian random fluctuations compared with that of the surging demand model (3.1), we provide a simulation illustration in Fig. 5.

We impose the following assumption on the variance of the demand forecast.

Assumption 4.1. The (standard) variance function $\lambda \to \sigma(\lambda)$ is continuous on I_K. Moreover, there exists a constant $\sigma_0 > 0$ such that $\sigma(\lambda) > \sigma_0$ for all $\lambda \in I_K$.

The (standard) variance function $\sigma(\cdot)$ here is assumed to have a strictly (positive) lower boundary, in what we refer to the non-degenerate case. If this condition is violated, then we refer to it as the degenerate case. For example, both the continuous

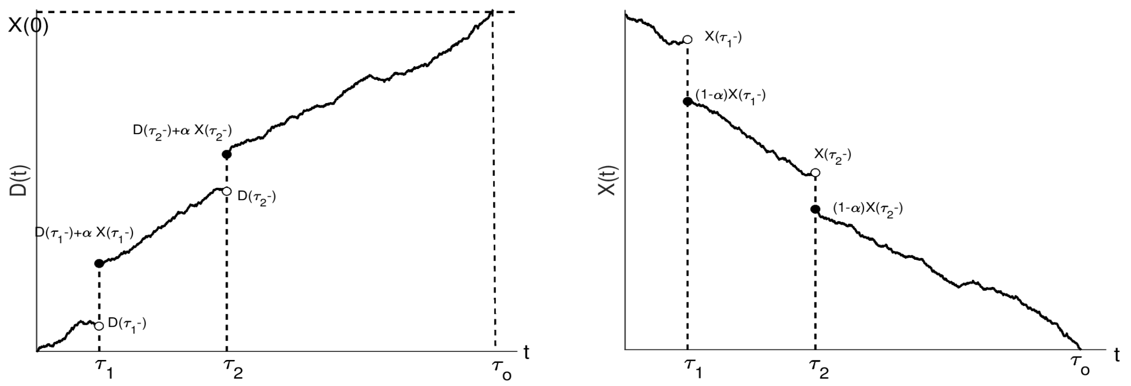

Figure 5: Left panel: Sample path of the cumulative demand process $t \to D_t$ with a linear surging demand $\alpha(p,\gamma,x)=\alpha_0 x$ ($\alpha_0 \in [0,1]$). Right panel: Sample path of the inventory level $t \to X_t$. Parameters are set to be $\mu(\Gamma)=0.15, \sigma(\lambda)=1, \alpha_0=0.2, \lambda(p)=10-p, p_t=10-0.3t$, and $X_0=80$.

demand model (2.1) and the continuous surging demand model (3.1) (corresponding to $\sigma(\cdot)\equiv 0$) result in degenerate models. When we study the well-posedness of HJB equations raised by these models, the degeneracy and non-degeneracy of the demand models would lead to different proofs of the well-posedness of the corresponding HJB equations (see Section 3.1).

In this extended demand model with surging demand, the corresponding value function V defined by (3.2) has the following structural properties.

Proposition 4.1. *Let Assumptions* 2.1, 3.1 *and* 4.1 *hold. Then, the value function V defined by* (3.2) *under the demand model* (4.1) *satisfies that*

(i) $x\to V(x)$ *is increasing on* $\mathbb{R}_+$,

(ii) $x\to V(x)$ *satisfies the linear growth condition on* $\mathbb{R}_+$,

(iii) $x\to V(x)$ *is right continuous at* $x=0$.

The above structural properties satisfied by the value function in the extended model are similar to those of Lemma 3.2 in the degenerate demand case. For item (iii), however, the proof is rather different and more challenging for the appearance of Brownian forecast errors. Equipped with surging demand model (4.1) with Brownian forecast errors and using the dynamic program, the value function V defined by (3.2) formally satisfies the following integro-differential HJB equation:

$$\sup_{\lambda\in I_K}\left[\tilde{\mathcal{L}}^{\lambda}V(x)+r(\lambda)+\phi\big(p(\lambda),x\big)\right]=\rho V(x),\quad x\in\mathbb{R}_+ \tag{4.2}$$

with the boundary condition $V(0)=0$. Here, $\tilde{\mathcal{L}}^{\lambda}$ is an integro-differential operator defined as, for $(\lambda,x)\in I_K\times\mathbb{R}_+$,

$$\tilde{\mathcal{L}}^{\lambda}f(x):=\frac{1}{2}\sigma^2(\lambda)f''(x)-\lambda f'(x)+\int_{\Gamma}\left\{f\big(x-\alpha\big(p(\lambda),\gamma,x\big)\big)-f(x)\right\}\mu(d\gamma) \tag{4.3}$$

for $f\in C^2(\mathbb{R}_+)$.

The following lemma helps to characterize the optimal demand rate (and, hence, the optimal price) if the value function is truly a classical solution to the HJB equation, which we prove in the next section.

Lemma 4.1. *Let Assumptions* 2.1, 3.1 *and* 4.1 *hold. If the value function V under the demand model* (4.1) *is a classical solution of the integro-differential HJB equation* (4.2), *there exists a measurable function* $\lambda^*:\mathbb{R}_+\to I_K$, *so that*

$$\mathcal{H}_3\big(x;\lambda^*(x)\big)=\sup_{\lambda\in I_K}\mathcal{H}(x;\lambda):=\sup_{\lambda\in I_K}\left[\tilde{\mathcal{L}}^{\lambda}V(x)+\lambda p(\lambda)+\phi\big(p(\lambda),x\big)\right],\quad x\in\mathbb{R}_+, \tag{4.4}$$

where the operator $\tilde{\mathcal{L}}^{\lambda}$ *is defined by* (4.3). *Moreover, if* $\lambda\to\mathcal{H}_3(x;\lambda)$ *is concave, then* λ^* *is unique.*

4.1 The well-posedness of the integro-differential HJB equation

This section establishes the well-posedness, in a classical sense, of the integro-differential HJB equation (4.2). The strategy for studying Eq. (4.2) is similar to that of the degenerate case in that it consists of two key steps:

(i) We prove that the value function $V(x)$ for $x\in\mathbb{R}_+$ defined by (3.2) is indeed the unique viscosity solution to a non-degenerate abstract equation (c.f. Eq. (4.5) below).

(ii) With the given value function V, we formulate Eq. (4.5) as a non-degenerate elliptic equation (c.f. Eq. (4.8)) that we prove has a unique classical solution.

Here, we emphasize that property (iii), satisfied by the value function documented in Proposition 4.1 above, plays a key role in the study of the well-posedness of the equation.

We rewrite the HJB equation (4.2) in the following equivalent abstract form:

$$\begin{cases} F(x,V,V',V'')=0, & x>0,\\ u(x)=0, & x=0, \end{cases} \tag{4.5}$$

where the functional F for any $u\in C^2(\mathbb{R}_+)$ is defined as

$$F(x,u,u',u''):=\rho u(x)-\sup_{\lambda\in I_K}\left[\frac{1}{2}\sigma^2(\lambda)u''(x)-\lambda u'(x)+d^u(x,\lambda)\right],\quad x\in\mathbb{R}_+. \tag{4.6}$$

In (4.6), the non-local term $d^u(x,\lambda)$ is given by (3.10). For the first step, we then have the following result.

Lemma 4.2. *Let Assumptions* 2.1, 3.1 *and* 4.1 *hold. If there is a sufficiently large discount factor* $\rho>0$, *then the value function* V *defined by* (3.2) *under the demand model* (4.1) *is a viscosity solution satisfying the linear growth to the abstract equation* (4.5).

As the second step, we next improve the smoothness of the viscosity solution the value function V of the non-degenerate HJB equation (4.2). To do this, for $(x,\lambda)\in\mathbb{R}_+\times I_K$, let $d^V(x,\lambda)$ be defined as (3.10) but with u replaced by the value function V under the demand model (4.1). Then, for any $u\in C^2(\mathbb{R}_+)$, let us define the following functional depending on the value function V:

$$F^V(x,u,u',u''):=\rho u(x)-\sup_{\lambda\in I_K}\left\{-\lambda u'(x)+\frac{1}{2}\sigma^2(\lambda)u''(x)+d^V(x,\lambda)\right\},\quad x\in\mathbb{R}_+. \tag{4.7}$$

Consider the following abstract equation given as

$$\begin{cases} F^V(x,u,u',u'')=0, & x>0,\\ u=0, & x=0. \end{cases} \tag{4.8}$$

Then, we have lemma.

Lemma 4.3. *Let Assumptions* 2.1, 3.1 *and* 4.1 *hold. If there is a sufficiently large discount factor* $\rho>0$ *then the value function* V *defined by* (3.2) *under the demand model* (4.1) *is the unique viscosity solution satisfying the linear growth of the abstract equation* (4.8)*. It is moreover continuous, that is,* $V\in C(\overline{\mathbb{R}}_+)$.

We next transform the abstract equation (4.8) into a standard one. To achieve this, we define the following function as

$$H^V(x,m,q):=\sup_{\lambda\in I_K}\frac{1}{\sigma^2(\lambda)}\left[-\rho m-\lambda q+d^V(x,\lambda)\right],\quad (x,m)\in\mathbb{R}_+\times\mathbb{R}. \tag{4.9}$$

Then, Eq. (4.8) above is equivalent to

$$\begin{cases} u''(x)-H^V\big(x,u(x),u'(x)\big)=0, & x>0,\\ u(x)=0, & x=0. \end{cases} \tag{4.10}$$

Observe that in Eqs. (4.9), (4.10), given the value function V, the standard equation (4.10) is of a class of non-degenerate HJB equations without the nonlocal (integral) term. Given value function V defined by (3.2) in the demand model (4.1), the following proposition indicates that Eq. (4.10) has the unique classical solution.

Proposition 4.2. *Under Assumptions* 2.1, 3.1 *and* 4.1*, the value function* V *defined by* (3.2) *in the demand model* (4.1)*, Eq.* (4.10) *admits a solution* $u\in C^2(\mathbb{R}_+)\cap C(\overline{\mathbb{R}}_+)$.

For the value function V defined by (3.2) in the demand model (4.1), let u be a classical solution of Eq. (4.8) whose existence can be guaranteed by Proposition 4.2. Then, it is obvious that this u is also a viscosity solution of (4.8). Therefore, it follows from Lemma 4.3 that

$$V=u\quad\text{on }\overline{\mathbb{R}}_+. \tag{4.11}$$

This yields the main result of this section.

Theorem 4.1. *Let Assumptions* 2.1, 3.1 *and* 4.1 *hold, then the value function* $V\in C^2(\mathbb{R}_+)\cap C(\overline{\mathbb{R}}_+)$ *is a unique classical solution of the HJB equation* (4.2) *that satisfies the linear growth.*

4.2 Example: Optimal price with linear demand and random fluctuations

We provide an illustrative example with Brownian forecast errors for the linear demand rate $p\to\lambda(p)$ defined in (2.8), and surging demand function $x\to\alpha(x)$ as defined in Section 2. We also assume that the random fluctuations of demand $\sigma(\cdot)$ does not depend on the control variable, or $\sigma(\cdot)\equiv\sigma>0$. Then, for the surging demand model (4.1) with Brownian forecast errors, the resulting HJB equation from (4.2) becomes that

$$\begin{aligned}\frac{\sigma^2}{2}V''(x)+&\sup_{\lambda\in[\epsilon,M]}\left[-\lambda V'(x)+\frac{(M-\lambda)\lambda}{m}-\frac{\lambda\mu\alpha(x)}{m}\right]\\ &+\frac{\mu M\alpha(x)}{m}+\mu V\big(x-\alpha(x)\big)-(\rho+\mu)V(x)=0.\end{aligned} \tag{4.12}$$

In view of Theorem 4.1, the corresponding Hamiltonian is read in the following manner:

$$\mathcal{H}_3(x;\lambda)=-\lambda V'(x)+\frac{(M-\lambda)\lambda}{m}-\frac{\lambda\mu\alpha(x)}{m},\quad (x,\lambda)\in\mathbb{R}_+\times I_K.$$

Note that $\lambda\to\mathcal{H}_3(x;\lambda)$ is concave. It follows from Lemma 4.1 that the unique optimal (feedback) demand rate is

$$\lambda^*(x)=\frac{1}{2}\{M-mV'(x)-\mu\alpha(x)\}\vee\epsilon,\quad x\in\mathbb{R}_+, \tag{4.13}$$

and so the unique optimal (feedback) price from Assumption 2.1 is given by

$$p^*(x)=\frac{1}{2m}\{M+mV'(x)+\mu\alpha(x)\}\wedge K,\quad x\in\mathbb{R}_+. \tag{4.14}$$

We then have furthermore the following lemma.

Lemma 4.4. *Under the surging demand model* (4.1) *with Brownian random fluctuations and linear demand rate* (2.8)*, the optimal price and demand rate are given as, respectively*

$$\begin{aligned}
p^*(x)&=\begin{cases}\dfrac{M-\epsilon}{m}, & x\in\mathcal{O},\\[2mm] \dfrac{M+\alpha(x)\mu+mV'(x)}{2m}, & x\in\mathcal{O}^c,\end{cases}\\
\lambda^*(x)&=\begin{cases}\epsilon, & x\in\mathcal{O},\\[2mm] \dfrac{M-\alpha(x)\mu-mV'(x)}{2}, & x\in\mathcal{O}^c,\end{cases}
\end{aligned}\tag{4.15}$$

where the set $\mathcal{O}$ is defined as

$$\mathcal{O}:=\left\{x\in\mathbb{R}_+;\frac{1}{2}\big(M-\mu\alpha(x)-mV'(x)\big)\le\epsilon\right\}. \tag{4.16}$$

The value function $V(x)$ for $x\in\mathbb{R}_+$ solves the following equation:

- *on $\mathcal{O}$,*

$$\begin{aligned}
&\frac{\sigma^2}{2}V''(x)-\epsilon V'(x)+\frac{(M-\epsilon)\epsilon}{m}-\frac{\epsilon\mu\alpha(x)}{m}\\
&\quad+\frac{\mu M\alpha(x)}{m}+\mu V[x-\alpha(x)]-(\rho+\mu)V(x)=0,
\end{aligned}\tag{4.17}$$

- *on $\mathcal{O}^c$,*

$$\begin{aligned}
&\frac{\sigma^2}{2}V''(x)+\frac{m}{4}V'(x)^2+\frac{1}{2}\big(\alpha(x)\mu-M\big)V'(x)+\mu V[x-\alpha(x)]\\
&\quad-(\rho+\mu)V(x)+\frac{\alpha(x)^2\mu^2}{4m}+\frac{\alpha(x)\mu M}{2m}+\frac{M^2}{4m}=0.
\end{aligned}\tag{4.18}$$

The boundary condition is given by $V(0)=0$.

Next, we show that for this surging demand model with Brownian random fluctuations, in an analogue to Proposition 3.2 in Section 3.2, optimal price may also fail to be monotone at the initial inventory level.

Proposition 4.3. *For a sufficiently large surging demand size $\Lambda > 0$ and a sufficiently small floor of the demand rate $\epsilon > 0$, there exists a nonempty interval $J \subset \mathbb{R}_+$ such that the optimal price $x \mapsto p^*(x)$ is strictly increasing for $x \in J$.*

Together with the abnormal interval J, the value function and optimal price with respect to the initial inventory level under different demand models are displayed in Fig. 6. We find that the firm always reduces the price with respect to the initial inventory level under the normal demand model, while sometimes it raises the price when there is surging demand.

We also note that the firm can earn higher expected revenue under surging demand since it increases suddenly, as compared to the a case of normal demand. This implies that surging demand is a critical factor in enabling the firm to formulate an optimal price strategy and thus increase its expected revenue. It should be kept in mind, however, that the demand variability (or random fluctuations of demand) may decrease the total expected maximum profit. Because of this, the firm must thoughtfully consider the possible random fluctuations of demand rates when devising an optimal price strategy.

Fig. 7 illustrates the impact of the variability of demand σ on the value function, indicating that a large enough demand forecast error can reduce the firm's total expected maximum profit.

Note that for $\epsilon \in [M/2, M]$ and $\Lambda \uparrow +\infty$ (i.e. the surging demand function $\alpha(x) = x$ for all $x > 0$), the region $\mathcal{O} = (0, \infty)$, hence, the HJB equation (4.12) becomes that

$$\frac{1}{2}\sigma^2 V''(x) - \epsilon V'(x) - (\rho + \mu)V(x) + \frac{\mu(M-\epsilon)}{m}x + \frac{(M-\epsilon)\epsilon}{m} = 0, \quad x > 0. \tag{4.19}$$

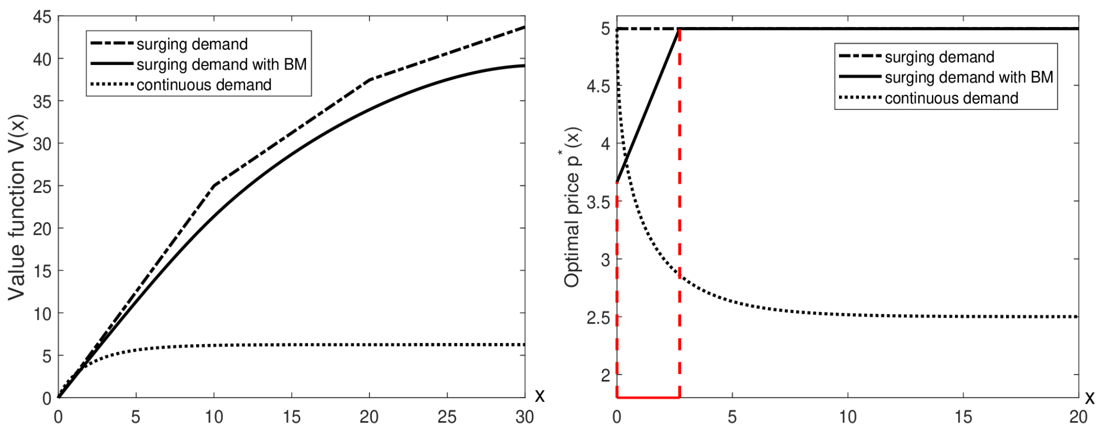

Figure 6: Left panel: Value function $x \to V(x)$. Right panel: Optimal price $x \to p^*(x)$. Parameters are set to be $\sigma = 10, \Lambda = 10, M = 5, m = 1, \rho = 1, \mu = 1$, and $\epsilon = 0.01$.

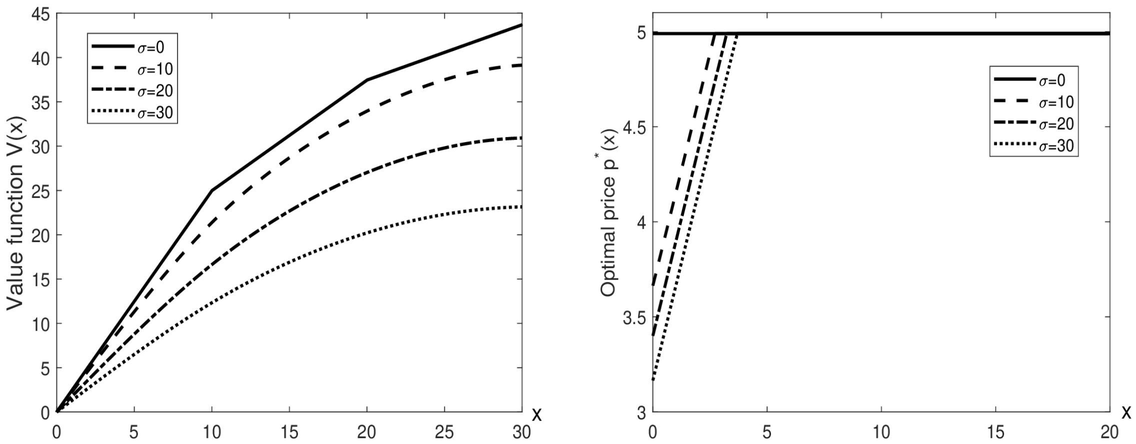

Figure 7: Left panel: Value function $x \to V(x)$. Right panel: Optimal price $x \to p^*(x)$. Parameters are set to be $\Lambda = 10, M = 5, m = 1, \rho = 1, \mu = 1$, and $\epsilon = 0.01$.

By matching the boundary condition $V(0)=0$, the above equation has the unique solution

$$V(x) = \frac{\mu(M-\epsilon)}{m(\mu+\rho)}x + \frac{\epsilon(M-\epsilon)\rho}{m(\mu+\rho)^2}\left[1 - \exp\left(\frac{\epsilon - \sqrt{\epsilon^2 + 2(\mu+\rho)\sigma^2}}{\sigma^2}x\right)\right]. \tag{4.20}$$

Therefore, it holds that

$$V(x) \le \frac{M-\epsilon}{m(\mu+\rho)^2}\left[\epsilon\rho + \mu(\mu+\rho)x - \epsilon\rho\exp\left(-\frac{\mu+\rho}{\epsilon}x\right)\right], \quad \forall x > 0.$$

Note that the right side of the above equation shows precisely the expected optimal profit without random fluctuations (i.e. $\sigma(\cdot) \equiv 0$). This supports our conclusions regarding the negative effects of demand random fluctuations on expected maximum profits, and is also consistent with our previous numerical analysis.

In the previous sections, we have studied the structural properties of optimal price strategy and value function under three different demand models. From these, our analysis reveals the following insights. First, the static price strategy is optimal in certain cases, because it can help the firm save the substantial effort involved in monitoring inventory levels and cost of price adjustments. Second, the firm that ignores the demand random fluctuations always overestimates its expected revenue. We find, in fact, that the Brownian random fluctuations usually has a negative effect on revenue. Lastly, it is usual that the firm with higher initial inventory level might reduce the price of the product, yet under the surging demand model, optimal price may increase with respect to the initial inventory level. This suggests that the firm ought to embrace the surging demand caused by rare events and be more cautious in making price decisions.

5 Conclusion

This paper studies the impact and implications of surging demand in the context of a dynamic pricing problem. A surge in demand may occur due to rare events, whose "arrival" and "place occurred" are jointly modelled by a marked point process. We derive the fully nonlinear integro-differential HJB equation for the dynamic pricing problem formulated as a stochastic control problem with an exit time. In general, it is difficult to verify that the value function is sufficiently smooth to satisfy the HJB equation in the classical sense. Inspired by the viscosity solution technique, we prove that the value function is the unique classical solution to the HJB equation. We assume that the firm is familiar with the initial demand environment – that is, it is aware of the demand rate function. When the demand rate is linear on the price variable, we make a comparison of the optimal price and the value function between the case with and without surging demand. We find that the resulting optimal price decreases as the initial inventory level increases in the absence of surging demand, which is consistent with the findings in the existing literature (e.g. [11,18,38]). Interestingly, in the case of surging demand, the optimal price may increase with respect to the inventory level, which provides the insight into how the firm should adjust its pricing strategy should rare events occur in order to make its maximal expected profit. From this, several topics emerge that are worthy of further research. For one, it would be interesting to consider surging demand functions different from (3.16). It could also be beneficial to explore whether our dynamic pricing problem can be extended to cases of holding cost and replenishment.

Appendix A. Proofs and auxiliary lemmas

This appendix collects the proofs of all results in the previous sections, except Lemmas 4.2 and 4.3 (see Appendix B).

Proof of Lemma 2.1. Note that the continuous demand model is a special case of the surging demand model by taking $\alpha(\cdot)\equiv 0$. Then, this lemma can be immediately followed by applying Theorem 3.1 and Lemma 4.1. □

Proof of Proposition 2.1. We prove the proposition inspired by the viscosity solution technique. To do it, we first define $\hat{V}(x):=-V(x)$ for $x\in\overline{\mathbb{R}}_+$. It follows from (2.7) that $\hat{V}$ obeys that

$$\begin{cases} \hat{F}(x,u,u')=0, & x>0, \\ u=0, & x=0, \end{cases} \tag{A.1}$$

where, for any $u\in C^1(\overline{\mathbb{R}}_+)$,

$$\hat{F}(x,u,u'):=\rho u(x)+\sup_{\lambda\in I_K}\{\lambda u'(x)+r(\lambda)\}$$

with $x>0$. Then, we can show that $\hat{V}$ is the unique classical solution to the abstract equation (A.1) (c.f. the proof of Theorem 3.1). In addition, we introduce the convex envelope $\hat{V}_{**}$ of $\hat{V}$ – that is, the largest convex function below $\hat{V}$ [2]

$$\hat{V}_{**}(x):=\inf\left\{k_1\hat{V}(x_1)+k_2\hat{V}(x_2);x=k_1x_1+k_2x_2,x_1,x_2\in\overline{\mathbb{R}}_+,k_1+k_2=1,k_1,k_2>0\right\}. \quad \text{(A.2)}$$

Note that the value function $x\to V(x)$ is bounded. In fact, Assumption 2.1 yields that

$$V(x)=\sup_{p\in\mathcal{P}_K}\int_0^{\tau_o}e^{-\rho s}p_s\lambda(p_s)ds\le\int_0^\infty e^{-\rho s}K\lambda(0)ds\le\frac{K\lambda(0)}{\rho}.$$

In the sequel, we show that $x\to V(x)$ is a concave function. It suffices to prove that, for all $x\in\overline{\mathbb{R}}_+$, there exist $x_1,x_2\in\overline{\mathbb{R}}_+$ and $k_1,k_2\ge 0$ such that

$$k_1+k_2=1,\quad x=k_1x_1+k_2x_2,\quad \hat{V}_{**}(x)=k_1\hat{V}(x_1)+k_2\hat{V}(x_2). \quad \text{(A.3)}$$

Since $x\to\hat{V}(x)=-V(x)$ is bounded, the convex envelope $x\to\hat{V}_{**}(x)$ is hence well-defined. We fix $x>0$. In light of (A.2), there exist

$$\left(x_1^n\right)_{n\ge1},\left(x_2^n\right)_{n\ge1}\subset\overline{\mathbb{R}}_+,\quad \left(k_1^n\right)_{n\ge1},\left(k_2^n\right)_{n\ge1}\subset\mathbb{R}_+$$

such that

$$k_1^n+k_2^n=1,\quad x=k_1^nx_1^n+k_2^nx_2^n,\quad \hat{V}_{**}(x)+\frac{1}{n}>k_1^n\hat{V}\left(x_1^n\right)+k_2^n\hat{V}\left(x_2^n\right)\ge\hat{V}_{**}(x). \quad \text{(A.4)}$$

Without loss of generality, we may assume that $x_1^n\le x_2^n$ for all $n\in\mathbb{N}$. Therefore, $0<x_1^n\le x$ and $x_2^n\ge x$. There exists a subsequence of $(x_1^n,x_2^n,k_1^n,k_2^n)_{n\ge1}$ (still denoted by $(x_1^n,x_2^n,k_1^n,k_2^n)_{n\ge1}$), such that (x_1^n,k_1^n,k_2^n) converges to $(x_1,k_1,k_2)\in\overline{\mathbb{R}}_+\times[0,1]^2$, as $n\to\infty$.

- If $k_2>0$, there exists an $N_1\in\mathbb{N}$ such that $k_2^n>0$ for all $n\ge N_1$. It follows from $x_2^n=(x-k_1^nx_1^n)/k_2^n$ that x_2^n also converges to some $x_2\ge0$ as $n\to\infty$. Taking $n\to\infty$ on both sides of (A.4), and using the continuity of $x\to\hat{V}(x)$, we arrive at

$$\hat{V}_{**}(x)=k_1\hat{V}(x_1)+k_2\hat{V}(x_2).$$

- If $k_2=0$, it follows from $k_1^n+k_2^n=1$ for all $n\in\mathbb{N}$ that $k_1=1$. Letting $n\to\infty$ on both sides of (A.4) again. Then, the boundedness of $x\to\hat{V}(x)$ yields that

$$\hat{V}(x)\ge\hat{V}_{**}(x)=\hat{V}(x_1).$$

  Note that $\hat{V}(x)$ is strictly decreasing in x. Then, we obtain $x_1\ge x$, which implies that $x_1=x$. Thus, we can take $x_1=x,x_2=x,k_1=1$ and $k_2=0$, which yields (A.3).

We next apply (A.3) to show $\hat{V}_{**}\in\mathsf{LSC}(\overline{\mathbb{R}}_+)$. In fact, for fixed $x\geq 0$, consider a sequence $(x_n)_{n\geq 1}\subset\overline{\mathbb{R}}_+$ satisfying $x_n\to x$, as $n\to\infty$. For each $n\geq 1$, there exist $x_1^n,x_2^n\in\overline{\mathbb{R}}_+$ and $k_1^n,k_2^n\geq 0$ such that

$$k_1^n+k_2^n=1,\quad x_n=k_1^n x_1^n+k_2^n x_2^n,\quad \hat{V}_{**}(x_n)=k_1^n\hat{V}\left(x_1^n\right)+k_2^n\hat{V}\left(x_2^n\right).$$

Since x is the convex combination of the limit point (x_1,x_2,k_1,k_2) of a converging subsequence $(x_1^{n'},x_2^{n'},k_1^{n'},k_2^{n'})$, we deduce that

$$\liminf_{n'\to\infty}\hat{V}_{**}(x_{n'})=\liminf_{n'\to\infty}\left(k_1^{n'}\hat{V}\left(x_1^{n'}\right)+k_2^{n'}\hat{V}\left(x_2^{n'}\right)\right)\geq k_1\hat{V}(x_1)+k_2\hat{V}(x_2)\geq\hat{V}_{**}(x).$$

This easily gives that $\hat{V}_{**}\in\mathsf{LSC}(\overline{\mathbb{R}}_+)$. Then, by [2, Proposition 2], $\hat{V}_{**}$ is also a viscosity supersolution of Eq. (A.1). It follows from the comparison theorem of the viscosity solution (c.f. the proof of Lemma 4.3) with $\hat{V}(0)=\hat{V}_{**}(0)=0$ that $\hat{V}_{**}(x)\geq\hat{V}(x)$ for all $x\geq 0$. On the other hand, by (A.2), we have that $\hat{V}_{**}(x)\leq\hat{V}(x)$ for all $x\geq 0$. Therefore, $\hat{V}=V_{**}$, which yields that $x\to V(x)=-\hat{V}(x)$ is a concave function.

Lastly, by applying Lemma 2.1 and $r\in C^1(I_K)$, the unique optimal (feedback) demand rate $\lambda^*(x)$ is given by

$$\lambda^*(x)=(r')^{-1}\big(V'(x)\big)\vee\lambda(K),$$

where $(r')^{-1}(\cdot)$ represents the inverse function of $r'(\cdot)$. Thus, by Assumption 2.1, the concavity of $x\to V(x)$ and $\lambda\to r(\lambda)$, the optimal price $p^*(x)$ decreases with respect to the initial inventory level x. We complete the proof of the proposition. □

Proof of Lemma 2.2. We introduce the following set given by

$$\mathcal{O}:=\left\{x\in\mathbb{R}_+;\frac{1}{2}\big(M-mV'(x)\big)<\epsilon\right\}. \tag{A.5}$$

On the set $\mathcal{O}$. In view of (2.10), the optimal (feedback) demand rate and price are, respectively, given by

$$\lambda^*(x)=\epsilon,\quad p^*(x)=K(\epsilon)=\frac{M-\epsilon}{m},\quad \forall x\in\mathcal{O}. \tag{A.6}$$

Plugging them into (2.9), and it results in the following equation on $\mathcal{O}$:

$$-\epsilon V'(x)-\rho V(x)+K(\epsilon)\epsilon=0,\quad \forall x\in\mathcal{O}. \tag{A.7}$$

Hence, with the initial condition $V(0)=0$, Eq. (A.7) has the following closed-form solution:

$$V(x)=\frac{K(\epsilon)\epsilon}{\rho}\left(1-e^{-\frac{\rho}{\epsilon}x}\right),\quad x\in\mathcal{O}. \tag{A.8}$$

In turn, by substituting (A.8) into $(M-mV'(x))/2\leq\epsilon$, we obtain that $\mathcal{O}=\{x\in\mathbb{R}_+;\ 0<x<x^*\}$, where x^* is given by (2.13).

On the set $\mathcal{O}^c$. On the set $\mathcal{O}^c=\{x\in\mathbb{R}_+;\ x>x^*\}$, the optimal demand rate is $\lambda^*(x)=(M-mV'(x))/2$. Accordingly, the optimal price is

$$p^*(x)=\frac{1}{2m}\big(M+mV'(x)\big)$$

for $x\in\mathcal{O}^c$. This yields that the HJB equation (2.9) on $\mathcal{O}^c$ becomes that

$$V'(x)=\frac{M}{m}-\sqrt{\frac{4\rho V(x)}{m}},\quad x\geq x^* \tag{A.9}$$

with the boundary condition (continuous fit) $V(x^*)=\epsilon^2/(m\rho)$. Then, in view of (2.6), it follows that, for all $x\in\mathbb{R}_+$,

$$V(x)=\sup_{p\in\mathcal{P}_K}\int_0^{\tau_o}e^{-\rho s}r(\lambda_s)ds=\sup_{p\in\mathcal{P}_K}\int_0^{\tau_o}e^{-\rho s}p_s(M-mp_s)ds\leq\int_0^{\infty}e^{-\rho s}\frac{M}{4m^2}ds\leq\frac{M^2}{4m\rho}.$$

On the other hand, since $\epsilon\in(0,M/2)$, we have $K(\epsilon)>M/(2m)$. Then, we take the constant price strategy $p_t\equiv M/(2m)\in[0,K(\epsilon)]$. Hence, the corresponding selling period with initial inventory level x is given by $\tau_o=2x/M$. Thus, for all $x\in\mathbb{R}_+$,

$$V(x)\geq\int_0^{\frac{2x}{M}}e^{-\rho s}p_s(M-mp_s)ds=\int_0^{\frac{2x}{M}}e^{-\rho s}\frac{M^2}{4m}ds=\frac{M^2}{4m\rho}\left(1-e^{-\frac{2\rho x}{M}}\right).$$

That is,

$$\frac{M^2}{4m\rho}\left(1-e^{-\frac{2\rho x}{M}}\right)\leq V(x)\leq\frac{M^2}{4m\rho},\quad\forall x\in\mathbb{R}_+.$$

This yields that $\lim_{x\to\infty}V(x)=M^2/(4m\rho)$. By Eq. (A.9), we have that

$$p^*(x)=\frac{M}{m}-\sqrt{\frac{\rho V(x)}{m}},\quad x\geq x^*.$$

Together with Proposition 2.1, we obtain $p^*(x)\downarrow M/(2m),\lambda^*(x)\uparrow M/2$, as $x\to\infty$. □

Proof of Lemma 3.1. The proof is similar to the proof of Lemma 4.1. □

Proof of Lemma 3.2. The proof is similar to the proof of Proposition 4.1. □

Proof of Lemma 3.3. The proof is similar to the proof of Lemma 4.2. □

Proof of Lemma 3.4. The proof is similar to the proof of Lemma 4.3. □

Proof of Proposition 3.1. We apply [31, Lemma 4] to establish the existence of classical solutions to Eq. (3.14). To this end, recall that $H^V(x,m)$ which is defined by (3.13), i.e.

$$H^V(x,m)=\sup_{\lambda\in I_K}\frac{1}{\lambda}\big\{-\rho m+d^V(x,\lambda)\big\},\quad (x,m)\in\mathbb{R}_+\times\mathbb{R}. \tag{A.10}$$

We next verify that the function $H^V(x,m)$ satisfies the conditions of [31, Lemma 4]: for any compact subset $\mathcal{X}$ of $\overline{\mathbb{R}}_+$, there exist constants M_1 and M_2 such that, for all (x,m), $(x,\tilde{m})\in\mathcal{X}\times\mathbb{R}$,

(i) $|H(x,m)|\leq M_1(1+|m|)$,

(ii) $|H(x,m)-H(x,\tilde{m})|\leq M_2|m-\tilde{m}|$,

(iii) H is continuous in $x\in\mathcal{X}$ for each m.

First of all, it follows from the linear growth of $x\to V(x)$ (c.f. Lemma 3.2) that there exists a constant $C>0$ such that $V(x)\leq C(1+x)$ for all $x\geq 0$. Therefore, for all $(x,\lambda)\in\overline{\mathbb{R}}_+\times I_K$,

$$\begin{aligned}
\big|d^V(x,\lambda)\big|&=\left|\int_\Gamma\big\{V\big(x-\alpha\big(p(\lambda),\gamma,x\big)\big)-V(x)\big\}\mu(d\gamma)+\lambda p(\lambda)+\phi\big(p(\lambda),x\big)\right|\\
&\leq\int_\Gamma\big|V\big(x-\alpha\big(p(\lambda),\gamma,x\big)\big)\big|\mu(d\gamma)+\mu(\Gamma)|V(x)|+\lambda p(\lambda)+\phi\big(p(\lambda),x\big)\\
&\leq M\left\{1+\int_\Gamma\big(x-\alpha\big(p(\lambda),\gamma,x\big)\big)^2\mu(d\gamma)\right\}x^2+M(1+x^2)+\lambda p(\lambda)+Mx\\
&\leq M(1+x^2)+\lambda(0)K+Mx,
\end{aligned} \tag{A.11}$$

where $M>0$ is a generic constant (independent of (x,λ)) which may be different from line to line. For any compact subset $\mathcal{X}$ of $\overline{\mathbb{R}}_+$, using Assumptions 2.1 and 3.1, there exists a constant $\tilde{M}=\tilde{M}(\mathcal{X})>0$, such that $|d^V(x,\lambda)|\leq\tilde{M}$ for all $(x,\lambda)\in\mathcal{X}\times I_K$. As a consequence, by the assumption $\lambda(K)>0$, it holds that

$$\sup_{(x,\lambda)\in\mathcal{X}\times I_K}\left|\frac{d^V(x,\lambda)}{\lambda}\right|\leq\frac{\tilde{M}}{\lambda(K)}. \tag{A.12}$$

Then, it follows from (A.12) that, for all $m\in\mathbb{R}$,

$$\big|H^V(x,m)\big|\leq\frac{\tilde{M}+\rho}{\lambda(K)}\{1+|m|\}.$$

This yields the condition (i). On the other hand, it is easy to check that

$$\big|H^V(x,m)-H^V(x,\tilde{m})\big|\leq\frac{\rho}{\lambda(K)}|m-\tilde{m}|,\quad\forall(x,m),(x,\tilde{m})\in\mathcal{X}\times\mathbb{R},$$

which verifies the validity of the condition (ii). For any $m\in\mathbb{R}$, using Assumptions 2.1 and 3.1, the function

$$h(m;x,\lambda):=\frac{1}{\lambda}\big\{-\rho m+d^V(x,\lambda)\big\}$$

is jointly continuous in $(x,\lambda)\in\mathcal{X}\times I_K$. From the Berge's maximum theorem (see, e.g. [31, Proposition 8]), it follows that $H^V(x,m)=\sup_{\lambda\in I_K}h(m;x,\lambda)$ is continuous in $x\in\mathcal{X}$ for each $m\in\mathbb{R}$. This implies that (iii) holds.

It remains to show that the solution u satisfies the linear growth condition. We finish this proof by using the comparison theorem (c.f. [33, Theorem II.IX]). In fact, in view of Assumptions 2.1 and 3.1, there exists a constant $\tilde{M}>0$ such that $|d^V(x,\lambda)|\leq\tilde{M}(1+x)$ for all $(x,\lambda)\in\mathbb{R}_+\times I_K$. Then, we let $C:=\lambda(0)\tilde{M}/(\lambda(K)\rho)$, and define the following function given by

$$w_1(x)=C(1+x),\quad w_2(x)=-C(1+x),\quad \forall x\geq 0.$$

Thus, we can obtain that, for all $x\geq 0$,

$$\begin{aligned}&w_1'(x)-H^V\big(x,w(x)\big)\\=&w_1'(x)-\sup_{\lambda\in I_K}\left\{-\frac{\rho}{\lambda}w_1(x)+\frac{1}{\lambda}d^V(x,\lambda)\right\}\\\geq& C+\frac{\rho}{\lambda(0)}C(1+x)-\frac{\tilde{M}}{\lambda(K)}(1+x)>0.\end{aligned}$$

Then, by the comparison theorem, $u(x)\leq w_1(x)=C(1+x)$ for all $x\geq 0$. In a similar fashion, for all $x\geq 0$, it holds that

$$\begin{aligned}&w_2'(x)-H^V\big(x,h(x)\big)\\=&w_2'(x)-\sup_{\lambda\in I_K}\left\{-\frac{\rho}{\lambda}w_2(x)+\frac{1}{\lambda}d^V(x,\lambda)\right\}\\\leq&-C-\frac{\rho}{\lambda(0)}C(1+x)+\frac{\tilde{M}}{\lambda(K)}(1+x)<0.\end{aligned}$$

Hence, using the comparison theorem again, it follows that $u(x)\geq w_2(x)=-C(1+x)$ for all $x\geq 0$. This yields that $|u(x)|\leq C(1+x)$ for all $x\geq 0$, i.e. u satisfies the linear growth condition. Thus, the proof of the proposition is complete. □

Proof of Lemma 3.5. The proof is similar to the proof of Lemma 2.2. □

Before the proof of Proposition 3.2, we have the following observation. If $\epsilon\in[M/2,M]$, since the value function $x\to V(x)$ is increasing, it holds that

$$\frac{1}{2}\big(M-mV'(x)-\mu\alpha(x)\big)\leq\epsilon,\quad \forall x>0.$$

Hence, the set $\mathcal{O}=(0,\infty)$, i.e. the optimal price $p^*(x)=K(\epsilon)$ for $x>0$. Then, the HJB equation (3.17) becomes that

$$-\epsilon V'(x)+\frac{(M-\epsilon)\epsilon}{m}-\frac{\epsilon\mu\alpha(x)}{m}+\frac{\mu M\alpha(x)}{m}+\mu V\big(x-\alpha(x)\big)-(\rho+\mu)V(x)=0. \tag{A.13}$$

In this case, we can actually get the analytical solution to Eq. (A.13). Note that, for $x\le\Lambda$, $\alpha(x)=x$, by solving Eq. (A.13) on $[0,\Lambda]$ with the initial condition $V(0)=0$, we obtain that

$$V(x)=\frac{M-\epsilon}{m(\mu+\rho)^2}\left[\epsilon\rho+\mu(\mu+\rho)x-\epsilon\rho\exp\left(-\frac{\mu+\rho}{\epsilon}x\right)\right],\quad x\in[0,\Lambda]. \tag{A.14}$$

For $x\ge\Lambda$, $\alpha(x)=\Lambda$, the solution to (A.13) on $[\Lambda,\infty)$ is

$$\begin{aligned}V(x)=&\exp\left(-\frac{\mu+\rho}{\epsilon}x\right)\\&\times\left[-\int_0^x\exp\left(\frac{\mu+\rho}{\epsilon}y\right)\frac{\mu M\Lambda-\epsilon\mu\Lambda+(M-\epsilon)\epsilon+\mu V(x-\Lambda)}{\epsilon}dy+C\right].\end{aligned} \tag{A.15}$$

Here $C\in\mathbb{R}$ is a constant which can be determined by the continuity of $V(x)$ at $x=\Lambda$. Then, by (A.14) and (A.15), we can get the exact expression of the value function $V(x)$ on $[\Lambda,2\Lambda]$. Similarly, by using the expression of the value function $V(x)$ on $[\Lambda,2\Lambda]$, we can solve Eq. (A.13) explicitly again on $[2\Lambda,3\Lambda]$, then so do on $[3\Lambda,4\Lambda],[4\Lambda,5\Lambda],\ldots$.

Next, by applying Lemma 3.5 and the discussion above, we give the proof of Proposition 3.2.

Proof of Proposition 3.2. We note that, the condition $M<\mu\Lambda$ gives that

$$\frac{M-\mu\alpha(x)-mV'(x)}{2}\le 0,\quad x\ge\frac{M}{\mu}.$$

Then, the optimal price $p^*(x)=(M-\epsilon)/m$ (that is, the ceiling price) for all $x\ge M/\mu$. Hence, it is sufficient to prove $\mathcal{O}^c\neq\emptyset$. Next, we prove it by contradiction. Suppose $\mathcal{O}^c=\emptyset$, i.e. $\mathcal{O}=(0,\infty)$. It follows from (A.14) that

$$\begin{aligned}\Lambda(x):=&\frac{M-\mu x-mV'(x)}{2}-\epsilon\\=&\frac{1}{2}\left\{M-\mu x-\frac{M-\epsilon}{\mu+\rho}\left[\mu+\rho\exp\left(-\frac{\mu+\rho}{\epsilon}x\right)\right]\right\}-\epsilon,\quad\forall x\le\Lambda.\end{aligned}$$

Consequently, we have

$$\Lambda'(x)=-\frac{1}{2}\left[\mu-\frac{\rho(M-\epsilon)}{\epsilon}\exp\left(-\frac{\mu+\rho}{\epsilon}x\right)\right].$$

Let $\Lambda'(x)=0$, we then obtain that

$$x_0=\frac{\epsilon}{\mu+\rho}\ln\frac{\rho(M-\epsilon)}{\mu\epsilon}.$$

Note that for the sufficiently small normal demand rate $\epsilon>0$, we have $0<x_0<\Lambda$, hence, the following estimate holds:

$$\begin{aligned}\Lambda(x_0)&=\frac{1}{2}\left(M-\mu x_0-\frac{\mu}{\mu+\rho}M\right)-\epsilon\\&=\frac{1}{2}\left(\frac{\rho}{\mu+\rho}M-\frac{\mu\epsilon}{\mu+\rho}\ln\frac{\rho(M-\epsilon)}{\mu\epsilon}\right)-\epsilon\ \to\ \frac{\rho}{\mu+\rho}M>0\quad\text{as}\quad\epsilon\ \to\ 0.\end{aligned}$$

This means that, if the normal demand rate $\epsilon>0$ small enough, then $\Lambda(x_0)>0$. This results in a contradiction with $\mathcal{O}=(0,\infty)$. Thus, we complete the proof of the proposition. □

Proof of Proposition 4.1. We first prove the conclusion (i). To this purpose, let $x,y\geq 0$ satisfy $x>y$. For $p=(p_t)_{t\geq 0}\in\mathcal{P}_K$, denote by $X^{x,p}=(X_t^{x,p})_{t\geq 0}$ the inventory process with the initial inventory level $X_0=x$ and the control p. It follows from [25, Theorem 3.1] that, for all $t\geq 0, X_t^x\geq X_t^y,\mathbb{P}$-a.s. Moreover, let $\tau_o(x,p)$ be the stopping time defined by (2.3) with X_t replaced by $X_t^{x,p}$. Therefore, we obtain that $\tau_o(x,p)\geq\tau_o(y,p),\mathbb{P}$-a.s. By the Assumption 3.1 and the nonnegativity of $\mu(\cdot)$, we get that, for any $p\in\mathcal{P}_K$,

$$\begin{aligned}&\mathbb{E}\left[\int_0^{\tau_o(x,p)}e^{-\rho s}\left\{r(p_s)+\phi\left(p_s,X_s^{x,p}\right)\right\}ds\right]\\&\geq\mathbb{E}\left[\int_0^{\tau_o(y,p)}e^{-\rho s}\left\{r(p_s)+\phi\left(p_s,X_s^{y,p}\right)\right\}ds\right].\end{aligned}$$

Thus, together with (3.2), it results in $V(x)\geq V(y)$ when $x>y\geq 0$. In other words, $x\to V(x)$ is increasing on $\mathbb{R}_+$. This shows the validity of the claim (i).

Next, we prove the conclusion (ii). For any $x\geq 0$, it follows from (3.1) that, for all $p\in\mathcal{P}_K$,

$$\begin{aligned}X_t^{x,p}\mathbf{1}_{t\leq\tau_o(x,p)}&=\left\{x-\int_0^t\lambda(p_s)ds-\int_0^t\sigma\big(\lambda(p_s)\big)dW_s-\int_0^t\int_\Gamma\alpha\big(p(s),\gamma,X_{s-}^{x,p}\big)N(d\gamma,ds)\right\}1_{t\leq\tau_o(x,p)}\\&\leq x+\left|\int_0^t\sigma\big(\lambda(p_s)\big)dW_s\right|,\quad t\geq 0.\end{aligned}$$

Then, by the Burkholder-Davis-Gundy inequality, we have that

$$\mathbb{E}[X_t^{x,p}\mathbf{1}_{t\leq\tau_o(x,p)}]\leq x+M\sqrt{t}$$

for some constant $M>0$, which is independent of (t,x,p). This immediately yields that, for all $(t,x)\in\overline{\mathbb{R}}_+^2$,

$$\sup_{p\in\mathcal{P}_K}\mathbb{E}\left[X_t^{x,p}\mathbf{1}_{t\leq\tau_o(x,p)}\right]\leq x+M\sqrt{t}. \tag{A.16}$$

In the sequel, we let $M>0$ be a generic positive constant independent of (t,x,p), which may be different from line to line. Then, it follows from (A.16), Assumptions 2.1 and 3.1 that

$$\begin{aligned}V(x)&\leq \sup_{p\in\mathcal{P}_K}\mathbb{E}\left[\int_0^\infty e^{-\rho s}r(p_s)ds\right]+\sup_{p\in\mathcal{P}_K}\mathbb{E}\left[\int_0^\infty e^{-\rho s}\phi(p_s,X_s^{x,p})\mathbf{1}_{s\leq\tau_o(x,p)}ds\right]\\&\leq\frac{M}{\rho}+M\int_0^\infty e^{-\rho s}\sup_{p\in\mathcal{P}_K}\mathbb{E}\left[X_s^{x,p}\mathbf{1}_{s\leq\tau_o(x,p)}\right]ds\\&\leq\frac{M}{\rho}+M\int_0^\infty e^{-\rho s}(x+M\sqrt{s})ds\\&=\frac{M}{\rho}(1+x)+M^2+M^2\int_0^\infty e^{-\rho s}\sqrt{s}ds\\&\leq\frac{M}{\rho}(1+x)+\frac{M^2}{\rho^{\frac{3}{2}}}\leq M(1+x).\end{aligned}$$

That is, $x\to V(x)$ satisfies the linear growth condition. Toward this end, we prove the validity of the conclusion (iii). To do it, note that $V(0-)=0$, let $(x_n)_{n\geq1}$ with $x_n\geq0$ satisfying $x_n\to0$ as $n\to\infty$. Then, by (2.5), it is enough to show that $V(x_n)\to V(0)=0$ as $n\to\infty$. In fact, first of all, we note that, for any $(t,x,p)\in\mathbb{R}_+^2\times\mathcal{P}_K$,

$$\begin{aligned}X_t^{x,p}\mathbf{1}_{t\leq\tau_o(x,p)}&=\left\{x-\int_0^t\lambda(p_s)ds-\int_0^t\sigma\big(\lambda(p_s)\big)dW_s-\int_0^t\int_\Gamma\alpha\big(p_s,\gamma,X_{s-}^{x,p}\big)N(d\gamma,ds)\right\}\mathbf{1}_{t\leq\tau_o(x,p)}\\&\leq Y^{x,p}(t)\mathbf{1}_{t\leq\tau_o(x,p)}.\end{aligned}\tag{A.17}$$

Here, we defined that

$$Y_t^{x,p}:=x-\int_0^t\sigma\big(\lambda(p_s)\big)dW_s,\quad\forall(t,x,p)\in\mathbb{R}_+^2\times\mathcal{P}_K.$$

Furthermore, we introduce that

$$\tilde{\tau}_o(x,p):=\inf\big\{t\geq0;Y_t^{x,p}\leq0\big\}.\tag{A.18}$$

As a consequence, for all $(x,p)\in\mathbb{R}_+\times\mathcal{P}_K$, it holds that $\tilde{\tau}_o(x,p)\geq\tau_o(x,p)$, $\mathbb{P}$-a.s. It follows from (A.17), Assumption 3.1 and Hölder inequality that

$$\begin{aligned}V(x)&=\sup_{p\in\mathcal{P}_K}\mathbb{E}\left[\int_0^\infty\mathbf{1}_{s\leq\tau_o(x,p)}e^{-\rho s}\big\{r(p_s)+\phi\big(p_s,X_s^{x,p}\big)\big\}ds\right]\\&\leq M\int_0^\infty e^{-\rho s}\sup_{p\in\mathcal{P}_K}\mathbb{E}\left[\mathbf{1}_{s\leq\tau_o(x,p)}\big(1+X_s^{x,p}\big)\right]ds\\&\leq M\int_0^\infty e^{-\rho s}\sup_{p\in\mathcal{P}_K}\mathbb{E}\left[\mathbf{1}_{s\leq\tilde{\tau}_o(x,p)}\big(1+Y_s^{x,p}\big)\right]ds\\&\leq M\int_0^\infty e^{-\rho s}\sup_{p\in\mathcal{P}_K}\sqrt{\mathbb{P}\big(\tilde{\tau}_o(x,p)\geq s\big)\mathbb{E}\big[\big(1+Y_s^{x,p}\big)^2\big]}ds.\end{aligned}\tag{A.19}$$

Using the BDG inequality with Assumption 4.1, it is not difficult to show that

$$\mathbb{E}\left[\left(1+Y_t^{x,p}\right)^2\right]\le 2\left\{\mathbb{E}\left[\left|Y_t^{x,p}\right|^2\right]+1\right\}\le 4(x^2+M^2t+1). \tag{A.20}$$

On the other hand, in light of the definition (A.18) of $\tilde{\tau}_o(x,p)$, this results in that

$$\tilde{\tau}_o(x,p)=\inf\left\{t\ge 0;\ \int_0^t \sigma\big(\lambda(p_s)\big)dW_s\ge x\right\}.$$

This yields that for all $(x,p)\in\mathbb{R}_+\times\mathcal{P}_K$,

$$\mathbb{P}\big(\tilde{\tau}_o(x,p)\ge s\big)\le \mathbb{P}\left(\sup_{t\in[0,s]}\int_0^t \sigma\big(\lambda(p_v)\big)dW_v\le x\right).$$

Note that there exists a standard Brownian motion $B=(B_t)_{t\ge 0}$ such that

$$\int_0^t \sigma\big(\lambda(p_v)\big)dW_v\overset{d}{=}B\left(\int_0^t \sigma^2\big(\lambda(p_v)\big)dv\right),\quad t\ge 0,$$

where "$\overset{d}{=}$" denotes the equality in law. Then, using Assumption 4.1, it follows that, $\mathbb{P}$-a.s.

$$\sup_{t\in[0,s]} B\left(\int_0^t \sigma^2\big(\lambda(p_v)\big)dv\right)\ge \sup_{t\in[0,s]} B_{\sigma_0^2 t}=\sigma_0 \sup_{t\in[0,\sigma_0^2 s]} B_t^{\sigma_0}, \tag{A.21}$$

where $B_t^{\sigma_0}:=B_{\sigma_0^2 t}/\sigma_0$ for $t\ge 0$ is also a standard Brownian motion. For any $a>0$, let us define the stopping time given by

$$\tau_a^{\sigma_0}:=\inf\big\{t\ge 0;\ B_t^{\sigma_0}=a\big\}.$$

Then, from [23, Remark 8.3, Chapter 2], it follows that, $\tau_a^{\sigma_0}$ admits the following closed-form probability density given by

$$\mathbb{P}\big(\tau_a^{\sigma_0}\in dt\big)=\frac{a}{\sqrt{2\pi t^3}}\exp\left(-\frac{a^2}{2t}\right)\mathbf{1}_{t>0}dt.$$

Consequently, we obtain from (A.21) that

$$\begin{aligned}\mathbb{P}\big(\tilde{\tau}_o(x,p)\ge s\big)&\le \mathbb{P}\left(\sup_{t\in[0,s]} B_{\int_0^t \sigma^2(\lambda(p(s)))ds}\le x\right)\le \mathbb{P}\left(\sup_{t\in[0,s]} B_{\sigma_0^2 t}\le x\right)\\&=\mathbb{P}\left(\sigma_0\sup_{t\in[0,\sigma_0^2 s]} B_t^{\sigma_0}\le x\right)=\mathbb{P}\left(\tau_{\frac{x}{\sigma_0}}^{\sigma_0}\ge \sigma_0^2 s\right)\\&=\int_{\sigma_0^2 s}^{+\infty}\frac{x}{\sigma_0\sqrt{2\pi t^3}}\exp\left(-\frac{x^2}{2\sigma_0^2 t}\right)dt.\end{aligned} \tag{A.22}$$

It follows from (A.19), (A.20) and (A.22) that, for all $n\geq 1$,

$$0\leq V(x_n)\leq M\int_0^\infty e^{-\gamma s}\left\{4\left(x_n^2+M^2s+1\right)\int_{\sigma_0^2 s}^{+\infty}\frac{x_n}{\sigma_0\sqrt{2\pi t^3}}\exp\left(-\frac{x_n^2}{2\sigma_0^2 t}\right)dt\right\}^{\frac{1}{2}}ds. \tag{A.23}$$

By noting that $x_n\to 0$ with $x_n\geq 0$ as $n\to\infty$, there exists $N\geq 1$ such that $|x_n|\leq 1$ for all $n\geq N$. Therefore, for all $t>0$,

$$\frac{x_n}{\sigma_0\sqrt{2\pi t^3}}\exp\left(-\frac{x_n^2}{2\sigma_0^2 t}\right)\leq\frac{1}{\sigma_0\sqrt{2\pi t^3}},\quad \forall n\geq N.$$

It follows from the dominated convergence theorem (DCT) that, for all $s>0$,

$$\lim_{n\to\infty}\int_{\sigma_0^2 s}^{\infty}\frac{x_n}{\sigma_0\sqrt{2\pi t^3}}\exp\left(-\frac{x_n^2}{2\sigma_0^2 t}\right)dt=\int_{\sigma_0^2 s}^{\infty}\lim_{n\to\infty}\frac{x_n}{\sigma_0\sqrt{2\pi t^3}}\exp\left(-\frac{x_n^2}{2\sigma_0^2 t}\right)dt=0.$$

For all $s\geq 0$, it holds that

$$\int_{\sigma_0^2 s}^{\infty}\frac{x_n}{\sigma_0\sqrt{2\pi t^3}}\exp\left(-\frac{x_n^2}{2\sigma_0^2 t}\right)dt\leq\int_0^{\infty}\frac{x_n}{\sigma_0\sqrt{2\pi t^3}}\exp\left(-\frac{x_n^2}{2\sigma_0^2 t}\right)dt=1.$$

As a consequence, for all $n\geq N$,

$$e^{-\gamma s}\left\{4\left(x_n^2+M^2s+1\right)\int_{\sigma_0^2 s}^{\infty}\frac{x_n}{\sigma_0\sqrt{2\pi t^3}}\exp\left(-\frac{x_n^2}{2\sigma_0^2 t}\right)\right\}^{\frac{1}{2}}\leq 2e^{-\gamma s}(2+M^2s)^{\frac{1}{2}}.$$

Using DCT again, it follows that

$$\lim_{n\to\infty}\int_0^\infty e^{-\rho s}\left\{4\left(x_n^2+M^2s+1\right)\int_{\sigma_0^2 s}^{\infty}\frac{x_n}{\sigma_0\sqrt{2\pi t^3}}\exp\left(-\frac{x_n^2}{2\sigma_0^2 t}\right)dt\right\}^{\frac{1}{2}}ds=0.$$

Then, the desired result follows from (A.23). □

Proof of Lemma 4.1. Using Assumptions 2.1, 3.1 and 4.1, and the continuity of $x\to V(x)$, it follows that, for any $x>0$, the function $\mathcal{H}_3(x;\lambda)$ is continuous in $\lambda\in I_K$. Note that the set I_K is a compact set. Then, by [19, Proposition D.5], there exists a measurable $\lambda^*=\lambda^*(x)\in I_K$ such that $\mathcal{H}_3(x;\lambda^*)=\sup_{\lambda\in I_K}\mathcal{H}_3(x;\lambda)$. Thus, we complete the proof of the lemma. □

Proof of Proposition 4.2. We apply [31, Proposition 1] to prove the desired result. To do it, it is enough to verify that the function H^V defined by (4.9) satisfies the following conditions:

I. For any compact subset $\mathcal{X}$ of $\overline{\mathbb{R}}_+$, there exist constants M_1,M_2 such that, for all $(x,m,q),(x,\tilde{m},\tilde{q})\in\mathcal{X}\times\mathbb{R}^2$,

- (a) $|H(x,m,q)|\leq M_1\{1+|m|+|q|\}$,
- (b) $|H(x,m,q)-H(x,\tilde{m},\tilde{q})|\leq M_2\{|m-\tilde{m}|+|q-\tilde{q}|\}$,
- (c) H is continuous in $x\in\mathcal{X}$ for each (m,q).

II. For all $(x,q)\in\overline{\mathbb{R}}_+\times\mathbb{R}, m\to H(x,m,q)$ is non-increasing.

III. For any $\overline{K}>0$, there exist constants $K_1,K_2>\overline{K}$ such that, for all $x\in\overline{\mathbb{R}}_+$ and $\varepsilon\in\{-1,1\}$,

$$H(x,K_1+K_2x,\varepsilon K_2)<0, \quad H(x,-K_1-K_2x,\varepsilon K_2)>0.$$

We first verify that H^V satisfies the condition I. First of all, we have from Assumptions 2.1 and 4.1 that

$$\frac{2\lambda}{\sigma^2(\lambda)}\leq\frac{2\lambda(0)}{\sigma_0^2}, \quad \frac{2\rho}{\sigma^2(\lambda)}\leq\frac{2\rho}{\sigma_0^2}, \quad \forall\lambda\in I_K. \tag{A.24}$$

Moreover, it follows from the linear growth of $x\to V(x)$ that there exists a constant $C>0$ such that $V(x)\leq C(1+x)$ for all $x\geq 0$. Then, by the similar argument in the proof of Proposition 3.1, using Assumptions 2.1, 3.1 and 4.1, there exists a constant $\tilde{M}=\tilde{M}(\mathcal{X})>0$ such that $|d^V(x,\lambda)|\leq\tilde{M}$ for all $(x,\lambda)\in\mathcal{X}\times I_K$. As a consequence

$$\sup_{(x,\lambda)\in\mathcal{X}\times I_K}\left|\frac{d^V(x,\lambda)}{\sigma^2(\lambda)}\right|\leq\frac{\tilde{M}}{\sigma_0}. \tag{A.25}$$

Then, it follows from (4.9), (A.24) and (A.25) that, for all $(m,q)\in\mathbb{R}^2$,

$$|H(x,m,q)|\leq M_1\{1+|m|+|q|\}$$

with

$$M_1:=\max\left\{\frac{2\tilde{M}}{\sigma_0},\frac{2\lambda(0)}{\sigma_0^2},\frac{2\rho}{\sigma_0^2}\right\}.$$

This yields I(a). Moreover, by (4.9) and Assumption 4.1, it holds that, for all (x,m,q), $(x,\tilde{m},\tilde{q})\in\mathcal{X}\times\mathbb{R}^2$,

$$\begin{aligned}&|H^V(x,m,q)-H^V(x,\tilde{m},\tilde{q})|\\ \leq&\sup_{\lambda\in I}\left\{\frac{2\lambda}{\sigma^2(\lambda)}|m-\tilde{m}|+\frac{2\rho}{\sigma^2(\lambda)}|q-\tilde{q}|\right\}\\ \leq& M_2\{|m-\tilde{m}|+|q-\tilde{q}|\}\end{aligned}$$

with $M_2=\max\{2\lambda(0)/\sigma_0^2,2\rho/\sigma_0^2\}$. This shows property I(b). Analogously to the proof of Proposition 3.1, for any $(m,q)\in\mathbb{R}^2$, by Assumptions 2.1, 3.1, and 4.1, the function

$$h(m,q;x,\lambda):=\frac{2}{\sigma^2(\lambda)}\{-\rho m-\lambda q+d^V(x,\lambda)\}$$

is jointly continuous in $(x,\lambda)\in\mathcal{X}\times I_K$. Then, from Berge's maximum theorem (see, e.g. [31, Proposition 8]), it follows that $H^V(x,m,q)=\sup_{\lambda\in I_K}h(m,q;x,\lambda)$ is continuous in $x\in\mathcal{X}$ for each $(m,q)\in\mathbb{R}^2$. This yields I(c).

In view of the expression (4.9) of H^V, it is not difficult to see that $m\to H(x,m,q)$ is non-increasing since $\gamma>0$. Hence, the condition II is satisfied by H^V. Finally, by Assumptions 2.1, 3.1 and 4.1, there exists a constant $\tilde{C}>0$ such that $d^V(x,\lambda)\le\tilde{C}(1+x)$ for all $(x,\lambda)\in\overline{\mathbb{R}}_+\times I_K$. For any $\overline{K}>0$, let us define that

$$K_2=\max\left\{\frac{\tilde{C}}{\rho},\overline{K}\right\}+1,\quad K_1=\max\left\{\frac{\lambda(0)}{\rho}K_2+\frac{\tilde{C}}{\rho},\overline{K}\right\}+1. \tag{A.26}$$

Then, we arrive from Assumptions 4.1 and 3.1 at

$$\begin{aligned}
&H^V(x,K_1+K_2x,-K_2)\\
\le&\sup_{\lambda\in I_K}\frac{2}{\sigma^2(\lambda)}\left\{-\rho(K_1+K_2x)+\lambda K_2+\tilde{C}(1+x)\right\}\\
\le&\sup_{\lambda\in I_K}\frac{2}{\sigma^2(\lambda)}\left\{-\rho\left(\frac{\lambda}{\rho}K_2+\frac{\tilde{C}}{\rho}+1+\left(\frac{\tilde{C}}{\rho}+1\right)x\right)+\lambda K_2+\tilde{C}(1+x)\right\}\\
\le&-\inf_{\lambda\in I_K}\frac{2\rho}{\sigma^2(\lambda)}<0,\quad\forall x\ge 0.
\end{aligned}$$

In a similar fashion, we also have that, for all $x\ge 0$,

$$H^V(x,-K_1-K_2x,-K_2)\ge\sup_{\lambda\in I_K}\frac{2}{\sigma^2(\lambda)}\left\{\rho(K_1+K_2x)+\lambda K_2-\tilde{C}(1+x)\right\}\ge\sup_{\lambda\in I_K}\frac{2\rho}{\sigma^2(\lambda)}>0.$$

Consequently, the condition III is satisfied by H^V. Thus, the desired result follows from [31, Proposition 1]. Then, the proof of the proposition is complete. □

Proof of Lemma 4.4. The proof is similar to the proof of Lemma 2.2. □

Proof of Proposition 4.3. For $x\ge M/\mu$, it follows from $M<\mu\Lambda$ that

$$\frac{M-\mu\alpha(x)-mV'(x)}{2}\le 0.$$

Then, the optimal price $p^*(x)=(M-\epsilon)/m$ for all $x\ge M/\mu$, and hence it is sufficient to prove $\mathcal{O}^c\neq\emptyset$. We next prove it by contradiction. Suppose $\mathcal{O}^c=\emptyset$, i.e. $\mathcal{O}=(0,\infty)$. The value function $V(x)$ satisfies the following equation:

$$\begin{aligned}
&\frac{1}{2}\sigma^2V''(x)-\epsilon V'(x)+\frac{(M-\epsilon)\epsilon}{m}-\frac{\epsilon\mu\alpha(x)}{m}\\
&\quad+\frac{\mu M\alpha(x)}{m}+\mu V[x-\alpha(x)]-(\rho+\mu)V(x)=0
\end{aligned} \tag{A.27}$$

with boundary condition $V(0)=0$. For $x\le\Lambda$, the surging demand function $\alpha(x)=x$, then Eq. (A.27) reduces to

$$\frac{1}{2}\sigma^2V''(x)-\epsilon V'(x)-(\rho+\mu)V(x)+\frac{\mu(M-\epsilon)}{m}x+\frac{(M-\epsilon)\epsilon}{m}=0,$$

whose solution is given by

$$V(x)=\frac{\mu(M-\epsilon)}{m(\mu+\rho)}x+\frac{\epsilon(M-\epsilon)\rho}{m(\mu+\rho)^2}+C_1\exp(Ax)+C_2\exp(Bx). \tag{A.28}$$

Here C_1,C_2 are constants to be determined, while the constants A,B are defined by

$$A:=\frac{\epsilon+\sqrt{\epsilon^2+2(\mu+\rho)\sigma^2}}{\sigma^2},\quad B:=\frac{\epsilon-\sqrt{\epsilon^2+2(\mu+\rho)\sigma^2}}{\sigma^2}. \tag{A.29}$$

The boundary condition $V(0)=0$ gives

$$\frac{\epsilon(M-\epsilon)\rho}{m(\mu+\rho)^2}+C_1+C_2=0.$$

As $D\to\infty, C_1\to 0$ and $C_2\to(\epsilon(M-\epsilon)\rho)/(m(\mu+\rho)^2)$. It follows from (A.28) that

$$V'(x)=\frac{\mu(M-\epsilon)}{m(\mu+\rho)}+AC_1\exp(Ax)+BC_2\exp(Bx).$$

Then, we have that

$$\begin{aligned}
&\frac{M-\mu\alpha(x)-mV'(x)}{2}-\epsilon\\
=&\frac{M-\mu\alpha(x)}{2}-\frac{\mu(M-\epsilon)}{2(\mu+\rho)}+\frac{1}{2}AC_1\exp(Ax)+\frac{1}{2}BC_2\exp(Bx)-\epsilon\\
\to&\ \frac{M}{2}-\frac{\mu(M-\epsilon)}{2(\mu+\rho)}+\frac{B\epsilon(M-\epsilon)\rho}{2m(\mu+\rho)^2}-\epsilon \quad \text{as} \quad D\to\infty, \quad x\to 0,\\
=&\frac{M}{2}-\frac{\mu(M-\epsilon)}{2(\mu+\rho)}+\frac{1}{\epsilon+\sqrt{\epsilon^2+2(\mu+\rho)\sigma^2}}\frac{\epsilon(M-\epsilon)\rho}{m(\mu+\rho)}-\epsilon\\
\to&\ \frac{M}{2}-\frac{\mu M}{2(\mu+\rho)}=\frac{\rho M}{2(\mu+\rho)}>0 \quad \text{as} \quad \epsilon\to 0.
\end{aligned}$$

This means that, for a sufficiently large surging demand size $\Lambda>0$ and a sufficiently small floor of the normal demand rate $\epsilon>0$, there exist a sufficiently small $x^*>0$ such that

$$\frac{M-\mu\alpha(x^*)-mV'(x^*)}{2}>\epsilon.$$

This results in a contradiction with $\mathcal{O}=(0,\infty)$. □

Appendix B. Proof of Lemmas 4.2 and 4.3

Proof of Lemma 4.2. To get the existence of viscosity solution, it is enough to prove that the value function V defined by (3.2) is a viscosity solution of the abstract equation (4.5). This result follows from a standard argument, hence we omit the proof here. □

Proof of Lemma 4.3. In view of Proposition 3.2, Lemma 4.2, and [29, Proposition 5.4], we can see that the value function V defined by (3.2) is a viscosity solution of the abstract equation (4.8). To get uniqueness, it is sufficient to prove the following comparison principle: Let $u\in\mathsf{USC}(\overline{\mathbb{R}}_+)$ (respectively $v\in\mathsf{LSC}(\overline{\mathbb{R}}_+)$) be a viscosity subsolution (respectively supersolution) satisfying the linear growth of Eq. (4.8). Then, $u\le v$ on $\overline{\mathbb{R}}_+$. We show it by contradiction, and assume that there exists $\overline{x}\in\overline{\mathbb{R}}_+, u(\overline{x})-v(\overline{x})\ge 2\delta>0$ with $\delta>0$. It follows from $u(0)\le 0$ and $v(0)\ge 0$ that $\overline{x}>0$. Define

$$\Psi(x,y):=u(x)-v(y)-\psi(x,y), \quad (x,y)\in\overline{\mathbb{R}}_+^2,$$

where

$$\psi(x,y)=k|x-y|^2+\varepsilon(x^2+y^2)$$

with $(k,\varepsilon)\in\mathbb{R}_+\times(0,1]$. Moreover, define $M_k:=\sup_{x,y\ge 0}\Psi(x,y)$. Since u,v satisfy the linear growth condition, by the upper semi-continuity of Ψ, we have that $M_k<\infty$, and there exists $(x_k,y_k)\in\overline{\mathbb{R}}_+^2$ such that $M_k=\Psi(x_k,y_k)$. As a consequence

$$M_k\ge u(\overline{x})-v(\overline{x})-\psi(\overline{x},\overline{x})\ge 2\delta-2\varepsilon\overline{x}^2.$$

This yields that there exists $\varepsilon_0\in(0,\delta/(2\overline{x}^2))$ such that $M_k>\delta$ for all $\varepsilon\in(0,\varepsilon_0]$. Using $\Psi(0,0)\le\Psi(x_k,y_k)$ and the linear growth of u,v, it follows that

$$\begin{aligned}
&k|x_k-y_k|^2+\varepsilon\left(x_k^2+y_k^2\right)\\
&\le u(0)-v(0)+u(x_k)-y(y_k)\\
&\le u(0)-v(0)+2C\{1+|x_k|+|y_k|\}
\end{aligned} \tag{B.1}$$

for some $C>0$. Therefore, we can find a positive constant $C_\varepsilon>0$ such that $|x_k|,|y_k|\le C_\varepsilon$ for all $k\ge 1$. By this, there exists a subsequence, still denoted by (x_k,y_k), which converges to some $(x_\varepsilon,y_\varepsilon)\in\overline{\mathbb{R}}_+^2$. Hence, the estimate (B.1) also yields the following estimate given by

$$k|x_k-y_k|^2\le 2C(1+2C_\varepsilon)+u(0)-v(0),$$

from which we can conclude that $x_k-y_k\to 0$ as $k\to\infty$, and hence $x_\varepsilon=y_\varepsilon$. On the other hand, it follows from $\Psi(x_\varepsilon,y_\varepsilon)\le\Psi(x_k,y_k)$ for all $k\ge 1$, we obtain that

$$k|x_k-y_k|^2\le u(x_k)-u(x_\varepsilon)+v(y_k)-v(y_\varepsilon)+\varepsilon(x_\varepsilon^2+y_\varepsilon^2)-\varepsilon(x_k^2+y_k^2).$$

By the semi-continuity of u,v, we can get

$$k|x_k-y_k|^2 \ \to\ 0 \quad \text{as} \quad k\ \to\ \infty. \tag{B.2}$$

We next prove that, for all $\varepsilon_1>0$, there exists $\varepsilon\in(0,\varepsilon_1]$ such that $x_\varepsilon=y_\varepsilon>0$. We also show it by contradiction, and assume that there exists $\varepsilon_1>0$ such that $x_\varepsilon=y_\varepsilon=0$ for any $\varepsilon\in(0,\varepsilon_1]$. Since $M_k\ge\delta$, it holds that

$$u(x_k)-v(x_k)-k|x_k-y_k|^2-\varepsilon(x_k^2+y_k^2)\ge\delta>0.$$

Letting $k\to\infty$ and by (B.2), it results in $\liminf_{k\to\infty}(u(x_k)-v(x_k))\geq\delta$. Moreover, by the semi-continuity of u,v, it follows that

$$\begin{aligned}&\liminf_{k\to\infty}\big(u(x_k)-v(x_k)\big)\leq\limsup_{k\to\infty}\big(u(x_k)-v(x_k)\big)\\&\leq\limsup_{x\to 0}\big(u(x)-v(x)\big)\leq u(0)-v(0)\leq 0.\end{aligned}$$

Thus, we get a contradiction, which means that $x_\varepsilon=y_\varepsilon>0$ for some $\varepsilon\in(0,\varepsilon_1]$. Consequently, we may choose a sequence $(\varepsilon_n)_{n=1}^\infty\subset(0,\varepsilon_0]$ such that $\varepsilon_n\to 0$ as $n\to\infty$ and $x_{\varepsilon_n}=y_{\varepsilon_n}>0$. Then, for any ε_n, there exists a $N_n\geq 1$ such that $x_k>0,y_k>0$ for all $k\geq N_n$. By [14, Theorem 1] and [26, Lemma 4.4.5], there exist q_1^k,q_2^k such that

$$\begin{bmatrix}q_1^k & 0\\ 0 & -q_2^k\end{bmatrix}\leq 2k\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}+\left\{\frac{4\big((k+\varepsilon_n)^2+k^2\big)}{k^3}+2\varepsilon_n\right\}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \tag{B.3}$$

$$\sup_{\lambda\in I_K}\left\{\frac{1}{2}\sigma^2(\lambda)q_1^k-\lambda[2k(x_k-y_k)+2\varepsilon_n x_k]+d^u(x_k,\lambda)\right\}\geq\gamma u(x_k), \tag{B.4}$$

$$\sup_{\lambda\in I_K}\left\{\frac{1}{2}\sigma^2(\lambda)q_2^k-\lambda[2k(x_k-y_k)-2\varepsilon_n y_k]+d^v(y_k,\lambda)\right\}\leq\gamma v(y_k). \tag{B.5}$$

Subtract (B.5) from (B.4), we have that

$$\begin{aligned}\gamma\big(u(x_k)-v(y_k)\big)\leq&\sup_{\lambda\in I_K}\left\{\frac{1}{2}\sigma^2(\lambda)q_1^k-\lambda[2k(x_k-y_k)+2\varepsilon_n x_k]+d^u(x_k,\lambda)\right\}\\&-\sup_{\lambda\in I_K}\left\{\frac{1}{2}\sigma^2(\lambda)q_2^k-\lambda[2k(x_k-y_k)-2\varepsilon_n y_k]+d^v(y_k,\lambda)\right\}.\end{aligned}$$

Note that $\gamma(u(x_k)-v(y_k))\geq\gamma M_k\geq\gamma\delta>0$. Therefore,

$$\gamma\delta\leq\sup_{\lambda\in I_K}\left\{\frac{1}{2}\sigma^2(\lambda)\big(q_1^k-q_2^k\big)+2\varepsilon_n\lambda|x_k-y_k|+\big(d^u(x_k,\lambda)-d^v(y_k,\lambda)\big)\right\}. \tag{B.6}$$

Using (B.3), it yields that

$$q_1^k-q_2^k\leq\frac{4\big((k+\varepsilon_n)^2+k^2\big)}{k^3}+2\varepsilon_n.$$

By the definition of M_k and Assumption 3.1, it holds that

$$\begin{aligned}&d^u(x_k,\lambda)-d^v(y_k,\lambda)\\=&\int_\Gamma\big(\phi(\lambda,\gamma,x_k)-\phi(\lambda,\gamma,y_k)\big)\mu(d\gamma)-\mu(\Gamma)\big(u(x_k)-v(y_k)\big)\\&+\int_\Gamma\Big(u\big(x_k-\alpha\big(p(\lambda),\gamma,x_k\big)\big)-v\big(y_k-\alpha\big(p(\lambda),\gamma,y_k\big)\big)\Big)\mu(d\gamma)\end{aligned}$$

$$
\begin{aligned}
&\le C|x_k-y_k|+\int_\Gamma\Big(M_k+\psi\big(x_k-\alpha\big(p(\lambda),\gamma,x_k\big),y_k-\alpha\big(p(\lambda),\gamma,y_k\big)\big)\Big)\mu(d\gamma)\\
&\qquad -\mu(\Gamma)(M_k+\psi(x_k,y_k))\\
&=C|x_k-y_k|+\varepsilon_n\int_\Gamma\Big[\big(x_k-\alpha\big(p(\lambda),\gamma,x_k\big)\big)^2-x_k^2+\big(y_k-\alpha\big(p(\lambda),\gamma,y_k\big)\big)^2-y_k^2\Big]\mu(d\gamma)\\
&\qquad +k\int_\Gamma\Big[\big(x_k-\alpha\big(p(\lambda),\gamma,x_k\big)\big)^2-x_k^2+\big(y_k-\alpha\big(p(\lambda),\gamma,y_k\big)\big)^2-y_k^2\Big]\\
&\qquad -2\big(x_k-\alpha\big(p(\lambda),\gamma,x_k\big)\big)\big(y_k-\alpha\big(p(\lambda),\gamma,y_k\big)\big)\mu(d\gamma)\\
&\le C|x_k-y_k|
\end{aligned}
$$

for some constant $C>0$ which is independent of (x_k,y_k). It follows from Assumption 4.1 that, there exists a positive constant $M>0$, such that

$$
\limsup_{k\to\infty}\sup_{\lambda\in I_K}\left\{\frac{1}{2}\sigma^2(\lambda)\big(q_1^k-q_2^k\big)+2\varepsilon_n\lambda|x_k-y_k|+\big(d^u(x_k,\lambda)-d^v(y_k,\lambda)\big)\right\}\le M^2\varepsilon_n. \tag{B.7}
$$

Using (B.6) and (B.7), it results in $0<\gamma\delta\le M^2\varepsilon_n$. Letting $n\to\infty$, we get the desired contradiction, and hence the comparison principle holds. Let u,v be viscosity solutions of Eq. (4.8), which satisfy the linear growth condition. Then, we have $u^*\le v^*$ and $v^*\le u_*$ on $\overline{\mathbb{R}}_+$. Note that $v_*\le v\le v^*$ and $u_*\le u\le u^*$ on $\overline{\mathbb{R}}_+$. Therefore, $u=v=v^*=v_*=u^*=u_*$. This shows that the viscosity solution is unique and continuous on $\overline{\mathbb{R}}_+$. □

Acknowledgements

We would like to thank the referee's helpful comments and suggestions to improve the presentation of the paper. We also thank Meng Li for his insightful discussions on the dynamic pricing problems.

This work is partially supported by the Natural Science Basic Research Program of Shaanxi (Grant No. 2023-JC-JQ-05), by the National Natural Science Foundation of China (Grant No. 11971368) and by the Fundamental Research Funds for the Central Universities (Grant No. 20199235177).

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