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Stability Analysis of an SIS Epidemic Model in Heterogeneous Environment

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Abstract. This paper studies an susceptible-infected-susceptible reaction-diffusion model in spatially heterogeneous environment proposed in [Allen *et al.*, Discrete Contin. Dyn. Syst., 21, 2008], where the existence and uniqueness of the endemic equilibrium are established and its stability is proposed as an open problem. However, till now, there is no progress in the stability analysis except for special cases with either equal diffusion coefficients or constant endemic equilibrium. In this paper, we demonstrate the first criterion in determining the stability of the non-constant endemic equilibrium with different diffusion coefficients. Thanks to this criterion, when one of the diffusion rates is small or large, the impact of spatial heterogeneity on the stability can be characterized based on the asymptotic behavior of the endemic equilibrium.

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1 Introduction

Partial differential equations are widely used in the modeling and analysis of the spread of infectious diseases. The impact of spatially heterogeneous environment and individual movement on the persistence or extinction of a disease has attracted a lot of studies in the

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literature. We may refer to, e.g. [1–6, 10–12, 16] and the references therein. In particular, the susceptible-infected-susceptible (SIS) model is one of the most basic mathematical models for infectious disease dynamics.

Allen *et al.* [1] proposed the following frequency dependent SIS reaction-diffusion model in spatially heterogeneous environment:

$$\begin{cases} \dfrac{dS}{dt}=d_S\Delta S-\beta(x)\dfrac{SI}{S+I}+\gamma(x)I, & x\in\Omega, \quad t>0, \\ \dfrac{dI}{dt}=d_I\Delta I+\beta(x)\dfrac{SI}{S+I}-\gamma(x)I, & x\in\Omega, \quad t>0, \\ \dfrac{\partial S}{\partial \nu}=\dfrac{\partial I}{\partial \nu}=0, & x\in\partial\Omega, \quad t>0, \\ S(x,0)=S_0(x), \quad I(x,0)=I_0(x), & x\in\partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset R^n, n\geq 1$, is a bounded domain with smooth boundary $\partial\Omega$, ν represents the unit outer normal vector on $\partial\Omega$ and $I_0(x), S_0(x)\in C(\bar{\Omega})$ are nonnegative functions satisfying $\int_\Omega I_0 dx>0$. Here $S(x,t)$ and $I(x,t)$ denote the densities of susceptible and infected individuals at location x and time t respectively, d_S and d_I are the corresponding diffusion coefficients for the susceptible and infected populations, $\beta(x)$ and $\gamma(x)$ are positive Hölder continuous on $\bar{\Omega}$ and represent the rates of disease transmission and recovery at x respectively.

Let $(\hat{S},\hat{I})$ denote, if exists, a nonnegative equilibrium solution of the problem (1.1), i.e. $(\hat{S},\hat{I})$ satisfies

$$\begin{cases} d_S\Delta S-\beta(x)\dfrac{SI}{S+I}+\gamma(x)I=0, & x\in\Omega, \\ d_I\Delta I+\beta(x)\dfrac{SI}{S+I}-\gamma(x)I=0, & x\in\Omega, \\ \dfrac{\partial S}{\partial \nu}=\dfrac{\partial I}{\partial \nu}=0, & x\in\partial\Omega. \end{cases} \tag{1.2}$$

Obviously, there are only two possibilities:

- $\hat{I}\equiv 0$ in Ω, then $(\hat{S},0)$ is called a disease-free equilibrium of the problem (1.1).
- $\hat{I}>0$ for some $x\in\Omega$, then $(\hat{S},\hat{I})$ is called an endemic equilibrium of (1.1).

The main purpose of this paper is to analyze the stability of the endemic equilibrium when it exists. The existence and uniqueness of the endemic equilibrium is thoroughly investigated in [1]. To be more specific, the basic reproduction number can be defined as follows:

$$R_0=\sup_{\varphi\in H^1(\Omega),\varphi\neq 0}\left\{\frac{\displaystyle\int_\Omega \beta\varphi^2 dx}{\displaystyle\int_\Omega d_I|\nabla\varphi|^2+\gamma\varphi^2 dx}\right\}.$$

It is shown that if $R_0<1$, then the disease-free equilibrium, which always exists, is globally asymptotically stable and there is no endemic equilibrium, while if $R_0>1$, then the disease-free equilibrium is unstable and there exists an unique endemic equilibrium, denoted by $(\hat{S},\hat{I})$. It is further proved that the endemic equilibrium exists if and only if $\{x\in\Omega,\ \beta(x)>\gamma(x)\}$ is nonempty and one of the following conditions is valid:

- $\int_\Omega\beta(x)dx\geq\int_\Omega\gamma(x)dx$ for all $d_I>0$,
- $\int_\Omega\beta(x)dx<\int_\Omega\gamma(x)dx$ with d_I smaller than some critical value.

The stability of the endemic equilibrium $(\hat{S},\hat{I})$ is proposed as an open problem in [1]. However, till now, this problem remains unknown except for the following two special cases:

- $d_S=d_I$, where the stationary problem (1.2) can be transformed into a single equation [14],
- $\beta=r\gamma$ with the constant $r>1$, where the equilibria are constant [15].

In this paper, we further the studies in the stability analysis of the endemic equilibrium. To be specific, we will focus on the local stability of the non-constant endemic equilibrium when the diffusion coefficients are allowed to be different.

Our first main result provides a sufficient condition to guarantee the stability of the endemic equilibrium $(\hat{S},\hat{I})$.

Theorem 1.1. *Assume that the endemic equilibrium $(\hat{S},\hat{I})$ exists, then it is locally stable when the following inequality holds:*

$$\gamma(x)>\beta(x)\frac{\hat{S}^2}{(\hat{S}+\hat{I})^2},\quad x\in\bar{\Omega}. \tag{1.3}$$

Notice that the criterion (1.3) involves the endemic equilibrium itself and this makes it difficult to determine whether this criterion is valid.

On the basis of Theorem 1.1, to obtain stability criteria directly depending on the diffusion rates and the spatially heterogenous environment, we analyze the asymptotic behavior of $(\hat{S},\hat{I})$ with either small or large diffusion rates and obtain some restrictions on $\beta(x),\gamma(x)$ such that the corresponding endemic equilibrium satisfies the criterion (1.3). The corresponding result is stated as follow.

Theorem 1.2. *Assume that the endemic equilibrium $(\hat{S},\hat{I})$ of the system* (1.1) *exists for all $d_S>0$, $d_I>0$, we have the following statements:*

(1) If

$$\sup_{x\in\Omega}\frac{\beta(x)}{\gamma(x)}<\left(\inf_{x\in\Omega}\frac{\beta(x)}{\gamma(x)}\right)^2, \tag{1.4}$$

then there exists a constant $K_1>0$ large enough such that for any $d_I>0$, $(\hat{S},\hat{I})$ is locally stable whenever $d_S\geq K_1$.

(2) *If* (1.4) *is valid, then there exists a constant* $K_2 > 0$ *large enough such that for any* $d_S > 0$, $(\hat{S},\hat{I})$ *is locally stable whenever* $d_I \geq K_2$.

(3) *If*

$$\beta(x) > \gamma(x), \quad x \in \bar{\Omega}, \tag{1.5}$$

then there exists a constant $\delta > 0$ *sufficiently small such that for any* $d_S > 0$, $(\hat{S},\hat{I})$ *is locally stable whenever* $d_I \leq \delta$.

Theorem 1.2 directly reflects the effect of spatial heterogeneity and movement of individuals on the stability of endemic equilibrium solutions of (1.1). To prove Theorem 1.2, it is crucial to analyze the asymptotic profiles of the endemic equilibrium and then we can determine the conditions on $\beta(x),\gamma(x)$ such that the criterion (1.3) is valid. It is worth pointing out that, among other things, the asymptotic profiles of the endemic equilibrium are analyzed in [13], when one diffusion rate is fixed and the other one goes to infinity. However, either the smallness or the largeness of one diffusion rate imposed in Theorem 1.2 is always independent of the other diffusion rate. In other words, to prove Theorem 1.2, we need derive some uniform estimates related to the asymptotic profiles and naturally more delicate analysis is required.

As introduced in [1], x is called a low-risk site if $\beta(x) < \gamma(x)$, and x is called a high-risk site if $\beta(x) > \gamma(x)$. Also, we say Ω is a low-risk domain if $\int_\Omega \beta(x)dx < \int_\Omega \gamma(x)dx$ and a high-risk domain if $\int_\Omega \beta(x)dx \geq \int_\Omega \gamma(x)dx$.

It is routine to check that the condition (1.4) guarantees the condition (1.5). This indicates that the stability criteria derived in Theorem 1.2 always require the high-risk site everywhere in Ω, and thus Ω is a high-risk domain. Moreover, as remarked earlier, it is proved in [1] that when

$$\int_\Omega \beta(x)dx > \int_\Omega \gamma(x)dx,$$

the system (1.1) admits a unique endemic equilibrium $(\hat{S},\hat{I})$ for all $d_S > 0, d_I > 0$. Simply speaking, the stability criteria (1.4) or (1.5) derived in Theorem 1.2 automatically guarantees the existence of the endemic equilibrium. More importantly, the conditions (1.4) and (1.5) reveal that the stability of the endemic equilibrium requires higher risk of being infected. Therefore, the stability criteria in Theorem 1.2 are relatively natural, although they might not be optimal.

Although Theorems 1.1 and 1.2 shed some light on the stability criteria of the non-constant endemic equilibrium with different diffusion coefficients, more natural and in-depth questions still remain open. For example, the stability criteria derived in Theorem 1.2 always require the high-risk site everywhere in Ω, while the existence and uniqueness result is proved in [1] when Ω is a high-risk domain. This indicates that the local stability in a high-risk domain with nonempty low-risk site is an interesting question. Furthermore, even when the conditions simultaneously guarantee the existence, uniqueness and local stability of the non-constant endemic equilibrium with different diffusion coefficients, the global stability is still completely unknown. This is closely related to the fundamental mathematical problem whether the system (1.1) allows the possibility of time-periodic solutions. We will return to it in a future paper.

The rest of this paper is organized as follows. Section 2 is devoted to the proof of Theorem 1.1. Theorem 1.2 is proved in Section 3.

2 Proof of Theorem 1.1

The linearized problem of the model (1.2) at the endemic equilibrium $(\hat{S},\hat{I})$ is as follows:

$$\begin{cases} d_S\Delta\phi-\beta(x)\dfrac{\hat{I}^2}{(\hat{S}+\hat{I})^2}\phi+\left(\gamma(x)-\beta(x)\dfrac{\hat{S}^2}{(\hat{S}+\hat{I})^2}\right)\psi=\lambda\phi, & x\in\Omega,\\ d_I\Delta\psi-\left(\gamma(x)-\beta(x)\dfrac{\hat{S}^2}{(\hat{S}+\hat{I})^2}\right)\psi+\beta(x)\dfrac{\hat{I}^2}{(\hat{S}+\hat{I})^2}\phi=\lambda\psi, & x\in\Omega,\\ \dfrac{\partial\phi}{\partial\nu}=\dfrac{\partial\psi}{\partial\nu}=0, & x\in\partial\Omega, \end{cases} \tag{2.1}$$

where ϕ and ψ satisfy

$$\int_\Omega \big(\phi(x)+\psi(x)\big)dx=0. \tag{2.2}$$

The extra condition (2.2) is due to the property that the total population in the problem (1.1) is preserved, i.e. if

$$\int_\Omega \big(S_0(x)+I_0(x)\big)dx=N,$$

then it is routine to check that

$$\int_\Omega \big(S(x,t)+I(x,t)\big)dx=N, \quad t>0.$$

To prove Theorem 1.1, it suffices to show that all the eigenvalues of the linearized problem (2.1)-(2.2) have negative real parts under the assumption (1.3). For this purpose, we consider a more general eigenvalue problem

$$\begin{cases} d_S\Delta\phi+a(x)\phi+p(x)\psi=\lambda\phi, & x\in\Omega,\\ d_I\Delta\psi+\alpha(x)\psi+q(x)\phi=\lambda\psi, & x\in\Omega,\\ \dfrac{\partial\phi}{\partial\nu}=\dfrac{\partial\psi}{\partial\nu}=0, & x\in\partial\Omega, \end{cases} \tag{2.3}$$

and establish the following result, which is crucial in the proof of Theorem 1.1.

Proposition 2.1. *Assume $a(x),\alpha(x),p(x),q(x)\in C(\bar{\Omega})$ and $p(x),q(x)>0$ in $\bar{\Omega}$. Then*

(i) The eigenvalue problem (2.3) *admits a simple eigenvalue, denoted by λ_p, with a corresponding eigenfunction (ϕ_p,ψ_p) satisfying $\phi_p,\psi_p\in C(\bar{\Omega})$ and $\phi_p,\psi_p>0$ on $\bar{\Omega}$*

(ii) If there exists an eigenpair $(\lambda,(\phi,\psi))$ with $\phi,\psi>0$ in $\bar{\Omega}$, then $\lambda=\lambda_p$ and (ϕ,ψ) is a constant multiple of (ϕ_p,ψ_p).

(iii) For any other eigenvalue λ of the problem (2.3)*, we have $\lambda_p>\mathrm{Re}\,\lambda$.*

It is known that Propositions 2.1(i) and 2.1(ii) follow directly from Krein-Rutman theorem [7], while extra effort is needed to handle Proposition 2.1(iii), see [8] for example, where the proof relies on analytic semigroup theory and the introduction of Poincaré map. In this paper, we provide an original and elementary method to demonstrate Proposition 2.1(iii). This method has its independent value, since it is constructive and only requires strong maximum principle.

Suppose that Proposition 2.1 is valid, we complete the proof of Theorem 1.1 first.

Proof of Theorem 1.1. Obviously, due to the condition (1.3), the eigenvalue problem (2.1) satisfies the assumptions in Proposition 2.1. Moreover, since the endemic equilibrium $(\hat{S},\hat{I})$ satisfies (1.2), it is routine to verify that for the eigenvalue problem (2.1), $\lambda_p=0$ and $(\hat{S},\hat{I})$ is the corresponding eigenfunction. Hence, it follows from Proposition 2.1 that except for $\lambda_p=0$, the real parts of all the eigenvalues to the eigenvalue problem (2.1) are negative. However, the restriction (2.2) indicates that $\lambda_p=0$ is not an eigenvalue of the problem (2.1)-(2.2). Therefore, Theorem 1.1 is proved. ☐

The rest of this section is devoted to the proof of Proposition 2.1.

Proof of Proposition 2.1. (i) and (ii) follows directly from Krein-Rutman theorem. The details are omitted since they are standard. We prove (iii) by an original elementary method.

For convenience, let $\lambda=\lambda_1+i\lambda_2$ denote an eigenvalue of the problem (2.3) and $(\phi,\psi)=(\phi_1+i\phi_2,\psi_1+i\psi_2)$ denote the corresponding eigenfunction, i.e.

$$\begin{cases}\Delta\phi_1+a\phi_1+p\psi_1=\lambda_1\phi_1-\lambda_2\phi_2, & x\in\Omega,\\ \Delta\phi_2+a\phi_2+p\psi_2=\lambda_1\phi_2+\lambda_2\phi_1, & x\in\Omega,\\ \Delta\psi_1+\alpha\psi_1+q\phi_1=\lambda_1\psi_1-\lambda_2\psi_2, & x\in\Omega,\\ \Delta\psi_2+\alpha\psi_2+q\phi_2=\lambda_1\psi_2+\lambda_2\psi_1, & x\in\Omega,\\ \dfrac{\partial\phi_1}{\partial\nu}=\dfrac{\partial\phi_2}{\partial\nu}=\dfrac{\partial\psi_1}{\partial\nu}=\dfrac{\partial\psi_2}{\partial\nu}=0, & x\in\partial\Omega.\end{cases} \tag{2.4}$$

Also for the convenience of readers, we recall that

$$\begin{cases}\Delta\phi_p+a\phi_p+p\psi_p=\lambda_p\phi_p, & x\in\Omega,\\ \Delta\psi_p+\alpha\psi_p+q\phi_p=\lambda_p\psi_p, & x\in\Omega,\\ \dfrac{\partial\phi_p}{\partial\nu}=\dfrac{\partial\psi_p}{\partial\nu}=0, & x\in\partial\Omega.\end{cases} \tag{2.5}$$

Without loss of generality, we assume that $|\phi|\le\phi_p, |\psi|\le\psi_p$ in $\bar{\Omega}$ and there exists $x_0\in\bar{\Omega}$ such that $|\phi(x_0)|=\phi_p(x_0)$. We may also assume that

$$\phi_p(x_0)=|\phi(x_0)|=\phi_1(x_0), \quad \phi_2(x_0)=0. \tag{2.6}$$

This can be achieved by replacing $\phi(x),\psi(x)$ with $\overline{\phi(x_0)}/|\phi(x_0)|\phi(x),\overline{\phi(x_0)}/|\phi(x_0)|\psi(x)$, respectively.

We claim that $\lambda_p\ge\mathrm{Re}\lambda=\lambda_1$ and

$$\psi_p(x_0)=|\psi(x_0)|=\psi_1(x_0), \quad \psi_2(x_0)=0. \tag{2.7}$$

Suppose that this is not true, i.e. either $\lambda_p<\lambda_1$ or (2.7) is invalid. Two cases will be discussed separately.

Case 1. Assume that $x_0\in\Omega$. It follows from (2.4) and (2.5) that

$$\begin{aligned}\Delta(\phi_p-\phi_1)&=-a(\phi_p-\phi_1)-p(\psi_p-\psi_1)+\lambda_p\phi_p-\lambda_1\phi_1+\lambda_2\phi_2\\&=-p(\psi_p-\psi_1)+(\lambda_1-a)(\phi_p-\phi_1)+(\lambda_p-\lambda_1)\phi_p+\lambda_2\phi_2.\end{aligned} \tag{2.8}$$

According to the choice of x_0, one sees $\Delta(\phi_p-\phi_1)(x_0)\ge 0$. However, if $\lambda_p<\lambda_1$ or (2.7) is invalid, then at $x=x_0$,

$$-p(\psi_p-\psi_1)+(\lambda_1-a)(\phi_p-\phi_1)+(\lambda_p-\lambda_1)\phi_p+\lambda_2\phi_2<0,$$

where the condition that $p>0$ in $\bar{\Omega}$ is required. This contradicts to (2.8).

Case 2. Assume ϕ_p touches $|\phi|$ only somewhere on $\partial\Omega$, i.e. $\phi_p>|\phi|$ in Ω and $x_0\in\partial\Omega$. Thanks to (2.6) and (2.8), we have $\Delta(\phi_p-\phi_1)<(\lambda_1-a)(\phi_p-\phi_1)$ in a small neighborhood of x_0, if either $\lambda_p<\lambda_1$ or (2.7) is invalid. A contradiction arises at $x=x_0$ because of the Hopf boundary lemma.

The claim is proved, i.e. $\lambda_p\ge\mathrm{Re}\lambda=\lambda_1$ and (2.6), (2.7) hold simultaneously.

Now we consider the case that $\lambda_p=\mathrm{Re}\lambda$. To prove (iii), it suffices to show that if $\lambda_p=\mathrm{Re}\lambda$, then $\lambda_p=\lambda$.

When $|\phi|\ne 0$, direct computation yields that

$$\begin{aligned}\Delta|\phi|&=\frac{\phi_1\Delta\phi_1+\phi_2\Delta\phi_2}{|\phi|}+\frac{|\phi_2\nabla\phi_1-\phi_1\nabla\phi_2|^2}{|\phi|^3}\\&=\frac{1}{|\phi|}[\phi_1(-a\phi_1-p\psi_1+\lambda_p\phi_1-\lambda_2\phi_2)+\phi_2(-a\phi_2-p\psi_2+\lambda_p\phi_2+\lambda_2\phi_1)]\\&\quad+\frac{|\phi_2\nabla\phi_1-\phi_1\nabla\phi_2|^2}{|\phi|^3}\\&=(-a+\lambda_p)|\phi|-p\frac{\phi_1\psi_1+\phi_2\psi_2}{|\phi|}+\frac{|\phi_2\nabla\phi_1-\phi_1\nabla\phi_2|^2}{|\phi|^3},\end{aligned}$$

and hence in view of Cauchy-Schwarz inequality,

$$\begin{aligned}&\Delta|\phi|+(a-\lambda_p)|\phi|+p|\psi|\\&=p\left[|\psi|-\frac{\phi_1\psi_1+\phi_2\psi_2}{|\phi|}\right]+\frac{|\phi_2\nabla\phi_1-\phi_1\nabla\phi_2|^2}{|\phi|^3}\geq 0.\end{aligned} \tag{2.9}$$

Similarly, if $|\psi|\neq 0$, we can derive

$$\begin{aligned}&\Delta|\psi|+(\alpha-\lambda_p)|\psi|+q|\phi|\\&=q\left[|\phi|-\frac{\phi_1\psi_1+\phi_2\psi_2}{|\psi|}\right]+\frac{|\psi_2\nabla\psi_1-\psi_1\nabla\psi_2|^2}{|\psi|^3}\geq 0.\end{aligned} \tag{2.10}$$

Define

$$\Omega_+=\{x\in\bar{\Omega}\,|\,|\phi|>0,|\psi|>0\}.$$

Ω_+ is not empty due to the existence of x_0. Let Ω_1 denote the connected component of Ω_+ containing x_0. It follows from (2.5)-(2.7), (2.9) and (2.10) that

$$\begin{cases}\Delta(\phi_p-|\phi|)+(a-\lambda_p)(\phi_p-|\phi|)+p(\psi_p-|\psi|)\leq 0, & x\in\Omega_1,\\ \Delta(\psi_p-|\psi|)+(\alpha-\lambda_p)(\psi_p-|\psi|)+q(\phi_p-|\phi|)\leq 0, & x\in\Omega_1,\\ |\phi|\leq\phi_p,|\psi|\leq\psi_p, & x\in\Omega_1.\end{cases} \tag{2.11}$$

Thanks to the choice of x_0,

$$|\phi|(x_0)-\phi_p(x_0)=|\psi|(x_0)-\psi_p(x_0)=0.$$

Suppose that $|\phi|\not\equiv\phi_p$ in Ω_1. The strong maximum principle guarantees that $|\phi|(x)-\phi_p(x)>0$ in the interior of Ω_1. This indicates that $x_0\in\partial\Omega_1$. No matter whether $x_0\in\Omega$ or $x_0\in\partial\Omega$, a contradiction can be derived based on the Hopf boundary lemma. Hence, $|\phi|\equiv\phi_p$ in $\bar{\Omega}_1$, and $|\psi|\equiv\psi_p$ in $\bar{\Omega}_1$ can be derived similarly. This implies that $|\phi|>0,|\psi|>0$ in $\bar{\Omega}_1$. According to the continuity of $\phi_p,\psi_p,|\phi|$ and $|\psi|$, it is routine to demonstrate $\Omega_1=\bar{\Omega}$. Therefore,

$$|\phi|=\phi_p,\quad |\psi|=\psi_p,\quad x\in\bar{\Omega}. \tag{2.12}$$

This conclusion, together with (2.9) and (2.10), implies that

$$\phi_1\psi_1+\phi_2\psi_2=|\phi||\psi|,\quad |\phi_2\nabla\phi_1-\phi_1\nabla\phi_2|=|\psi_2\nabla\psi_1-\psi_1\nabla\psi_2|=0,\quad x\in\bar{\Omega}. \tag{2.13}$$

Let Ω_ϕ denote a connected component of the set $\{x\in\bar{\Omega}\,|\,\phi_1>0\}$ containing x_0, where we recall that in (2.6)

$$|\phi(x_0)|=\phi_1(x_0)=\phi_p(x_0)>0,\quad \phi_2(x_0)=0.$$

According to (2.13), we see that

$$\nabla\left(\frac{\phi_2}{\phi_1}\right)=0,\quad x\in\Omega_\phi.$$

This implies that

$$\frac{\phi_2}{\phi_1}(x)=\frac{\phi_2}{\phi_1}(x_0)=0, \quad x\in\Omega_\phi,$$

and it follows from (2.12) that

$$\phi_2(x)=0, \quad \phi_1(x)=\phi_p(x), \quad x\in\Omega_\phi.$$

Due to the continuity of $\phi_1(x)$ and $\phi_p(x)$, it is standard to show that $\Omega_\phi=\bar{\Omega}$. Similarly, we obtain

$$\psi_2(x)=0, \quad \psi_1(x)=\psi_p(x), \quad x\in\bar{\Omega}.$$

Therefore, $\lambda_1=\lambda_p$ and $\lambda_2=0$, i.e. $\lambda_p=\lambda$. The proof is complete. □

3 The stability of the endemic equilibrium with large or small diffusion rates

This section is devoted to studying the local stability of the endemic equilibrium when one of the diffusion rates of the susceptible and infected individuals is either small or large. From Theorem 1.1, we know that the condition (1.3) as follows:

$$\gamma(x)>\beta(x)\frac{\hat{S}^2}{(\hat{S}+\hat{I})^2}, \quad x\in\bar{\Omega},$$

where $(\hat{S},\hat{I})$ denotes the unique endemic equilibrium of the problem (1.1) whenever it exists, is sufficient for the local stability of the endemic equilibrium. However, this condition depends on the solution itself. Therefore, in view of the continuous dependence on parameters, the key point is to verify whether the asymptotic profiles of the endemic equilibrium $(\hat{S},\hat{I})$ satisfy the condition (1.3).

Proof of Theorem 1.2. Case 1. Thanks to Theorem 1.1, we only need to show that there exists K_1 such that the condition (1.3) is valid for all $d_S>K_1$. Suppose that this is not true, i.e. there exist $d_S^{(k)}>0$ and $d_I^{(k)}>0$ with $\lim_{k\to\infty}d_S^{(k)}=\infty$ such that

$$\gamma(x)\le\beta(x)\frac{{S_k}^2}{(S_k+I_k)^2} \quad \text{somewhere in } \bar{\Omega}, \tag{3.1}$$

where (S_k,I_k) denotes the unique endemic equilibrium of the problem

$$\begin{cases} d_S^{(k)}\Delta S-\beta(x)\dfrac{SI}{S+I}+\gamma(x)I=0, & x\in\Omega, \\ d_I^{(k)}\Delta I+\beta(x)\dfrac{SI}{S+I}-\gamma(x)I=0, & x\in\Omega, \\ \dfrac{\partial S}{\partial\nu}=\dfrac{\partial I}{\partial\nu}=0, & x\in\partial\Omega, \\ \displaystyle\int_\Omega(I+S)dx=N. \end{cases}$$

Up to a subsequence, we may assume in addition that $d_I^{(k)} \to d \in [0,\infty]$. Next, we will obtain contradictions according to the different situations of d.

Case 1.1. $d \in (0,\infty)$. Direct computation shows that for large k we have

$$\frac{1}{d_I^{(k)}} \left\| \frac{\beta S_k}{S_k+I_k} - \gamma \right\|_{L^\infty(\Omega)} \le \frac{2}{d}(\|\beta\|_{L^\infty(\Omega)} + \|\gamma\|_{L^\infty(\Omega)}), \tag{3.2}$$

by the fact that $d_I^{(k)} > d/2$ for large k. By means of Harnack inequality (see, e.g. [9]), we can make sure that there exists a constant $C_1 > 0$ independent of k such that

$$\sup_\Omega I_k \le C_1 \inf_\Omega I_k. \tag{3.3}$$

This, together with $\int_\Omega I_k dx \le N$, indicates that there exists a positive constant C_2 independent of k such that

$$\|I_k\|_{L^\infty(\Omega)} \le C_2.$$

By standard elliptic regularity theory, we can further derive a positive constant C_3 independent of k such that

$$\|I_k\|_{W^{2,n+1}(\Omega)} \le C_3. \tag{3.4}$$

Based on the fact $d_S^{(k)} \to \infty$, we invoke (3.4) and thereby find a positive constant C_4 independent of k such that

$$\frac{1}{d_S^{(k)}} \left\| -\frac{\beta S_k I_k}{S_k+I_k} + \gamma I_k \right\|_{L^\infty(\Omega)} \le C_4.$$

Similarly, the fact $\int_\Omega S_k dx \le N$ and standard elliptic regularity theory imply that there exists $C_5 > 0$ independent of k such that

$$\|S_k\|_{W^{2,n+1}(\Omega)} \le C_5.$$

Therefore, we can find a subsequence of (S_k, I_k), still denoted by itself, and $(S^*, I^*) \in W^{2,n+1}(\Omega) \times W^{2,n+1}(\Omega)$ such that $(S_k, I_k) \to (S^*, I^*)$ weakly in $W^{2,n+1}(\Omega) \times W^{2,n+1}(\Omega)$, strongly in $C^1(\bar{\Omega}) \times C^1(\bar{\Omega})$ and $S^* \ge 0, I^* \ge 0$. The estimate (3.3) implies that $I^* > 0$ for all $x \in \bar{\Omega}$ unless $I^* \equiv 0$.

Next, we claim that $I^* > 0$ for all $x \in \bar{\Omega}$. Suppose that $I^* \equiv 0$, i.e. $I_k \to 0$ weakly in $W^{2,n+1}(\Omega)$ and strongly in $C^1(\bar{\Omega})$. It is routine to show that $S^* > 0$ in $\bar{\Omega}$ by strong maximal principle. Define

$$\bar{I}_k = \frac{I_k}{\|I_k\|_{L^\infty(\Omega)}},$$

then it satisfies

$$\begin{cases} d_I^{(k)} \Delta \bar{I}_k + \left(\dfrac{\beta S_k}{S_k+I_k} - \gamma \right) \bar{I}_k = 0, & x \in \Omega, \\ \dfrac{\partial \bar{I}_k}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{3.5}$$

Similar to the previous discussion, one sees that $\overline{I}_k \to \overline{I}$ weakly in $W^{2,n+1}(\Omega)$ and strongly in $C^1(\bar{\Omega})$, where $\overline{I}>0$ is the solution of

$$\begin{cases} d\Delta\overline{I}+(\beta-\gamma)\overline{I}=0, & x\in\Omega, \\ \dfrac{\partial\overline{I}}{\partial\nu}=0, & x\in\partial\Omega. \end{cases}$$

Therefore, we have

$$\int_\Omega(\beta-\gamma)dx=-d\int_\Omega\left|\frac{\nabla\overline{I}}{\overline{I}}\right|^2 dx\le 0,$$

which implies that

$$\inf_{x\in\Omega}\frac{\beta(x)}{\gamma(x)}\le 1,$$

and thus

$$\left(\inf_{x\in\Omega}\frac{\beta(x)}{\gamma(x)}\right)^2\le\sup_{x\in\Omega}\frac{\beta(x)}{\gamma(x)}.$$

This leads to a contradiction to the assumption (1.4). The claim is proved.

Now, due the above claim, it is routine to check that (S^*,I^*) is a positive solution of

$$\begin{cases} \Delta S^*=0, & x\in\Omega, \\ d\Delta I^*+\beta(x)\dfrac{S^*I^*}{S^*+I^*}-\gamma(x)I^*=0, & x\in\Omega, \\ \dfrac{\partial S^*}{\partial\nu}=\dfrac{\partial I^*}{\partial\nu}=0, & x\in\partial\Omega, \\ \displaystyle\int_\Omega(I^*+S^*)dx=N. \end{cases} \tag{3.6}$$

Obviously, S^* is a constant. Therefore, the problem (3.6) can be reduced to

$$\begin{cases} d\Delta I^*+\left(\beta(x)\dfrac{S^*}{S^*+I^*}-\gamma(x)\right)I^*=0, & x\in\Omega, \\ S^*=\dfrac{1}{|\Omega|}\left(N-\displaystyle\int_\Omega I^*dx\right), \\ \dfrac{\partial I^*}{\partial\nu}=0, & x\in\partial\Omega. \end{cases}$$

We consider the set of the minimum points of $I^*(x)$. If $I^*(x)>\min_{x\in\bar{\Omega}}I^*(x)$ for all $x\in\Omega$, then a contradiction arises by Hopf boundary lemma. Hence, there exists $x_1\in\Omega$ such that $I^*(x_1)=\min_{x\in\bar{\Omega}}I^*(x)$ and obviously

$$\left(\frac{\beta S^*}{S^*+I^*}-\gamma\right)(x_1)\le 0.$$

It follows that

$$I^*(x)\geq \min_{\bar{\Omega}} I^*(x)=I^*(x_1)\geq \left(\frac{\beta(x_1)}{\gamma(x_1)}-1\right)S^*\geq \left(\inf_{x\in\Omega}\frac{\beta(x)}{\gamma(x)}-1\right)S^*,\quad x\in\Omega.$$

This yields that

$$\left(\frac{I^*+S^*}{S^*}\right)^2\geq \left(\inf_{x\in\Omega}\frac{\beta(x)}{\gamma(x)}\right)^2>\sup_{x\in\Omega}\frac{\beta(x)}{\gamma(x)},\quad x\in\Omega,$$

where the assumption (1.4) is needed. Since $(S_k,I_k)\to(S^*,I^*)$ strongly in $C^1(\bar{\Omega})\times C^1(\bar{\Omega})$, we obtain for large k that

$$\left(\frac{I_k+S_k}{S_k}\right)^2>\sup_{x\in\Omega}\frac{\beta(x)}{\gamma(x)}\geq\frac{\beta(x)}{\gamma(x)},\quad x\in\Omega.$$

This contradicts to (3.1).

Case 1.2. $d=+\infty$. The proof is similar to Case 1.1. We only point out the difference. It is easy to show that $(S_k,I_k)\to(S^*,I^*)$ weakly in $W^{2,n+1}(\Omega)\times W^{2,n+1}(\Omega)$ and strongly in $C^1(\bar{\Omega})\times C^1(\bar{\Omega})$. Moreover, it is routine to check that both S^* and I^* are positive constants. Then obviously

$$\int_\Omega\left(\beta(x)\frac{S_k}{S_k+I_k}-\gamma(x)\right)I_k dx=0$$

guarantees that

$$\int_\Omega\left(\beta(x)\frac{S^*}{S^*+I^*}-\gamma(x)\right)I^* dx=0.$$

Therefore, we derive

$$\begin{aligned}\left(\frac{I^*+S^*}{S^*}\right)^2&=\left(\frac{\int_\Omega\beta dx}{\int_\Omega\gamma dx}\right)^2\\&\geq\left(\int_\Omega\left(\inf_{x\in\Omega}\frac{\beta}{\gamma}\right)\cdot\gamma(x)dx\right)^2\left(\int_\Omega\gamma(x)dx\right)^{-2}\\&=\left(\inf_{x\in\Omega}\frac{\beta(x)}{\gamma(x)}\right)^2>\sup_{x\in\Omega}\frac{\beta(x)}{\gamma(x)}.\end{aligned}$$

Therefore, for large k we can obtain

$$\left(\frac{I_k+S_k}{S_k}\right)^2>\sup_{x\in\Omega}\frac{\beta(x)}{\gamma(x)}\geq\frac{\beta(x)}{\gamma(x)},\quad x\in\Omega,$$

which is a contradiction to (3.1).

Case 1.3. $d=0$. It is routine to check that $d_S^{(k)}S+d_I^{(k)}I$ satisfies

$$\begin{cases} \Delta\big(d_S^{(k)}S_k+d_I^{(k)}I_k\big)=0, & x\in\Omega, \\ \dfrac{\partial\big(d_S^{(k)}S_k+d_I^{(k)}I_k\big)}{\partial\nu}=0, & x\in\partial\Omega. \end{cases}$$

Thus, there exists a constant $M_k>0$ such that $d_S^{(k)}S_k+d_I^{(k)}I_k=M_k$. By integrating over Ω, we get

$$M_k|\Omega|=\int_\Omega\big(d_S^{(k)}S_k+d_I^{(k)}I_k\big)dx\le d_S^{(k)}\int_\Omega(S_k+I_k)dx=d_S^{(k)}N.$$

Thus, we derive an uniform upper bound of S_k as follows:

$$S_k\le\frac{M_k}{d_S^{(k)}}\le\frac{N}{|\Omega|}.$$

Combining with

$$-d_I^{(k)}\Delta I_k+\gamma I_k\le\|\beta\|_{L^\infty(\Omega)}S_k, \tag{3.7}$$

we can derive $C_6>0$ independent of k such that

$$\|I_k\|_{L^\infty(\Omega)}\le C_6.$$

Based on the uniform bound of S_k and I_k, similar to Case 1.1, we can find a subsequence, still denoted by itself, and a constant $S^*\ge 0$ such that $S_k\to S^*$ weakly in $W^{2,n+1}(\Omega)$ and strongly in $C^1(\bar\Omega)$. Indeed, the constant S^* should be positive, since $S^*=0$ will lead to $I_k\to 0$ in $C(\bar\Omega)$ by using (3.7) and the boundary condition, and this contradicts to the fact that $\int_\Omega(S_k+I_k)dx=N$.

Fix $\varepsilon\in(0,1)$ small such that

$$\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^2\cdot\left(\inf_{x\in\Omega}\frac{\beta(x)}{\gamma(x)}\right)^2>\sup_{x\in\Omega}\frac{\beta(x)}{\gamma(x)}, \tag{3.8}$$

and then choose K_1 large such that

$$(1+\varepsilon)S^*\ge S_k\ge(1-\varepsilon)S^*,\quad k>K_1.$$

Then, for $k>K_1$, we have

$$d_I^{(k)}\Delta I_k+\left(\frac{\beta(1-\varepsilon)S^*}{(1-\varepsilon)S^*+I_k}-\gamma\right)I_k\le 0.$$

We construct a subsolution as follows:

$$\underline{I}:=(1-\varepsilon)\left(\inf_{x\in\Omega}\frac{\beta(x)}{\gamma(x)}-1\right)S^*.$$

It is routine to check that

$$d_I^{(k)}\Delta\underline{I}+\left(\frac{\beta(1-\varepsilon)S^*}{(1-\varepsilon)S^*+\underline{I}}-\gamma\right)\underline{I}\geq 0.$$

By comparison principle, we get

$$I_k\geq\underline{I}=(1-\varepsilon)\left(\inf_{\Omega}\frac{\beta(x)}{\gamma(x)}-1\right)S^*,\quad k\geq K_1.$$

Then it follows from the choice of ε in (3.8) that

$$\begin{aligned}\left(\frac{I_k+S_k}{S_k}\right)^2&\geq[(1+\varepsilon)S^*]^{-2}\left[(1-\varepsilon)\left(\inf_{\Omega}\frac{\beta(x)}{\gamma(x)}-1\right)S^*+(1-\varepsilon)S^*\right]^2\\&\geq\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^2\left(\inf_{\Omega}\frac{\beta(x)}{\gamma(x)}\right)^2>\sup_{x\in\Omega}\frac{\beta(x)}{\gamma(x)}\geq\frac{\beta(x)}{\gamma(x)},\quad x\in\Omega.\end{aligned}$$

This is a contradiction to (3.1). The proof of Theorem 1.2(1) is complete.

Case 2. Similar to the proof of Theorem 1.2(1), we assume by contradiction that there exist $d_S^{(k)}>0$ and $d_I^{(k)}>0$ with $\lim_{k\to\infty}d_S^{(k)}=d\in[0,\infty],\lim_{k\to\infty}d_I^{(k)}=\infty$ such that

$$\gamma(x)\leq\beta(x)\frac{{S_k}^2}{(S_k+I_k)^2}\quad\text{for some}\quad x_k\in\bar{\Omega},\tag{3.9}$$

where (S_k,I_k) denotes the unique endemic equilibrium of the problem

$$\begin{cases}d_S^{(k)}\Delta S-\beta(x)\dfrac{SI}{S+I}+\gamma(x)I=0, & x\in\Omega,\\ d_I^{(k)}\Delta I+\beta(x)\dfrac{SI}{S+I}-\gamma(x)I=0, & x\in\Omega,\\ \dfrac{\partial S}{\partial\nu}=\dfrac{\partial I}{\partial\nu}=0, & x\in\partial\Omega,\\ \displaystyle\int_{\Omega}(I+S)dx=N.\end{cases}$$

Now we try to derive contradictions according to the different situations of d.

Case 2.1. $d\in(0,\infty)$. Similar to Case 1.1, we can find a subsequence of (S_k,I_k), still denoted by itself, and $(S^*,I^*)\in W^{2,n+1}(\Omega)\times W^{2,n+1}(\Omega)$ such that $(S_k,I_k)\to(S^*,I^*)$ weakly in $W^{2,n+1}(\Omega)\times W^{2,n+1}(\Omega)$, strongly in $C^1(\bar{\Omega})\times C^1(\bar{\Omega}),S^*\geq 0,I^*\geq 0$ and (S^*,I^*) satisfy the following equation:

$$\begin{cases}d\Delta S^*-\left(\beta(x)\dfrac{S^*}{S^*+I^*}-\gamma(x)\right)I^*=0, & x\in\Omega,\\ I^*=\dfrac{1}{|\Omega|}\left(N-\displaystyle\int_{\Omega}S^*dx\right),\\ \dfrac{\partial S^*}{\partial\nu}=0, & x\in\partial\Omega,\\ \displaystyle\int_{\Omega}(I^*+S^*)dx=N.\end{cases}\tag{3.10}$$

Obviously, I^* is a constant.

We first verify that $S^*(x)>0$ in $\bar{\Omega}$ and $I^*>0$. Suppose that $I^*=0$, then by the problem (3.10), it is easy to see that

$$S^*=\frac{1}{|\Omega|}\int_\Omega S^* dx=\frac{N}{|\Omega|}.$$

According to the equation satisfied by I_k, one has

$$d_I^{(k)}\int_\Omega \frac{|\nabla I_k|^2}{I_k^2}dx+\int_\Omega\left(\beta(x)\frac{S_k}{S_k+I_k}-\gamma(x)\right)dx=0.$$

By letting $k\to\infty$, we have

$$\int_\Omega(\beta(x)-\gamma(x))dx\le 0.$$

This contradicts to the condition (1.4). Hence, the constant I^* is positive. Next, suppose that $S^*(x)$ touches zero somewhere in $\bar{\Omega}$. Then based on the problem (3.10), a contradiction can be derived by the maximum principle and the Hopf boundary lemma. Thus, $S^*(x)>0$ in $\bar{\Omega}$.

Now we consider the set of the maximum points of $S^*(x)$. If $S^*(x)<\max_{x\in\bar{\Omega}}S^*(x)$ for all $x\in\Omega$, one sees a contradiction by the Hopf boundary lemma. Then assume that there exists $x_2\in\Omega$ such that

$$S^*(x_2)=\max_{x\in\bar{\Omega}}S^*(x),\quad \left(\frac{\beta S^*}{S^*+I^*}-\gamma\right)(x_2)\le 0.$$

It follows that

$$\frac{I^*+S^*(x)}{S^*(x)}\ge\frac{I^*+S^*(x_2)}{S^*(x_2)}\ge\frac{\beta(x_2)}{\gamma(x_2)}\ge\inf_{x\in\Omega}\frac{\beta(x)}{\gamma(x)},\quad x\in\Omega,$$

where the fact that I^* is a constant has been used. This, together with the assumption (1.4), yields that

$$\left(\frac{I^*+S^*}{S^*}\right)^2\ge\left(\inf_{x\in\Omega}\frac{\beta(x)}{\gamma(x)}\right)^2>\sup_{x\in\Omega}\frac{\beta(x)}{\gamma(x)},\quad x\in\Omega.$$

A contradiction to (3.9) follows immediately since $(S_k,I_k)\to(S^*,I^*)$ strongly in $C^1(\bar{\Omega})\times C^1(\bar{\Omega})$ as $k\to\infty$.

Case 2.2. $d=+\infty$. This case is the same as Case 1.2 and the proof is omitted.

Case 2.3. $d=0$. Similar to the arguments in the proof of Case 1.3, we obtain that there exists a constant $I^*>0$ such that $I_k\to I^*$ weakly in $W^{2,n+1}(\Omega)$ and strongly in $C^1(\bar{\Omega})$ as $k\to\infty$. Then notice that the condition (1.4) guarantees that there exists $\varepsilon>0$ small enough such that

$$\left[(1+\varepsilon)^{-2}\left(\inf_{x\in\Omega}\frac{\beta(x)}{\gamma(x)}-1\right)+1\right]^2>\sup_{x\in\Omega}\frac{\beta(x)}{\gamma(x)},$$

and we can choose K_2 large enough such that

$$(1+\varepsilon)^{-1}I^*<I_k<(1+\varepsilon)I^*,\quad k>K_2. \tag{3.11}$$

Recall that

$$\begin{cases} d_S^{(k)}\Delta S_k-\beta(x)\dfrac{S_kI_k}{S_k+I_k}+\gamma(x)I_k=0, & x\in\Omega, \\ \dfrac{\partial S_k}{\partial\nu}=0, & x\in\partial\Omega. \end{cases}$$

Similar to the discussion in Case 2.1, we consider the set of the maximum sets of $S_k(x)$ and conclude that there exists $\hat{x}_k\in\Omega$ such that $S_k(\hat{x}_k)=\max\limits_{x\in\bar{\Omega}}S_k(x)$ and that

$$\frac{S_k(x)+I_k(\hat{x}_k)}{S_k(x)}\geq\frac{S_k(\hat{x}_k)+I_k(\hat{x}_k)}{S_k(\hat{x}_k)}\geq\frac{\beta(\hat{x}_k)}{\gamma(\hat{x}_k)}\geq\inf_{x\in\Omega}\frac{\beta(x)}{\gamma(x)},\quad x\in\Omega.$$

This, combined with (3.11), indicates that

$$\frac{(1+\varepsilon)^2I_k(x)}{S_k(x)}\geq\frac{(1+\varepsilon)I^*}{S_k(x)}\geq\frac{I_k(\hat{x}_k)}{S_k(x)}\geq\inf_{x\in\Omega}\frac{\beta(x)}{\gamma(x)}-1.$$

Then by direct computation, we get a contradiction to (3.9) as follows:

$$\left(\frac{I_k(x)+S_k(x)}{S_k(x)}\right)^2\geq\left[(1+\varepsilon)^{-2}\left(\inf_{x\in\Omega}\frac{\beta(x)}{\gamma(x)}-1\right)+1\right]^2>\sup_{x\in\Omega}\frac{\beta(x)}{\gamma(x)}.$$

Case 3. Since $\beta(x)>\gamma(x),x\in\bar{\Omega}$, we choose $\alpha_*(x)\in C^2(\bar{\Omega})$ such that

$$1<\sqrt{\frac{\beta(x)}{\gamma(x)}}<\alpha_*<\frac{\beta(x)}{\gamma(x)},\quad x\in\bar{\Omega},$$
$$\frac{\partial\alpha_*}{\partial\nu}=0,\quad x\in\partial\Omega.$$

Notice that

$$\begin{cases} -d_S\Delta\hat{S}-d_I\Delta\hat{I}=0, & x\in\Omega, \\ \dfrac{\partial\hat{S}}{\partial\nu}=\dfrac{\partial\hat{I}}{\partial\nu}=0, & x\in\partial\Omega, \end{cases}\tag{3.12}$$

thus there exists $C>0$ such that $d_S\hat{S}+d_I\hat{I}=C>0$. Without loss of generality, we assume that $C=1$, otherwise, we simply replace I and S by $\hat{I}/C$ and $\hat{S}/C$, respectively.

Now, due to the relation that $d_S\hat{S}+d_I\hat{I}=1$, the inequality

$$\left(\frac{\hat{S}(x)}{\hat{S}(x)+\hat{I}(x)}\right)^2<\frac{\gamma(x)}{\beta(x)},\quad x\in\bar{\Omega}$$

is equivalent to

$$\left(\frac{1-d_I\hat{I}(x)}{1+(d_S-d_I)\hat{I}(x)}\right)^2<\frac{\gamma(x)}{\beta(x)},\quad x\in\bar{\Omega}.\tag{3.13}$$

Also it is routine to check that $\hat{I}$ satisfies

$$\begin{cases} d_I\Delta\hat{I}+\left(\beta(x)\dfrac{1-d_I\hat{I}}{1+(d_S-d_I)\hat{I}}-\gamma(x)\right)\hat{I}=0, & x\in\Omega,\\ \dfrac{\partial\hat{I}}{\partial\nu}=0, & x\in\partial\Omega. \end{cases} \tag{3.14}$$

Define

$$I_*:=\frac{\alpha_*(x)-1}{d_I(\alpha_*(x)-1)+d_S}.$$

We claim that there exists $\delta>0$ sufficiently small such that I_* satisfies

$$\begin{cases} d_I\Delta I_*+\left(\beta(x)\dfrac{1-d_I I_*}{1+(d_S-d_I)I_*}-\gamma(x)\right)I_*\geq 0, & x\in\Omega,\\ \dfrac{\partial I_*}{\partial\nu}=0, & x\in\partial\Omega. \end{cases}$$

provided that $d_S>0, d_I\leq\delta$.

Assume that the claim is valid. Observe that

$$\beta(x)\frac{1-d_I I}{1+(d_S-d_I)I}-\gamma(x)$$

is decreasing in $0<I<1/d_I$. Then by the comparison principle, it is standard to verify that $I_*\leq\hat{I}$ in $\bar{\Omega}$ and thus

$$\left(\frac{1-d_I\hat{I}(x)}{1+(d_S-d_I)\hat{I}(x)}\right)^2\leq\left(\frac{1-d_I I_*(x)}{1+(d_S-d_I)I_*(x)}\right)^2,\quad x\in\Omega.$$

Moreover, according to the choices of α_* and I_*, we have

$$\left(\frac{1-d_I I_*(x)}{1+(d_S-d_I)I_*(x)}\right)^2=\alpha_*^{-2}<\frac{\gamma(x)}{\beta(x)},\quad x\in\bar{\Omega}.$$

Therefore, (3.13) is valid when $d_S>0, d_I\leq\delta$ and the desired conclusion follows.

It remains to verify the claim. Direct computation yields that

$$\begin{aligned} d_I|\Delta I_*| &= d_I\left|\Delta\left(\frac{\alpha_*(x)-1}{d_I(\alpha_*(x)-1)+d_S}\right)\right| \\ &= \left|\Delta\left(1-\frac{d_S}{d_I(\alpha_*-1)+d_S}\right)\right| \\ &= \left|\frac{d_S d_I\Delta\alpha_*}{[d_I(\alpha_*-1)+d_S]^2}-\frac{2d_S d_I^2|\nabla\alpha_*|^2}{[d_I(\alpha_*-1)+d_S]^3}\right| \\ &= \left|\frac{d_S d_I(\alpha_*-1)}{[d_I(\alpha_*-1)+d_S]^2}\cdot\left(\frac{\Delta\alpha_*}{\alpha_*-1}-\frac{d_I(\alpha_*-1)}{d_I(\alpha_*-1)+d_S}\cdot\frac{2|\nabla\alpha_*|^2}{(\alpha_*-1)^2}\right)\right| \end{aligned}$$

$$\leq d_I I_* \left(\frac{\|\alpha_*\|_{C^2(\bar{\Omega})}}{\alpha_*-1} + \frac{2\|\alpha_*\|^2_{C^2(\bar{\Omega})}}{(\alpha_*-1)^2} \right), \tag{3.15}$$

and

$$\begin{aligned} &\left(\beta(x)\frac{1-d_I I_*}{1+(d_S-d_I)I_*}-\gamma(x)\right)I_* \\ =&\beta(x)\left(\frac{1-d_I I_*}{1+(d_S-d_I)I_*}-\frac{\gamma(x)}{\beta(x)}\right)I_* = \beta\left(\frac{1}{\alpha_*(x)}-\frac{\gamma(x)}{\beta(x)}\right)I_*. \end{aligned} \tag{3.16}$$

Therefore, we can choose $\delta > 0$ small such that for $d_I \leq \delta$,

$$d_I \left(\frac{\|\alpha_*\|_{C^2(\bar{\Omega})}}{\alpha_*-1} + \frac{2\|\alpha_*\|^2_{C^2(\bar{\Omega})}}{(\alpha_*-1)^2} \right) < \beta(x)\left(\frac{1}{\alpha_*(x)}-\frac{\gamma(x)}{\beta(x)}\right), \quad x \in \bar{\Omega}.$$

This, together with (3.15) and (3.16), completes the proof of the claim. □

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